On The Lattice of Trail Sets of a Connected Graph

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Abstract— *In this paper Trail set is defined for a finite connected graph G and it is found that the set of all trail sets $\tau(G)$ together with empty set partially ordered by set inclusion relation forms a lattice. Also derived graph of G, denoted by G_d is defined such that lattices $\tau(G)$ and $\tau(G_d)$ are isomorphic. Some of the properties of the lattices so obtained are studied. The definition of trail sets is extended to directed graphs and studied.*

Keywords — *connected graph, lattice, tree, pendant vertex, Connected digraph, Path, Circuit.*

I. INTRODUCTION

In many research papers, lattices are constructed using graphs and their properties are studied. In fact in [9], it is shown that the set of all convex subgraphs of a directed graph G together with empty set partially ordered by set inclusion relation forms a complete semimodular, A-regular lattice. In [7], it is shown that the set of all convex sets of a finite connected graph together with empty set partially ordered by set inclusion relation forms a lattice. In [8],lattice of path sets of a connected graph is discussed. In [1], [2], lattice of convex edge sets of a connected directed graph is discussed.

Motivated by the above studies, in this paper we have defined trail sets of a finite connected graph G and found that the set of all trail sets of G together with empty set partially ordered by set inclusion relation forms a lower semi modular lattice. In section 2 after introducing some basic concepts and notations we have shown that the set of all trail sets of a finite connected graph G together with empty set forms a lattice with respect to the partial order set inclusion and is denoted by $< \tau(G), \subseteq >$. In section 3 some of the properties of such lattices are studied.

In section 4, derived graph G_d is defined such that $< \tau(G), \subseteq >$ and $< \tau(G_d), \subseteq >$ are isomorphic. Also we found that for any connected graph G, G_d is a tree. Some of the properties of $< \tau(G_d), \subseteq >$ are also studied.

In section 5, the definition of trail sets is extended to directed graphs and it is shown that the set of all trail sets of a finite connected digraph G together with empty set also forms a lattice with respect to the partial order set inclusion with an example. The conditions under which $< \tau(G), \subseteq >$ forms a chain and $< \tau(G), \subseteq >$ is a complemented lattice are studied for a connected digraph G. Necessary and sufficient condition for an element A in $< \tau(G), \subseteq >$ to be doubly irreducible is established. Also it is observed that for certain digraphs there is unique $< \tau(G), \subseteq >$ depending upon its underlying graphs.

II. PRELIMINARIES

A walk of a graph G is an alternating sequence of vertices and edges $v_0,e_1,v_1,\ldots v_{n-1},e_n,v_n$ beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it. This walk joins v_0 and v_n, and may also be denoted by $v_0,v_1,\ldots v_n$, called v_0-v_n walk. It is closed if $v_0 = v_n$ and is open otherwise. An open walk in which no edge is repeated is a trail. A closed walk in which no edge is repeated is a circuit. A circuit containing all the vertices of G is called a spanning circuit. A walk in which no vertex is repeated is a path. A closed path is called a cycle. A path containing n vertices is denoted by P_n. A graph is acyclic, if it has no cycles. A tree is a connected acyclic graph.

An element 'a' of a lattice L is join irreducible if $a = b \vee c$ implies that $a = b$ or $a = c$. 'a' is meet irreducible if $a = b \wedge c$ implies that $a = b$ or $a = c$. An element which is both meet and join irreducible is called doubly irreducible. A lattice L is said to satisfy the lower covering condition if for $a, b \in L$ $a \wedge b \prec b$ implies $a \prec a \vee b$. A lattice L is lower semimodular(LSM) if $a \vee b$ covers both a and b implies that both a and b cover $a \wedge b$.

Let G be a finite connected graph. V(G) be the vertex set of G. A set T $\subseteq$ V(G) is said to be trail set in G if it satisfies the following two conditions (i)for every two vertices u, v $\in$ T, the vertex set of every u-v trail is contained in T, (ii) for every vertex v $\in$ T , if there exist a circuit containing v, then the set of all vertices of a maximal circuit containing v is contained in T.

For a finite connected graph G, let the set of all trail sets in G together with empty set be denoted by $\tau(G)$. Define a binary relation $\leq$ on $\tau(G)$ by, for A, B $\in \tau(G)$, A $\leq$ B if and only if A $\subseteq$ B. Then clearly $\leq$ is a partial order on $\tau(G)$. Moreover $< \tau(G),\ \subseteq >$ forms a lattice where for A, B $\in \tau(G)$, A $\wedge$ B = A $\cap$ B and A $\vee$ B = < A $\cup$ B >, where < A $\cup$ B > is the trail set generated by A $\cup$ B or equivalently the smallest trail set containing A $\cup$ B.

