

# Estimation of Integrals Associated with Generalized Galue' Type Struve Function

Mohd. Saif<sup>#1</sup>, Kottakkaran Sooppy Nisar<sup>\*2</sup>, Abdul Hakim Khan<sup>#3</sup>

Research Scholar, Mohd. Saif, Department of Applied Mathematics, Aligarh Muslim University,  
Aligarh-202002, UP, India

Assistant Professor, K. S. Nisar Department of Mathematics, College of Arts and Science at Wadi Al-dawaser  
Prince Sattam bin Abdulaziz University,  
Riyadh region 11991, Saudi Arabia  
Professor, A.H. Khan, Department of Applied Mathematics, Aligarh Muslim University  
Aligarh-202002, UP, India

**Abstract -** The aim of this paper is to derive some fascinating integrals involving the generalized Galue' type Struve function in term of Fox-Wright function. Based on the new results, some integral formulas involving different special functions are established as special cases of our main result for different values of parameters.

**Keyword -** Fox-Wright function, Galue' type Struve function, Mittag Leffler function, Wright function, Struve function [2010]33B15, 33C10, 33C15.

## I. INTRODUCTION

In 1882, Herman Struve investigated the Struve function of order  $p$  which is defined as:

$$H_p(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+3/2)\Gamma(k+p+3/2)} \left(\frac{z}{2}\right)^{2k+p+1}. \quad (1.1)$$

for all  $z \in \mathbb{C}$ , which is a particular solution of non-homogeneous differential equation

$$z^2 w''(z) + zw'(z) + (z^2 - p^2)w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+1/2)}, \quad (1.2)$$

and its homogeneous part is Bessel equation. Since, several generalization of Struve function have been developed by many author's, see for references ([5, 6, 14, 15, 16, 22]). For more details about Struve function and it various applications one may be referred to the recent papers[1, 2, 18, 19, 20, 23]. A useful generalization of Struve function called Galue' type Struve function given in [17] and is defined as:

$$a' W_{p,b',c',\delta}^{\lambda,\mu}(z) = \sum_{k=0}^{\infty} \frac{(-c')^k \left(\frac{z}{2}\right)^{2k+p+1}}{\Gamma(\lambda k + \mu) \Gamma(a' k + \frac{p}{\delta} + \frac{b'+2}{2})}, \quad (a' \in \mathbb{N}, p, b', c' \in \mathbb{C}), \quad (1.3)$$

where  $\lambda > 0, \delta > 0$  and  $\mu$  is an arbitrary parameters.

For  $\lambda = a' = 1, \mu = 3/2$  and  $\delta = 1$ , (1.3) gives a generalization of Struve function, which is defined as:

$$H_{p,b',c'}(z) = \sum_{k=0}^{\infty} \frac{(-c')^k}{\Gamma(k+3/2)\Gamma(k+p+\frac{b'+2}{2})} \left(\frac{z}{2}\right)^{2k+p+1} \quad (b', c' \in \mathbb{C}). \quad (1.4)$$

The series representation of Bessel function of the first kind of order  $v$  is defined as (see [9]):

$$J_v(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+v+1)} \left(\frac{x}{2}\right)^{2k+v}. \quad (1.5)$$

L. Galue' [10] introduced a generalization of the Bessel function of order  $p$  which is defined as:

$$a J_v(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(a' k + v + 1) k!} \left(\frac{x}{2}\right)^{2k+p}, \quad (x \in \mathbb{R}, a' \in \mathbb{N}). \quad (1.6)$$

In 2010, Baricz [4] investigated Galue-type generalization of modified Bessel function given as:

$$a' I_p(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(a' k + p + 1) k!} \left(\frac{z}{2}\right)^{2k+p} \quad (z \in \mathbb{C}, a' \in \mathbb{N}). \quad (1.7)$$

The Wright function ([12, 13]) introduced for the first time by E. M. Wright is defined by the series representation as:

$$W_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta) k!} \quad (\alpha > -1, \beta \in \mathbb{C}). \quad (1.8)$$

The Fox-Wright function [24] is defined by the series representation, valid in the whole complex plane

$${}_q\Psi_s[z] = {}_q\Psi_s \left[ \begin{matrix} (\alpha_1, A_1), \dots, (\alpha_q, A_q) \\ (\beta_1, B_1), \dots, (\beta_s, B_s) \end{matrix} ; z \right] = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 k) \dots \Gamma(\alpha_q + A_q k)}{\Gamma(\beta_1 + B_1 k) \dots \Gamma(\beta_s + B_s k)} \frac{(z)^k}{k!} \quad (1.9)$$

$\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_s \in \mathbb{R}$  such that  $1 + \sum_{i=1}^s B_i - \sum_{r=1}^q A_r > 0$ ,  
where  $\Gamma(z)$  denotes the gamma function and q and s are non negative integers.

