Three Parameter Laplace Type Bimodal Distribution

D V Ramana Murty[1], G Arti[2] and M. Vivekananda Murty[3]
[1]*Department of Statistics, VT College, Rajahmundry, East Godavari District, Andhra Pradesh, India.*
[2]*Department of Management, GITAM (Deemed to be University), Visakhapatnam, India*
[3]*Former Professor of Statistics, Andhra University, Visakhapatnam, India.*

Abstract - *This paper is on the Three parameter Laplace type Bimodal distribution. After discussing distributional properties, order statistics were developed and discussed. Inferential aspects were discussed and estimates of the parameters were obtained through Method of Moments and Maximum Likelihood Estimation techniques. Minimum unbiased estimator of the location parameter and best linear unbiased estimator of the location and scale parameter were also obtained.*

I. INTRODUCTION

The Laplace distribution has received considerable attention as an appropriate model in reliability theory and life testing models.The statistical data in the fields like agriculture, meteorology and population studiesare appearing as if it is generated from a Laplace distribution, but have the kurtosis lies between 3 and 6. In this paper we introduce a three parameter Laplace type bimodal distribution which suits the distribution arising out of the situations mentioned above. The various distributional properties and inferential aspects of this distribution are discussed.

II. THREE PARAMETER LAPLACE TYPE BIMODAL DISTRIBUTION

A random variable is said to follow a three parameter Laplace type bimodal distribution, if its probability density function is of the following form.

$$f(x,\mu,\beta)=\frac{\left(\frac{x-\mu}{\beta}\right)^{2r}}{2\beta[(2r)!]}e^{-\left|\frac{x-\mu}{\beta}\right|},\quad -\infty<x<\infty,\qquad -\infty<\mu<\infty,\quad \beta>0 \qquad \longrightarrow 1$$

For different values of $r=0,1,2,\ldots\infty$ we have different continuous distributions with three parametersμ,β and r($r$can be treated as an index parameter) which can be used to determine the specific distribution. This family includes Laplace distribution, when $r=0$[Johnson and Kotz (1972)]. Making the transformations,$y=\frac{x-\mu}{\beta}$ in (1), we get

$$f(y)=\frac{y^{2r}e^{-y}}{2\beta[(2r)!]},\quad -\infty<y<\infty \qquad \rightarrow 2$$

The distribution given in (1) is symmetric about μ and it is a generalized Laplace type distribution with three parameters.

III. PROPERTIES OF THREE PARAMETER LAPLACE TYPE BIMODAL DISTRIBUTION:

The distributions belonging to this family are having the following characteristics.
The Mean of the distribution is

$$E(X)=\int_{-\infty}^{\infty}xf(x,\mu,\beta)\,dx=\int_{-\infty}^{\infty}x\frac{\left(\frac{x-\mu}{\beta}\right)^{2r}}{2\beta[(2r)!]}e^{-\left|\frac{x-\mu}{\beta}\right|}\,dx$$

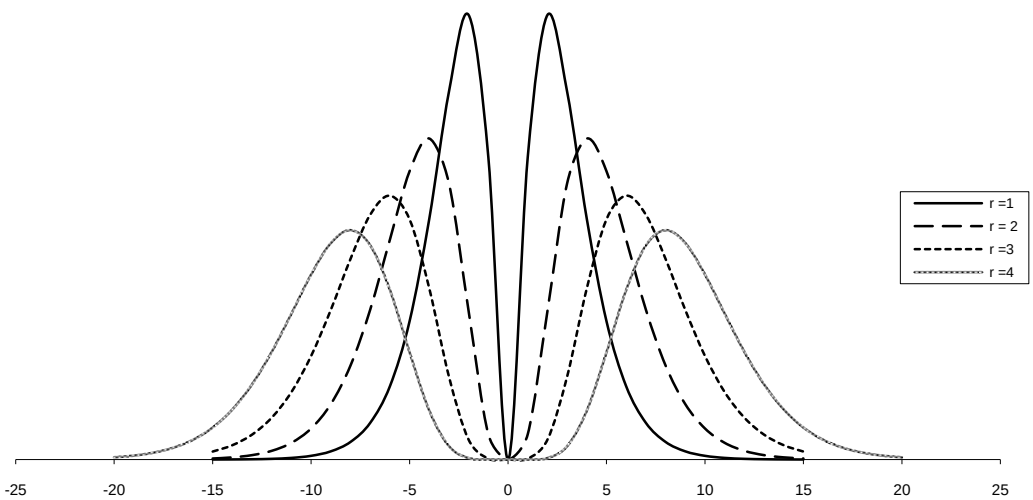
On simplification, one can get Mean $=\mu$
Distribution is symmetric aboutμ, which is the mean.
The Median of the distribution M is such that