For example, the lattice given in Fig 2.2 represents the lattice $< \tau(G),\ \subseteq >$ of the connected graph G given in Fig 2.1.

Fig 2.1

Fig 2.2
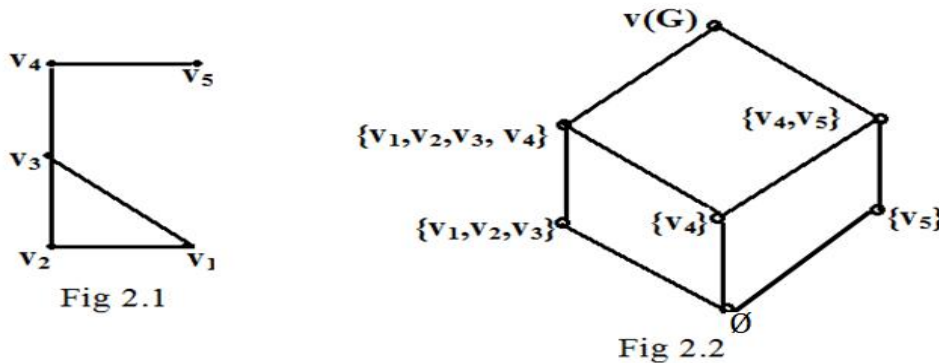
Throughout this paper we consider G as a non trivial graph and we use the notation $\tau(G)$ to represent the lattice $< \tau(G),\ \subseteq >$. The undefined terms and notations used in this paper are from [3], [4] and [5].

III. ON THE LATTICE $\tau(G)$

Remark 3. 1: The smallest element of $\tau(G)$ is empty set and the largest element is V(G).

Theorem 3.2: A singleton set $\{v_i\}$ is a trail set if and only if v_i does not belong to a circuit.
Proof: Let A = $\{v_i\}$ be a trail set. If v_i belong to a circuit, then all the vertices of that circuit also belong to A, a contradiction.
Conversely, If v_i does not belong to a circuit, then $A = \{v_i\}$ is a trail set.

Remark 3. 3: $\tau(G)$ is atomic where atoms are as follows.
  (i)   Trail sets containing single vertex.
  (ii) Trail sets containing all vertices of maximal circuits of G.

Theorem 3.4: $\tau(G)$ is a two element chain if and only if either G is P_2 or G contains a spanning circuit.
Proof: Let $\tau(G)$ be a two element chain. Then there are only two trail sets $\emptyset$ and $V(G)$ such that $\emptyset \prec V(G)$, which implies $V(G)$ is an atom. Therefore G contains a spanning circuit. Conversely, let G contains a spanning circuit, then $\emptyset, V(G)$ are the only two trail sets, such that $\emptyset \prec V(G)$. Thus $\tau(G)$ is a two element chain.

Remark 3.5: $\tau(G)$ is a two element chain for all graphs containing spanning circuit. Moreover $\tau(G)$ is a chain only in this case. Infact if G does not contain spanning circuit, then G contains atleast one bridge. Therefore $\tau(G)$ will contain atleast two atoms. Thus $\tau(G)$ will contain a sublattice as shown in Fig 3.1. Hence we can conclude that $\tau(G)$ is a chain if and only if G has a spanning circuit.
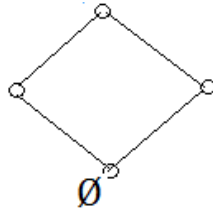
Fig 3.1

Example 3.6: We note that the graphs shown in figures 2.1 and 3.2 are not isomorphic but their lattices shown in figures 2.2 and 3.3 are isomorphic.
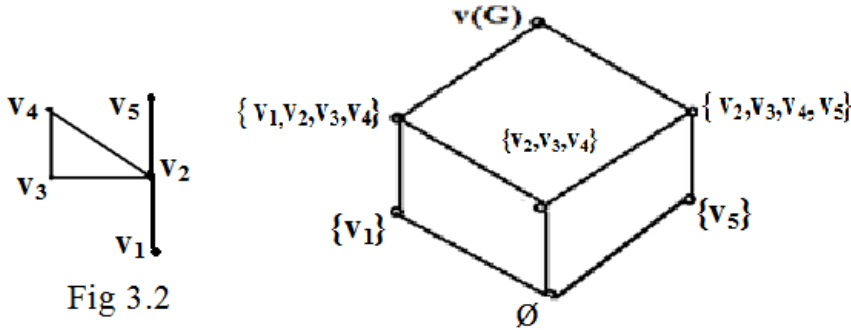

Fig 3.2

Fig 3.3

Theorem 3.7: G contains n-1 bridges if and only if $\tau(G)$ contains n atoms.
Proof: Let G contains n-1 bridges. Since G is a connected graph, the removal of these n-1 bridges results into a disconnected graph with n components. Each component is either a single vertex or a circuit. Therefore trail sets of vertices of these components form atoms of $\tau(G)$. Hence $\tau(G)$ contains n atoms. Conversely if $\tau(G)$ contains n atoms, then each atom is either a set with single vertex or a set containing vertices of a circuit. But then since G is connected, clearly G must contain n-1 bridges.