The Mittag-Leffler function  $E_{\alpha, \beta}(z)$  is a special function which depends on two complex parameter  $\alpha$  and  $\beta$  which is defined by [25] the following power series:

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha, \beta \in \mathbb{C}, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0). \quad (1.10)$$

Also we recall the following interesting and useful results.

- The results established by A. Erdelyi et.al. [8]

$$\int_0^1 x^{c-1} (1-x)^{-\frac{1}{2}} {}_2F_1(a, b; a+b+1/2; x) dx = \frac{\pi \Gamma(c) \Gamma(a+b+1/2) \Gamma(c-a-b+1/2)}{\Gamma(a+1/2) \Gamma(b+1/2) \Gamma(c-a+1/2) \Gamma(c-b+1/2)}, \quad (1.11)$$

where  $\operatorname{Re}(c) > 0, \operatorname{Re}(2c - a - b) > -1$ .

- Edward [7] established the following result:

$$\int_0^1 \int_0^1 u^\rho (1-v)^{\rho-1} (1-u)^{\sigma-1} (1-uv)^{1-\rho-\sigma} du dv = \frac{\Gamma(\rho) \Gamma(\sigma)}{\Gamma(\rho+\sigma)}, \quad (1.12)$$

provided  $\operatorname{Re}(\rho) > 0$  and  $\operatorname{Re}(\sigma) > 0$ .

## I. MAIN RESULTS

In this section, we deduce some new integral formulas involving Galu e' type Struve function. These integral formula are expressed in terms of Fox-Wright function as given in Theorems 1, 2 and 3.

### Theorem 1

Let  $a' \in \mathbb{N}, p, b', c' \in \mathbb{C}, \operatorname{Re}(\delta) > 0, \operatorname{Re}(p) > -1, \operatorname{Re}(c+p+2-a-b/2) > 0, \operatorname{Re}(\frac{a+b+1}{2}) > 0$ . The following integral formula holds true:

$$\int_0^1 x^{c-1} (1-x)^{-1/2} {}_2F_1(a, b; a+b+1/2; x) {}_{a'}W_{p, b', c', \delta}^{\lambda, \mu}(xy) dx = \left( \frac{y}{2} \right)^{p+1} \frac{\pi \Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2}) \Gamma(\frac{b+1}{2})} \times {}_3\Psi_4 \left[ \begin{matrix} (c+p+2-a-\frac{b}{2}, 1), (c+p+1, 2), (1, 1) \\ (c+p+2-\frac{a}{2}, 1), (c+p+2-\frac{b}{2}, 2), (\mu, \lambda), (\frac{p}{\delta} + \frac{b'+2}{2}, a') \end{matrix} ; -\frac{c'y^2}{4} \right]. \quad (2.1)$$

**Proof.** First, we designate the L.H.S of (2.1) by I and then using (1.3) and interchanging the order of integration and summation, we get

$$I = \sum_{k=0}^{\infty} \frac{(-c')^k}{\Gamma(\lambda k + \mu)} \frac{1}{\Gamma(a' k + \frac{p}{\delta} + \frac{b'+2}{2})} \left( \frac{y}{2} \right)^{2k+p+1} \times \int_0^1 x^{(c+2k+p+1)-1} (1-x)^{-1/2} {}_2F_1(a, b; a+b+1/2; x) dx. \quad (2.2)$$