$$\int_{-\infty}^{M} f(x)\,dx + \int_{M}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{M} x\frac{\left(\frac{x-\mu}{\beta}\right)^{2r}}{2\beta[(2r)!]}e^{-\left|\frac{x-\mu}{\beta}\right|}\,dx + \int_{M}^{\infty} x\frac{\left(\frac{x-\mu}{\beta}\right)^{2r}}{2\beta[(2r)!]}e^{-\left|\frac{x-\mu}{\beta}\right|}\,dx = 1$$

Solving the equation one can have Median $= M = \mu$
For the generalized three parameter Laplace type bimodal distribution, mean and median are equal.
For obtaining mode of the distribution, taking the derivatives of (1) with respect to x and equating to zero and solving for x, we get the modes as $\mu \pm 2r\beta$ ie. $\mu - 2\beta$ and $\mu + 2\beta$
Therefore, this distribution is bimodal.

Fig - 4.1



Distribution function is obtained as

$$F_X(x) = \int_{-\infty}^{x} \frac{\left(\frac{x-\mu}{\beta}\right)^{2r}}{2\beta[(2r)!]}e^{-\left|\frac{x-\mu}{\beta}\right|}\,dx$$

On simplification, one can get

$$F_X(x) = 1 - \frac{1}{2}\sum_{i=0}^{2r}\frac{\left(\frac{x-\mu}{\beta}\right)^{i}}{i!}e^{-\left(\frac{x-\mu}{\beta}\right)}, \qquad for\ x \geq \mu$$

$$= \frac{1}{2}\sum_{i=0}^{2r}\frac{\left(\frac{x-\mu}{\beta}\right)^{i}}{i!}e^{-\left(\frac{x-\mu}{\beta}\right)}, \qquad for\ x < \mu \qquad \longrightarrow 3$$

The central moments of this distribution are

$$\mu_{2n}\int_{-\infty}^{\infty}(x-\mu)^{2n}f(x)\,dx = \int_{-\infty}^{\infty}(x-\mu)^{2n}\frac{\left(\frac{x-\mu}{\beta}\right)^{2r}}{2\beta[(2r)!]}e^{-\left|\frac{x-\mu}{\beta}\right|}\,dx$$

Making necessary transformation and integrating, one can get

$$\mu_{2n} = \frac{(2r+2n)!\,\beta^{2n}}{\beta[(2r)!]}, \mu_{2n+1} = 0 \qquad \longrightarrow 4$$

In particular, the first four central moments of this distribution are

$$\mu_1 = 0$$

$$\mu_2 = \frac{(2r+2)!\,\beta^2}{(2r)!}$$

$$\mu_3 = 0$$

$$\mu_4 = \frac{(2r+4)!\,\beta^4}{(2r)!} = \frac{(2r+4)(2r+3)(2r+2)!\,\beta^4}{(2r)!}$$

Thus, the variance of this distribution is $\mu_2 = \frac{(2r+2)!\beta^2}{(2r)!}$

Since, the distribution is symmetric, its skewness is zero.