IV.DERIVED GRAPH

Let G be a connected graph with n-1 bridges, say $e_1, e_2, \ldots e_{n-1}$ and G^1 be the graph obtained from G by deleting all bridges of G. Then G^1 is a disconnected graph with n components. These components are either single vertices or sub graphs of G containing circuit. Let us denote the vertex set of components as $T_1, T_2, \ldots T_n$. These n components are connected by n-1 edges which are bridges of G. If there is an edge connecting one vertex of T_i to one vertex of T_j, then they are adjacent. Let G_d be the graph with vertex set $\{T_1, T_2, \ldots T_n\}$ and edge set $\{e_1, e_2, \ldots e_{n-1}\}$. We call G_d as the derived graph of G, which is a connected acyclic graph with n vertices and n-1 edges. Hence it is a tree.

For example, the graph given in Fig 4.2 represents the derived graph G_d of the graph G given in Fig 4.1.,
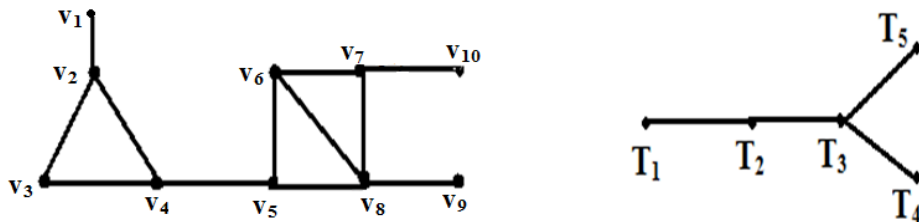

Fig 4.1

Fig 4.2

where $T_1=\{v_1\}$, $T_2=\{v_2,v_3,v_4\}$, $T_3=\{v_5,v_6,v_7,v_8\}$, $T_4=\{v_9\}$ and $T_5=\{v_{10}\}$
We use the notation $\tau(G_d)$ to represent the lattice $<\tau(G_d), \subseteq>$.

Remark4.1: If G itself is a tree , then G is isomorphic to G_d.

Remark4.2: Let G_d be the derived graph of G and $\tau(G_d)$ be the lattice of trail sets of G_d .Then the lattices $\tau(G)$ and $\tau(G_d)$ are isomorphic.

Remark4.3: Following statements are equivalent.
i)   G has spanning circuit
ii) G_d is a trivial graph
iii) $\tau(G)$ is a two element chain.

Theorem4.4: $\tau(G)$ is planar if and only if G_d is a path.

Proof: Let $\tau(G)$ be planar. If G_d is not a path, then there exist a vertex with degree at least three as shown in Fig 4.3. Then $\tau(G)$ contains a subposet as shown in Fig 4.4, which implies $\tau(G)$ can not be planar(see[6]).
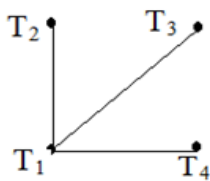
Fig 4.3
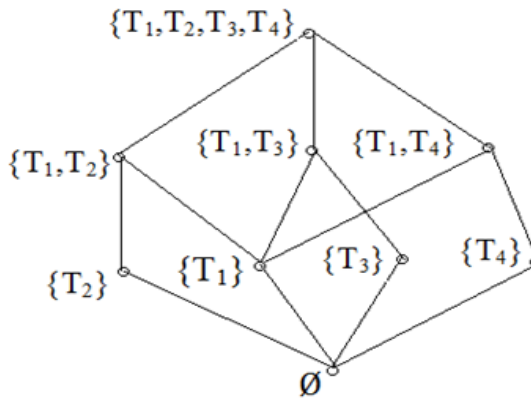
Fig 4.4

Conversely, if G_d is a path $T_1T_2…T_n$ as shown in Fig4.5, then $\tau(G)$ will be as shown in Fig 4.6 and hence it is planar.
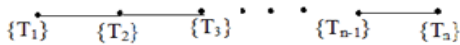