Now, we apply the integral formula (1.11) to the integral of (2.2) under the conditions given in Theorem 1, we obtain

$$I = \frac{\pi \Gamma(a+b+1/2) \Gamma(p+c+1+2k) \Gamma(p+c-a-b+3/2+2k)}{\Gamma(a+1/2) \Gamma(b+1/2) \Gamma(p+c+3/2-a+2k) \Gamma(p+c+3/2-b+2k)} \times \sum_{k=0}^{\infty} \frac{(-c')^k}{\Gamma(\lambda k + \mu)} \frac{1}{\Gamma(a k + \frac{p}{\delta} + \frac{b'+2}{2})}. \quad (2.3)$$

After solving the above equation with the help of (1.9), we get the required result (2.1). This completes the proof. ■

**Remark 2.1** If we setting  $\lambda = a' = \delta = 1, \mu = 3/2$  in (2.1), we get following new result as follows:

**Corollary 2.1** For  $p, b', c' \in \mathbb{C}$  with  $\operatorname{Re}(c + p + 2 - a - b/2) > 0, \operatorname{Re}(c + p + 1) > 0, \operatorname{Re}(c + p + 2 - a/2) > 0, \operatorname{Re}(\frac{a+b+1}{2}) > 0$ . The following integral formula holds good:

$$\int_0^1 x^{c'-1} (1-x)^{-1/2} {}_2F_1(a, b; a+b+1/2; x) H_{p,b',c'}(xy) dx = \left(\frac{y}{2}\right)^{p+1} \frac{\pi \Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2}) \Gamma(\frac{b+1}{2})} \\ \times {}_3\Psi_4 \left[ \begin{matrix} (c+p+2-a-\frac{b}{2}, 1), (c+p+1, 2), (1, 1) \\ (c+p+2-\frac{a}{2}, 1), (c+p+2-\frac{b}{2}, 2), (3/2, 1), (p+\frac{b'+2}{2}, 1) \end{matrix} ; \frac{-c'y^2}{4} \right]. \quad (2.4)$$

**Remark 2.2** For  $b' = c' = 1$  in (2.1), will give a new integral formula of classical Struve function.

**Theorem 2** For  $a' \in \mathbb{N}, p, b', c' \in \mathbb{C}, \lambda > 0, \delta > 0$  and  $\mu$  is arbitrary parameter with  $\operatorname{Re}(\beta) > 0, \operatorname{Re}(\alpha + p) > -1, \operatorname{Re}(\alpha + \beta + p) > -1, \operatorname{Re}(\delta) > 0$ , then following integral formula holds good:

$$\int_0^1 \int_0^1 y^\alpha (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{1-\alpha-\beta} {}_aW_{p,b',c',\delta}^{\lambda,\mu} (uy(1-x)(1-xy)) dx dy \\ = \Gamma(\beta) \left(\frac{u}{2}\right)^{p+1} {}_2\Psi_3 \left[ \begin{matrix} (\alpha+p+1, 2), (1, 1) \\ (\alpha+\beta+p+1, 2), (\mu, \lambda), (\frac{p}{\delta} + \frac{b'+2}{2}, a') \end{matrix} ; -\frac{c'u^2}{4} \right]. \quad (2.5)$$

**Proof.** denoting the L.H.S of (2.5) by I. Using (1.3), the R.H.S of (2.5) and then interchanging the order of integration and summation, we get

$$I = \sum_{k=0}^{\infty} \frac{(-c')^k}{\Gamma(\lambda k + \mu)} \frac{1}{\Gamma(a'k + \frac{p}{\delta} + \frac{b'+2}{2})} \left(\frac{u}{2}\right)^{2k+p+1} \\ \times \int_0^1 \int_0^1 y^{\alpha+2k+p+1} (1-x)^{(\alpha+2k+p+1)-1} (1-y)^{\beta-1} (1-xy)^{1-(\alpha+2k+p+1)-\beta} dx dy \quad (2.6)$$

Now we apply the integral formula (1.12) to the integral of (2.6) under the conditions given in (2.5).

$$I = \Gamma(\beta) \left(\frac{u}{2}\right)^{p+1} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+2k+p+1)}{\Gamma(\alpha+\beta+2k+p+1)} \frac{\left(\frac{-c'u^2}{4}\right)^k}{\Gamma(a'k + \frac{p}{\delta} + \frac{b'+2}{2})} \frac{\Gamma(1+k)}{k!}. \quad (2.7)$$

After solving the above result with the help of (1.9) we get the required result of Theorem 2. This completes the proof of Theorem. ■