The recurrence relation between the Central moments are

$$\mu_{2n} = \frac{(2r+2n)!\,\beta^2}{\beta[(2r)!]}\mu_{2n-2} \qquad \longrightarrow 5$$

The pth order absolute moments of the distribution are

$$E[|X-\mu|^p] = \int_{-\infty}^{\infty} |x-\mu|^p \frac{\left(\frac{x-\mu}{\beta}\right)^{2r}}{2\beta[(2r)!]} e^{-\left|\frac{x-\mu}{\beta}\right|}\, dx$$

On simplification, one can get,

$$E[|X-\mu|^p] = \frac{(2r+p)!\,\beta^p}{(2r)!} \qquad \longrightarrow 6$$

The characteristic function of this distribution is

$$\emptyset_X(t) = E[e^{itx}] \int_{-\infty}^{\infty} e^{itx} f(x)\, dx = \int_{-\infty}^{\infty} e^{itx} \frac{\left(\frac{x-\mu}{\beta}\right)^{2r}}{2\beta[(2r)!]} e^{-\left|\frac{x-\mu}{\beta}\right|}\, dx$$

Using the transformations $y = x - \mu$,and by simplification one can have

$$\emptyset_X(t) = \frac{e^{it\mu}\,(2r)!}{2}\left[\frac{1}{(1+i\beta t)^{2r+1}} + \frac{1}{(1-i\beta t)^{2r+1}}\right] \qquad \longrightarrow 7$$

Kurtosis of this distribution is

$$\beta_2 = \frac{\mu_4}{\mu_2^2} \; where\; \mu_2 = \frac{(2r+2)!\,\beta^2}{(2r)!} \; and\; \mu_4 = \frac{(2r+4)!\,\beta^4}{(2r)!}$$

Therefore,$\beta_2 = \frac{(2r+4)(2r+3)}{(2r+2)!}$

The distribution of the square of the variate:

Let $Y = X^2$ and $\beta = 1, \mu = 0$, then the probability density function of Y is,

$$f(y) = [(2r)!]^{-2r} \sum_{i=0}^{r} y^{i-\frac{1}{2}} e^{-\sqrt{y}},$$

by transforming$\sqrt{y} = z, f(z) = \sum_{i=0}^{r} 2^{[(2r)!]^{-2r}} z^{2i} e^{-z}$,

Which is a mixture ofγ- gamma variate with parameters $z^i, i = 0,1,2, \dots r$. and weights

$$w_i = 2(2i)!\,[(2r)!]^{-2r}$$

IV. ORDER STATISTICS OF THREE PARAMETER LAPLACE TYPE BIMODAL DISTRIBUTION

The simple explicit form of the distribution function as given in (3) leads us to derive the Order statistics connected with this three parameter Laplace type bimodal distribution. For simplicity let us assume$\mu = 0\; and\; \beta = 1$ then the probability density function (1) reduces to

$$f(x) = \frac{(x)^{2r} e^{-|x|}}{2[(2r)!]}, \qquad -\infty < x < \infty \qquad \longrightarrow 8$$

Let $X_{1:n} \le X_{2:n} \le \cdots X_{n:n}$denote the Order statistics obtained from a random sample of size n from the standardized Laplace type bimodal distribution having the probability density function of the form given (8).

The probability density function of mth order statistics is given by

$$f_{m:n}(x) = D_{m:n} \frac{(x)^{2r} e^{-|x|}}{2[(2r)!]} \sum_{q=0}^{n-m} \binom{n-m}{q} (-1)^q \left[1 - \frac{e^{-x}(2r)! \sum_{i=0}^{2k} \frac{x^i}{i!} e^{-|x|}}{2[(2r)!]}\right]^{m+q-1} \quad for\; x \ge 0$$

$$f_{m:n}(x) = D_{m:n}\frac{(x)^{2r}e^{-|x|}}{2[(2r)!]}\sum_{q=0}^{n-m}\binom{n-m}{q}(-1)^q\left[\frac{e^{-x}(2r)!\sum_{i=0}^{2k}\frac{x^i}{i!}e^{-|x|}}{2[(2r)!]}\right]^{m+q-1} \quad for\ x<0$$