**Corollary 2.2** The following integral formula holds good for  $\operatorname{Re}(\beta) > 0, \operatorname{Re}(\alpha + p) > -1, \operatorname{Re}(\alpha + \beta + p) > -1$ .

$$\int_0^1 \int_0^1 y^\alpha (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{1-\alpha-\beta} H_{p,b',c'}(uy(1-x)(1-xy)) dx dy \\ = \Gamma(\beta) \left(\frac{u}{2}\right)^{p+1} {}_2\Psi_3 \left[ \begin{matrix} (\alpha+p+1, 2), (1, 1) \\ (\alpha+\beta+p+1, 2), (\frac{3}{2}, 1), (p + \frac{b'+2}{2}; 1) \end{matrix} ; -\frac{c'u^2}{4} \right]. \quad (2.8)$$

It is easy to determine the above result with the help of Theorem 2 and substituting  $a' = \lambda = \delta = 1, \mu = 3/2$  and using equation(1.4).

**Remark 2.3** On setting  $b' = c' = 1$ , in (2.8) and using (1.1) we get a new result involving ordinary Struve function.

**Theorem 3** For  $a' \in \mathbb{N}p, b', c' \in \mathbb{C}, \lambda > 0, \delta > 0, \mu$  is arbitrary parameter with  $\operatorname{Re}(\alpha) > 0, \operatorname{Re}(\alpha + \beta + p) > -1$  then following integral formula holds true

$$\begin{aligned} & \int_0^1 \int_0^1 y^\alpha (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{1-\alpha-\beta} {}_{a'}W_{p,b',c',\delta}^{\lambda,\mu}(u(1-x)(1-xy)) dx dy \\ &= \Gamma(\alpha) \left(\frac{u}{2}\right)^{p+1} {}_2\Psi_3 \left[ \begin{matrix} (\beta+p+1; 2), (1, 1) \\ (\alpha+\beta+p+1; 2), (\mu, \lambda), \left(\frac{p}{\delta} + \frac{b'+2}{2}, a'\right) \end{matrix} ; -\frac{c'u^2}{4} \right]. \end{aligned} \quad (2.9)$$

**Proof.** First we denote the L.H.S of (2.9) by I and using (1.3) then interchanging the order of integration and summation ,we get

$$\begin{aligned} I &= \sum_{k=0}^{\infty} \frac{(-c')^k}{\Gamma(\lambda k + \mu) \Gamma(a' k + \frac{p}{\delta} + \frac{b'+2}{2})} (u(1-y)(1-xy)/2)^{2k+p+1} dx dy \\ &\times \int_0^1 \int_0^1 y^\alpha (1-x)^{\alpha-1} (1-y)^{(\beta+2k+p+1)-1} (1-xy)^{1-\alpha-(\beta+2k+p+1)} dx dy \end{aligned} \quad (2.10)$$

Now we apply the integral formula (1.12) to the integral of (2.9) under the condition given in Theorem 3, we obtain following equation

$$I = \Gamma(\alpha) \left(\frac{u}{2}\right)^{p+1} \sum_{k=0}^{\infty} \frac{\Gamma(\beta+2k+p+1)}{\Gamma(\alpha+\beta+p+2k+1)} \frac{(-c')^k}{\Gamma(\lambda k + \mu) \Gamma(a' k + \frac{p}{\delta} + \frac{b'+2}{2})} \left(\frac{u}{2}\right)^{2k}. \quad (2.11)$$

After solving the above result with the help of (1.9) we get the required result of Theorem 3.This completes the proof of Theorem. ■

**Corollary 2.3** For  $\operatorname{Re}(\alpha > 0), \operatorname{Re}(\beta + p) > -1, \operatorname{Re}(\alpha + \beta + p) > -1$  following integral formula hold true:

$$\begin{aligned} & \int_0^1 \int_0^1 y^\alpha (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{1-\alpha-\beta} H_{p,b',c'}(z) dx dy \\ &= \Gamma(\alpha) \left(\frac{u}{2}\right)^{p+1} {}_2\Psi_3 \left[ \begin{matrix} (\beta+p+1; 2), (1, 1) \\ (\alpha+\beta+p+1; 2), \left(\frac{3}{2}, 1\right), \left(p + \frac{b'+2}{2}, 1\right) \end{matrix} ; -\frac{c'u^2}{4} \right]. \end{aligned} \quad (2.12)$$

It is easy to determine the above result with the help of (1.4), Theorem 3 and setting  $a' = \lambda = \delta = 1, \mu = 3/2$  in (2.9).

**Remark 2.4** On setting  $b' = c' = 1$  in (2.12) and using (1.1), we get a new result involving ordinary struve function.

## II. SPECIAL CASES

As a direct consequence of our result by taking suitable values of parameters of generalized Galue' type Struve function, some special cases are obtained below.

**Example 3.1** On setting  $a' = \lambda = \mu = c' = \delta = 1, p = n - 1, b' = 2$  in Theorem 1, Theorem 2, Theorem 3 and using 1.5 we find new integral formulas involving different special functions asserted by the following corollaries.

**Corollary 3.1** For  $\operatorname{Re}(c+n) > 0, \operatorname{Re}(c+n+1-a/2) > 0, \operatorname{Re}(2c+2n+2-b) > 0, \operatorname{Re}(\frac{a+b+1}{2}) > 0, \operatorname{Re}(c+n+1-a-b/2) > 0, .$  Then the following integral formula holds good:

$$\begin{aligned} & \int_0^1 x^{c-1} (1-x)^{-1/2} {}_2F_1(a, b; a+b+1/2; x) J_n(xy) dx = \left(\frac{y}{2}\right)^n \frac{\pi \Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2}) \Gamma(\frac{b+1}{2})} \\ & \times {}_3\Psi_4 \left[ \begin{matrix} (c+n+1-a-\frac{b}{2}, 1), (c+n, 2), (1, 1) \\ (c+n+1-\frac{a}{2}, 1), (c+n+1-\frac{b}{2}, 1), (1, 1), (n+1, 2) \end{matrix} ; -\frac{y^2}{4} \right]. \end{aligned} \quad (3.1)$$

**Corollary 3.2** The following integral formula holds true for  $\operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0, \operatorname{Re}(\alpha+n) > 0, \operatorname{Re}(\alpha+\beta+n) > 0$

$n) > 0.$

$$\begin{aligned} & \int_0^1 \int_0^1 y^\alpha (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{1-\alpha-\beta} J_n(uy(1-x)(1-xy)) dx dy \\ &= \Gamma(\beta) \left(\frac{u}{2}\right)^n {}_2\Psi_3 \left[ \begin{matrix} (\alpha+n, 2), (1, 1) \\ (\alpha+\beta+n, 2), (1, 1), (n+1, 2) \end{matrix} ; -\frac{u^2}{4} \right]. \end{aligned} \quad (3.2)$$

**Corollary 3.3** For  $\operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta+n) > 0, \operatorname{Re}(\alpha+\beta+n) > 0$  the following integral formula holds true:

$$\begin{aligned} & \int_0^1 \int_0^1 y^\alpha (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{1-\alpha-\beta} J_n(u(1-x)(1-xy)) dx dy \\ &= \Gamma(\alpha) \left(\frac{u}{2}\right)^n {}_2\Psi_3 \left[ \begin{matrix} (\beta+n, 2), (1, 1) \\ (\alpha+\beta+n, 2), (1, 1), (n+1, 1) \end{matrix} ; -\frac{u^2}{4} \right]. \end{aligned} \quad (3.3)$$

**Example 3.2** On setting  $a' = \delta = p = 1, b' = 2, c' = -1$  in Theorem 1, Theorem 2, Theorem 3 and using (1.5), we find new integral formulas involving different special functions asserted by the following respective corollaries.