Where $D_{m:n} = m\binom{m}{n}$ $\longrightarrow 9$

The a th moment of $X_{m:n}$ is given by

$$\alpha^{(a)}{}_{m:n}$$

$$= D_{m:n}(-1)^a\sum_{q=0}^{n-m}\binom{n-m}{q}(-1)^q\sum_{j_1>j_2>\cdots j_r}^{m+q-1}\binom{m+q-1}{j_1}[D(0)]^{m+q-1-j_1}\left[\prod_{i=1}^{r-1}[D(i)]^{j_i-j_{i+1}}\binom{j_i}{j_{i+1}}\right][D(r)]^{j_r}\sum_j P_j\frac{R!}{(m+q)^{R+1}}$$

$$+D'_{m:n}\sum_{l=0}^{m-1}\binom{m-1}{l}(-1)^l\sum_{j_1>j_2>\cdots j_r}^{n+l-m}\binom{n+l-m}{j_1}[D(0)]^{n+l-m-j_1}\left[\prod_{i=1}^{r-1}[D(i)]^{j_i-j_{i+1}}\binom{j_i}{j_{i+1}}\right][D(r)]^{j_r}\sum_j P_j\frac{R!}{(n+l-m)^{R+1}}$$

Where $R = \sum_{y=1}^{r}\sum_{x=1}^{2_i} j_x + 2r + a, D'_{m:n} = \frac{D_{m:n}}{2(2r)!}, D(r) = \frac{(2r)!}{2(2r)!}$

And $P_j = \prod_{r_1=1}^{r-1}\binom{a}{j_{r_1}}\prod_{i=1}^{2r-1}\binom{j_{r_i}}{j_{r_i+1}}\prod_{i=1}^{2r}(i)^{-j_{r_i}}\binom{j_{2r}}{j_1}\prod_{i=1}^{2r-1}\binom{j_i}{j_{i+1}}\prod_{i=1}^{2r}(i)^{-j_i}$ $\longrightarrow 10$

From this (10) one can calculate the expected values of Order statistics.

Distribution of Median:

To obtain the distribution of Median, substitute $m = \frac{n+1}{2}$ if n is odd in (9).

Thus, one can get

$$f_{\frac{n+1}{2}:n}(x) = D_{\frac{n+1}{2}:n}\frac{(x)^{2r}e^{-|x|}}{2[(2r)!]}\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[1-\frac{e^{-|x|}(2r)!\sum_{i=0}^{2r}\frac{x^i}{i!}}{2[(2r)!]}\right]^{\frac{n+2q-1}{2}} \quad for\ x\geq 0$$

$$= D_{\frac{n+1}{2}:n}\frac{(x)^{2r}e^{-|x|}}{2[(2r)!]}\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{e^{-|x|}(2r)!\sum_{i=0}^{2r}\frac{x^i}{i!}}{2[(2r)!]}\right]^{\frac{n+2q-1}{2}} \quad for\ x<0$$

Where $D_{\frac{n+1}{2}:n} = \frac{n!}{\left[\left(\frac{n-1}{2}\right)!\right]^2}$ $\rightarrow 11$

when n is even then $m = \frac{2n+1}{4}$

$$f_{\frac{2n+1}{2}:n}(x) = D_{\frac{2n+1}{2}:n}\frac{x^{2r}e^{-|x|}}{2[(2r)!]}\sum_{q=0}^{\frac{2n-1}{4}}\binom{\frac{2n-1}{4}}{q}(-1)^q\left[1-\frac{e^{-|x|}(2r)!\sum_{i=0}^{2r}\frac{x^i}{i!}}{2[(2r)!]}\right]^{\frac{2n+4q-3}{4}} \quad for\ x\geq 0$$

$$= D_{\frac{2n+1}{2}:n}\frac{x^{2r}e^{-|x|}}{2[(2r)!]}\sum_{q=0}^{\frac{2n-1}{4}}\binom{\frac{2n-1}{4}}{q}(-1)^q\left[\frac{e^{-|x|}(2r)!\sum_{i=0}^{2r}\frac{x^i}{i!}}{2[(2r)!]}\right]^{\frac{2n+4q-3}{4}} \quad for\ x<0$$