**Corollary 3.4** For  $\operatorname{Re}(c+2-a-b) > 0, \operatorname{Re}(c+2-a) > 0, \operatorname{Re}(c+2-b) > 0, \operatorname{Re}(\frac{a+b+1}{2}) > 0, \operatorname{Re}(c+n+1-a-b/2) > 0$ . The following integral formula holds good:

$$\begin{aligned} & \int_0^1 x^{c-1} (1-x)^{-1/2} {}_2F_1(a, b; a+b+1/2; x) \phi(\mu, \lambda, xy) dx = \left(\frac{y}{2}\right)^2 \frac{\pi i \Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2}) \Gamma(\frac{b+1}{2})} \\ & \times {}_3\Psi_4 \left[ \begin{matrix} (c+1, 1), (\frac{c}{2} + 1 - \frac{a}{2} - \frac{b}{2}, \frac{1}{2}), (1, 1) \\ (\frac{c}{2} + 1 - \frac{a}{2}, \frac{1}{2}), (\frac{c}{2} + 1 - \frac{b}{2}, \frac{1}{2}), (\mu, \lambda), (3, 1) \end{matrix} ; y \right]. \end{aligned} \quad (3.4)$$

**Corollary 3.5** The following integral formula holds true for  $\operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0, \operatorname{Re}(\alpha+\beta) > 0, \lambda > 0, \mu > 0$ .

$$\begin{aligned} & \int_0^1 \int_0^1 y^\alpha (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{1-\alpha-\beta} \phi(\lambda, \mu, uy(1-x)(1-xy)) dx dy \\ &= \Gamma(\beta) {}_2\Psi_3 \left[ \begin{matrix} (\alpha; 1), (1, 1) \\ (\alpha+\beta, 1), (\mu, \lambda), (1, 1) \end{matrix} ; -u \right]. \end{aligned} \quad (3.5)$$

**Corollary 3.6** The following integral formula holds true for  $\operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0, \operatorname{Re}(\alpha+\beta) > 0, \lambda > 0, \mu > 0$

$$\begin{aligned} & \int_0^1 \int_0^1 y^\alpha (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{1-\alpha-\beta} \phi(\lambda, \mu, (u(1-x)(1-xy))) dx dy \\ &= \Gamma(\alpha) {}_2\Psi_3 \left[ \begin{matrix} (\beta, 1), (1, 1) \\ (\alpha+\beta, 1), (\mu, \lambda), (1, a') \end{matrix} ; u \right]. \end{aligned} \quad (3.6)$$

**Example 3.3** On setting  $\lambda = b' = 0, \mu = \delta = 1, c' = -1, p$  is replaced by  $p-1$  and  $z$  by  $(2\sqrt{z})$  in Theorem 1, Theorem 2 and Theorem 3 and using (1.10) we get with new integral formulas involving different special functions asserted by the following respective corollaries.

**Corollary 3.7** For  $\operatorname{Re}(c+p) > 0, \operatorname{Re}(2c+2p+2-2a-b) > 0, \operatorname{Re}(2c+2p+2-a) > 0, \operatorname{Re}(2c+2p+2-$

$b) > 0, Re(\frac{a+b+1}{2}) > 0, Re(c+n+1-a-b/2) > 0$ . The following integral formula holds good:

$$\int_0^1 x^{c-1} (1-x)^{-1/2} {}_2F_1(a, b; a+b+1/2; x) (\sqrt{z})^p E_{a', p}(xy) dx = \left(\frac{y}{2}\right)^p \frac{\pi \Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2}) \Gamma(\frac{b+1}{2})} \\ \times_3 \Psi_4 \left[ \begin{matrix} (c+p, 1), (\frac{c}{2} + \frac{p}{4} + \frac{1}{2} - \frac{a}{2} - \frac{b}{2}, \frac{1}{2}), (1, 1) \\ (\frac{c}{2} + \frac{p}{4} + \frac{1}{2} - \frac{a}{2}, \frac{1}{2}), (\frac{c}{2} + \frac{p}{4} + \frac{1}{2} - \frac{b}{2}, \frac{1}{2}), (0, 1), (p, a') \end{matrix} ; y \right]. \quad (3.7)$$

**Corollary 3.8** For  $Re(\alpha) > 0, Re(\beta) > 0, Re(\alpha) > -p/2, Re(\alpha+\beta) > -p/2, Re(u) > 0$ . Then the following integral formula holds true

$$\int_0^1 \int_0^1 y^\alpha (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{1-\alpha-\beta} (\sqrt{z})^p E_{a', p}(uy(1-x)(1-xy)) dx dy \\ = (\sqrt{u})^p \Gamma(\beta) {}_2\Psi_3 \left[ \begin{matrix} (\alpha + \frac{p}{2}, 1), (1, 1) \\ (\alpha + \beta + \frac{p}{2}, 1), (0, 1), (1, a') \end{matrix} ; u \right]. \quad (3.8)$$

**Corollary 3.9** For  $Re(\alpha) > 0, Re(\beta) > 0, Re(\alpha) > -p/2, Re(\alpha+\beta) > -p/2, Re(u) > 0$ . Then the following integral formula holds true

$$\int_0^1 \int_0^1 y^\alpha (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{1-\alpha-\beta} (\sqrt{z})^p E_{a', p}(u(1-x)(1-xy)) dx dy \\ = (\sqrt{u})^p \Gamma(\beta) {}_2\Psi_3 \left[ \begin{matrix} (\alpha + \frac{p}{2}, 1), (1, 1) \\ (\alpha + \beta + \frac{p}{2}, 1), (0, 1), (1, a') \end{matrix} ; u \right]. \quad (3.9)$$

**Example 3.4** On setting  $\lambda = \mu = \delta = b' = 1, c' = -1, p$  is replaced by  $p-1$  in Theorem 1, Theorem 2 and Theorem 3 and using (1.7) we get new integral formulas involving different special functions asserted by the following respective corollaries.

**Corollary 3.10** For  $Re(c+p) > 0, Re(2c+p+2-2a-2b) > 0, Re(2c+p+2-2a) > 0, Re(2c+p+2-2b) > 0, Re(\frac{a+b+1}{2}) > 0, Re(c+n+1-a-b/2) > 0$ . The following integral formula holds good:

$$\int_0^1 x^{c-1} (1-x)^{-1/2} {}_2F_1(a, b; a+b+1/2; x) {}_aI_p(xy) dx = \left(\frac{y}{2}\right)^p \frac{\pi \Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2}) \Gamma(\frac{b+1}{2})} \\ \times_3 \Psi_4 \left[ \begin{matrix} (c+p+1-a-b/2, 1), (c+p, 2), (1, 1) \\ (c+p+1-a/2, 1), (c+p+1-b/2, 1), (1, 1), (p + \frac{1}{2}, 1) \end{matrix} ; -\frac{y^2}{4} \right]. \quad (3.10)$$

**Corollary 3.11** The following integral formula holds true for  $Re(\alpha), Re(\beta) > 0, Re(\alpha) > -p, Re(\alpha+\beta) > -p$ .

$$\int_0^1 \int_0^1 y^\alpha (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{1-\alpha-\beta} {}_aI_p(uy(1-x)(1-xy)) dx dy \\ = \left(\frac{u}{2}\right)^p \Gamma(\beta) {}_2\Psi_3 \left[ \begin{matrix} (\alpha + p, 2), (1, 1) \\ (\alpha + \beta + p, 2), (1, 1), (p + 1/2, a') \end{matrix} ; u^2/4 \right]. \quad (3.11)$$

**Corollary 3.12** The following integral formula holds true for  $Re(\alpha), Re(\beta) > 0, Re(\alpha) > -p, Re(\alpha+\beta) > -p$ .

$$\int_0^1 \int_0^1 y^\alpha (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{1-\alpha-\beta} {}_aI_p(u(1-x)(1-xy)) dx dy$$

$$= \left(\frac{u}{2}\right)^p \Gamma(\alpha) {}_2\Psi_3 \left[ \begin{matrix} (\beta + p, 2), (1, 1) \\ (\alpha + \beta + p, 2), (1, 1), (p + 1/2, a') \end{matrix} ; u^2/4 \right]. \quad (3.12)$$

### III. CONCLUDING REMARK

The outcomes in the present paper is the computation of some integral formulae involving generalized Galu e' type Struve function in terms of Fox-Wright function. We observe that the generalized Galu e' type Struve function has a close relationship with some known special functions such as Struve function, Mittag-Leffler function, Bessel function, Wright function etc. As a consequence, we have attempted to compute the integrals in the form of different types of special functions by suitable replacement of parameters. Further, a lot of work can be put in the literature by writing the Fox-Wright function in terms of Fox H-function, Meijer G-function, etc. The results obtained here seems to be interesting and may potentially be useful in various applied problems.