$\rightarrow 12$

Joint Moments of the Order Statistics

The joint probability distribution of the Order Statistics $X_{m:n}$ and $X_{s:n}, m<s$ is given by

$f_{m,s:n}(x) = D_{m,s:n}[U(x)]^{m-1}[U(y)-U(x)]^{s-m-1}[1-U(y)]^{n-s}f(x)f(y)$

Where $U(x) = \frac{x^{2r}(2r)!e^{-|x|}\sum_{i=0}^{2r}\frac{x^i}{i!}}{2[(2r)!]}$ $\rightarrow 13$

Partition the range $0<x<y<\infty$ in to three mutually exclusive regions

$R_1: [(x,y): -\infty<x<y<0]$

$R_2: [(x,y): 0<x<y<\infty]$

$R_3: [(x,y): -\infty<x<0, 0<y<\infty]$

the product moments can be obtained as

$$E[X_{m:n}, X_{s:n}] = \int_{R_1 \cup R_2 \cup R_3} f(x,y)dxdy$$

$$= D_{m,s:n}\left[\sum_{i=0}^{s-m-1}\sum_{j=0}^{n-s}\binom{s-m-1}{i}\binom{n-s}{j}(-1)^{s-m-1+i+j}\Psi(s-2-i, i+j)\right.$$

$$+\sum_{i=0}^{m-1}\sum_{j=0}^{s-m-1}\binom{s-m-1}{j}\binom{m-1}{j}(-1)^{i+j}\Psi(s-m-1+i+j, n-s+j)$$

$$\left.-\sum_{i=0}^{s-m-1}\sum_{j=0}^{n-s}\binom{s-m-1-i}{j}\binom{s-m-1}{i}(-1)^{i+j}\int_0^x x[U(x)]^{m+i-1}f(x)dx\int_0^\infty y[U(y)]^{n-s-j}f(y)dy\right]$$

where

$$\Psi(a,b) = \int_0^\infty\int_0^y xy[U(x)]^a\,[U(y)]^b f(x)f(y)dxdy$$

$$= \int_0^\infty\int_0^y (2r)!^{-2}\sum_{v=0}^{s-m-1}\sum_{u=0}^{n-s}\binom{s-m-1}{v}\binom{n-s}{u}(-1)^{u+v}\sum_{j}^{m-i+v}\sum_{t}^{s-m-1-v-u}P(j)Q(t)\sum_{N=0}^{r}\binom{r}{N}$$

$$e^{(m-1+v)x+(1-m-1-v+u)y}x^{2r+1+\sum_{i=1}^{r}\sum_{l=0}^{j_i}j_{il}}y^{2r+1+\sum_{i=1}^{r}\sum_{l=0}^{2i}t_{il}}$$

$$P(j) = \binom{a}{j_1}\prod_{i=1}^{r-1}\left[\binom{j_i}{j_{i+1}}\right][(2r)!]^{j_i-j_{i+1}}\binom{j_i-j_{i+1}}{j_i}[(2r)!]^{j_r}\left\{\prod_{i=1}^{r}\left[\prod_{i=1}^{2i-j}\binom{j_i}{j_{i+1}}\right]\left[\prod_{i=1}^{2i}(l)^{-j_{il}}\right]\right\}\binom{j_r}{j_{rl}}$$

$$Q(t) = \binom{b}{t_1}\prod_{i=1}^{r-1}\left[\binom{t_i}{t_{i+1}}\right][(2r)!]^{t_i-t_{i+1}}\binom{t_i-t_{i+1}}{t_i}[(2r)!]^{t_r}\left\{\prod_{i=1}^{r}\left[\prod_{l=1}^{2i-1}\binom{t_{il}}{t_{il+1}}\right]\left[\prod_{i=1}^{2i}(l)^{-t_{il}}\right]\right\}\binom{t_r}{t_{rl}}$$

→14

V. ESTIMATION OF PARAMETERS:

Previously we have studied three parameter generalized Laplace type bimodal distribution and its distributional properties. Another aspect of any distributional study is to look in to the inferential aspects of the distribution, in particular the estimation of the parameters involved in the distribution under study. In this we will discuss the various methods of estimation by using Method of moments, maximum likelihood method of estimation, Best linear unbiased estimation in estimating the parameters of the three parameter Laplace type bimodal distribution and specially by using Numerical Analysis techniques namely Newton Raphson' s method & iterative method. The asymptotic behavior of these estimators is also studied. In this distribution the value of r is fixed, based on the Kurtosis of the distribution. After identifying r we estimate these moving parameters $\mu$and β.

A. Method of Moments:

According to this method, the moments of the population and the sample are equated correspondingly to deduce the estimators of the parameters. Let us consider a sample of size 'n' drawn from a population having the probability density function of the form given by

$$f(x,\mu,\beta) = \frac{\left(\frac{x-\mu}{\beta}\right)^{2r}}{2\beta[(2r)!]}e^{-\left|\frac{x-\mu}{\beta}\right|}$$

This distribution is having two parametersμ & β and r is identified through the sample kurtosis. Hence, we consider the first two moments of the sample and the population, which leads to the following equations.

$\mu = \bar{x},$

$\frac{(2r+2)!\,\beta^2}{(2r)!} = s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$→15

These equations give us the moment estimators as

$$\hat{\mu} = \bar{x}$$

$$and\ \hat{\beta}^2 = \frac{(2r)!\,s^2}{(2r+2)!}$$

where s^2 isthe sample variance.
The variance of $\hat{\mu}$ is

$$Var(\bar{x}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(x_i) = \frac{1}{n^2}\sum_{i=1}^{n}\frac{(2r)!\,s^2\,\beta^2}{(2r+2)!} = \frac{\beta^2}{n}\frac{(2r+2)!}{(2r)!} \qquad \rightarrow 16$$

and $\frac{1}{n}\sum_{i=1}^{n} E(x_i) = \mu$
That is the sample mean is an unbiased estimator ofμ→17
The unbiased estimator of $\beta^2$is $\hat{\beta}^2 = \frac{(2r)!s^2}{(2r+2)!}$ →18
where s^2 is the sample variance.
The variance of the estimators are given by
$Var(\hat{\mu}) = \frac{\beta^2}{n}\frac{(2r+2)!}{(2r)!}$ →19
and the unbiased estimator of β^2,

$$Var(\widehat{\beta^2}) = \beta^4\left[\frac{(2r)!}{(2r+2)!}\right]^2\left\{\left[\frac{(2r+4)!}{(2r)!}\right]\left(\frac{1}{n}-\frac{2}{n^2}-\frac{1}{n^3}\right)-\left[\left[\frac{(2r+4)!}{(2r)!}\right]^2\right]\left[\frac{1}{n}-\frac{4}{n^2}-\frac{3}{n^3}\right]\right\}$$

→20

As the variance tends to zero when n tends to ∞, the estimators $\bar{x}$ and$s^2$are consistent for μ and β^2 Similar to the simple Laplace distribution the median is the maximum likelihood estimators of the parameter μ. Hence estimation of all the parameters can be done sequentially withμbeing estimated by the median and then the other two parameters can be simultaneously estimated conditioned to $\mu = \hat{\mu}$

B. Maximum Likelihood Method of Estimation:

Let$x_1, x_2, \ldots . x_n$ be a sample of size n drawn from a population having the probability density function of the form given in equation (1). From equation (1) one can write the likelihood function of the sample is

$$L = \frac{\prod_{i=1}^{n}\left[\left(\frac{x-\mu}{\beta}\right)^{2r}\right]e^{-\left|\frac{x-\mu}{\beta}\right|}}{2\beta\,[(2r)!]} \qquad \rightarrow 21$$