### REFERENCES

- [1] R. M. Ali, Saiful R. Mondal, and K. S. Nisar, *Monotonicity properties of the generalized Struve functions*, J. Korean Math. Soc. 54 (2017), 575–598.
- [2] J. Choi, K.S. Nisar, *Certain families of integral formulas involving Struve function* Boletim da Sociedade Paranaense de Matematica, 37(2019), 27-35.
- [3] A.K. Agrawal, *A generalization of di-Bessel function*, Exton. Indian J. Pure Appl. Math, 15 (1984), 139-148.
- [4] A. Baricz, *Generalized Bessel function of the first kind*. Lecture Notes in Mathematics Springer, Berlin 2010.
- [5] K.N. Bhomick, *Some relation between a generalized Struve's function and hypergeometric functions*. Vijnana Parishad Anusandhan Patrika, 5(1962), 93-99.
- [6] K.N. Bhomick, *A generalized Struve's function and its recurrence formula*. Vijnana Parishad Anusandhan Patrika, 6 (1963), 1-11.
- [7] J. Edward, *A treatise on the integral calculus*, Chelsea Publication Company, New York, II (1922).
- [8] Erdelyi, *A Higher Transcendental function*, McGraw-Hill, New York, 1(1953).
- [9] H. Exton, *On a generalization of Bessel -Clifford equation and an application in quantum mechanics*. Riv Math Univ.Parma, 4 ( 1989), 41-16.
- [10] L. Galue (2003) *A generalized Bessel function*. Int Transform Spec Funct, 14 (2010), 395-401.
- [11] A. Gray ,G.B. Mathews,T.M. MacRobert *A treatise on Bessel function and their application to physics* Second ed.,Macmillan,London 1992.
- [12] R. Gorenflo, F. Mainardi, H.M. Srivastava, *Special function in fractional relaxation-oscillation and fractional diffusion-wave phenomena*,in:D.Baino(Ed.), Proceeding of the Eighth International Colloquium on Differential Equation, Plovdiv, August 18-23, 1997,VSP, Utrecht, (1998), 195-202.
- [13] R. Gorenflo, Y. Luckho, F. Mainardi, *Analytical properties and application of the Wright function*, Fract. Calc. Appl. Anal.2(1999), 383-414.
- [14] B.N. Kant, *Integral involving generalized Struve's function*,, Nepali Math Sci Rep, 6 (1981), 61-64.
- [15] O. Khan, M. Kamrujjama and N. U. Khan, *Certain Integral Transform Involving The Product of Galue type Struve Function and Jacobi polynomial*, Palestine J. Math. (Accepted) (2018).
- [16] K.S. Nisar, P. Agrwal, S.R. Mondal, *On fractional integral of generalized Struve function of first kind* Adv. Stud. Contemp. Math 26(2016), 63-70.
- [17] K.S. Nisar, D. Baleanu and M.M. Al Qurashi, Fractional calculus and application of generalized Struve function, SpringerPlus, 5, 910(2016), DOI 10.1186/s40064-016-2560-3.
- [18] K.S. Nisar, F.B.M. Belgacem, Dynamic k-Struve Sumudu solutions for fractional kinetic equations, Advances in Differential Equations, 2017 (2017), 340.
- [19] K.S. Nisar, Fractional kinetic equations involving Struve function using Sumudu transform, Mathematical Sciences International Research Journal, 6(2017), 13-17.
- [20] K.S. Nisar, S.R. Mondal, and J. Choi, J Inequal Appl (2017) 2017, 71. <https://doi.org/10.1186/s13660-017-1343-x>.
- [21] H. Orhan and N. Yagmur,*Geometric properties of generalized Struve function* Ann Alexandru Ioan Cuza Univ-Math,doi:10.2478/aicu-2014-0007.
- [22] R.P. Singh *Generalized Struve function and its recurrence relations*.Ranchi Univ Math, J 5 (1974), 67-75.
- [23] D.L. Suthar, S.D. Purohit, K.S. Nisar, Some unified integrals associated with generalized struve function, arXiv:1608.03134(2016).
- [24] E.M. Wright, *The asymptotic expansion of the generalized hypergeometric function*,J.London Math.Soc., 10(1935), 287-293.
- [25] A. Wiman, *Über den fundamental Satz in der Theorie der Funktionen* $E_\alpha(x)$ , Acta. Math. 29 (1905) 191-201.