Taking logarithms on both sides of (4.6.1) we get

$$logL = n + \frac{2r}{\beta}\sum_{i=0}^{n}\frac{(x_i-\mu)}{\left(\frac{x_i-\mu}{\beta}\right)^2} - \frac{1}{\beta}\sum_{i=1}^{n}|x_i-\mu| = 0 \qquad \longrightarrow 22$$

For obtaining the maximum likelihood estimators of the parameters we have to maximize L or $\log L$ with respect to the parametersμ andβ. The values of μ andβ can be obtained by solving the following equations.

$$\frac{(2r)!}{(2r+2)!} - r\sum_{i=0}^{n}\frac{1}{\left(\frac{x_i-\mu}{\beta}\right)^2} = 0$$

That is $\sum_{i=0}^{n}\frac{1}{\left(\frac{x_i-\mu}{\beta}\right)^2} - 2\beta[(2r)!] = 0$ →23

C. Best Linear Unbiased Estimators:

The best linear unbiased estimators of the location and scale parameters involved in the three parameter Laplace type bimodal distribution having the probability density function of the form given in (1) are obtained as follows. For that consider the family of distributions given in equation (1)
Let $X'_{m:n}$ and$X_{m:n}$be the m th order statistics drawn from the populations having the probability density function of the form given above and the corresponding standardized distribution respectively.
ie.$X_{m:n} = \frac{[X'_{m:n}-\mu]}{\beta}$, $1 \le m \le n$
Then the best linear unbiased estimators of μ and β are given by [Lioyd (1952)].

$$\mu^* = \frac{l'\Omega X}{l'\Omega l} \; and \; \beta^* = \frac{\alpha'\Omega X}{\alpha'\Omega\alpha}$$

where $l' = (1,1,\dots,1)$, $X' = [X_{1:n},\ X_{2:n},\dots X_{n:n}]$ and $\alpha' = [\alpha_{1:n},\ \alpha_{2:n},\dots \alpha_{n:n}]$

where,$\alpha_{1:n}$is the first moment of thei^{th} order statistics and Ω is the variance matrix of vector X

The variance of these estimators are $var(\mu^*) = \frac{\sigma}{l'\Omega l}, var(\beta^*) = \frac{\sigma}{\alpha'\Omega\alpha}$and $cov(\mu^*, \beta^*) = 0$

With the moments of the order statistics given in section 4 and 5, one can compute the$\mu^*$and$\beta^*$and their variances respectively.

REFERENCES:

[1] Cramer, H. (1946): Mathematical Methods of Statistics, Asia Publishing House.
[2] David. H.A. (1981): Order Statistics, John Wiley & Sons, Ins. New York.
[3] Eisenberger, I. (1964): 'Genesis of Bimodal distributions' techno metrics, p.357-363., vol.6.
[4] Feller, W. (1966). An Introduction to Probability Theory and Its Applications, Vol.I I, New York, John Wiley & sons, NC.
[5] Govindarajulu, Z. (1966): Best linear estimates under symmetric censoring of the parameters of a double exponential population, journal of the Statistical Association,61.
[6] Johnson, N.L. And kotz, S. (1970): Continuous Univariate Distributions, john Wiley Sons, New York.
[7] Kameda, T. (1928): On the reduction of frequency curves, SckandinaviskAktuarietidskrift, 11.
[8] Laplace, P.S. (1774): Memoire sur la pobabilite des causes par les evenemens, Memoiries de mathematique et de Physique, 6.
[9] Lloyd, E.H. (1952): Least square estimation of location and scale parameters using order statistics, Biometrica, 39.
[10] Rao, C.R. (1965): Linear Statistical Inference and its applications, Wiley Eastern.
[11] Rao, K.S. et al. (1988): A bimodal distribution, Bulletin.Cal.math. Soc.,80, No.4.
[12] Sharma, P.V.S, Rao, K.S.S, And Rao, R.P (1990): "On a family of bimodal distributions" Sankhya 52,3,287-292.
[13] D V Ramana Murty, G Arti and M. Vivekananda Murty, "Two Parameter Laplace Type Bimodal Distribution" (2018), International Journal of Mathematics Trends and Technology, May 2018.