Fig 4.5
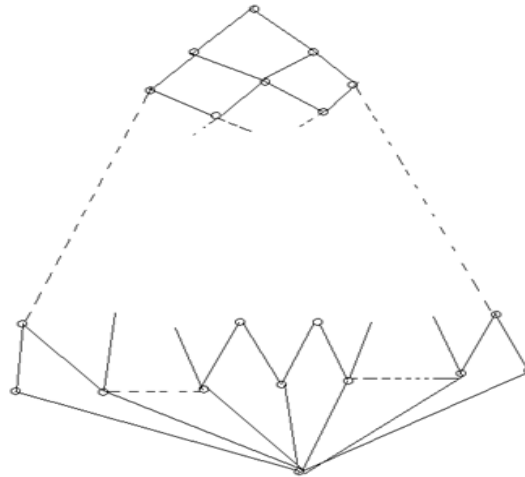
Fig 4.6

Theorem4.5: An element A $\in \tau(G_d)$ is doubly irreducible if and only if A = $\{T\}$ where T is a pendant vertex of G_d.

Proof: Let A $\in \tau(G_d)$ be doubly irreducible. If A = $\{T\}$ where T is a vertex with two edges incident on it, say TT_1 and TT_2. Then   A = $\{T,T_1\} \wedge \{T,T_2\}$, a contradiction. On the other hand if A contains more than one element, say A = $\{T_1,T_2,…T_n\} \in \tau(G_d)$,then A = $V_{i=1}^n\{T_i\}$. Therefore A = $\{T_i\}$ for some i, since A is join irreducible.

Conversely, for any vertex T in G_d, A = $\{T\} \in \tau(G_d)$ is an atom of $\tau(G_d)$. Therefore A is join irreducible. If A is meet reducible, say A = B $\wedge$ C = B $\cap$ C for some B, C $\in \tau(G_d)$ such that A $\neq$ B, A $\neq$ C ,then T $\in$ B $\cap$ C. Consider U $\in$ B , V $\in$ C where T $\neq$ U , T $\neq$ V. Let $T, U_1, U_2 …U$ be a

shortest path connecting T and U in B. Also let $T, V_1, V_2 \ldots V$ be a shortest path connecting T and V in C. If $U_1 = V_1$, then $U_1 \in B \cap C$, contradiction to $B \cap C = \{T\}$. Also, if $U_1 \neq V_1$, then $\{T, U_1\}$ and $\{T, V_1\}$ are two edges incident on T, contradiction to T is pendant vertex of G_d. Hence A must be irreducible.

Theorem4.6: For any connected graph G, $\tau(G)$ is lower semimodular.

Proof: Let G_d be the derived graph of G, with vertex set $\{T_1, T_2, \ldots T_n\}$.

Since G_d is a tree, if $A \vee B$ covers both A and B, then $A, B, A \vee B$ are of the form $\{T_i, T_j, \ldots T_k, T_l\}$, $\{T_i, T_j, \ldots T_k, T_m\}$ and $\{T_i, T_j, \ldots T_k, T_l, T_m\}$ respectively. Where $\{T_i, T_j, \ldots T_k\}$ are common elements in A and B. Clearly $A \wedge B = \{T_i, T_j, \ldots T_k\}$ and it is covered by both A and B. Hence $\tau(G)$ lower semi modular.

Remark 4.7: The following statements are equivalent.
 i) G is a graph with single bridge.
 ii) G_d is P_2.
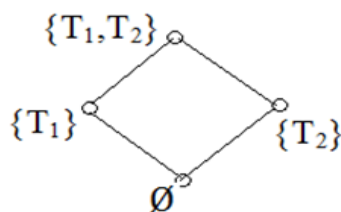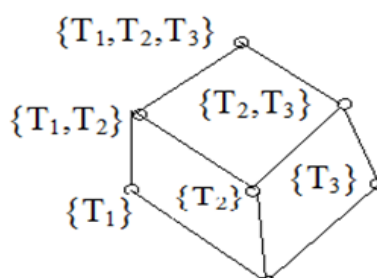 iii) $\tau(G)$ will be of the form as shown in Fig 4.7

Fig 4.7

Fig 4.8

Remark 4.8: The following statements are equivalent.
 i) G is a graph with two bridges.
 ii) G_d is P_3.
 iii) $\tau(G)$ will be of the form as shown in Fig 4.8.

Theorem 4.9: $\tau(G)$ contains n atoms if and only if G contains n-1 bridges.

Proof: If $\tau(G)$ contains n atoms, then $\tau(G_d)$ also contains n atoms. Therefore G_d is a tree with n vertices and hence it contains exactly n-1 edges which are nothing but bridges of G.

Conversely, if G contains n-1 bridges, then G_d contains n vertices. Therefore $\tau(G_d)$ as well as $\tau(G)$ will contain n atoms.

Theorem 4.10: A chain containing more than two elements cannot be realized as the lattice of trail sets.

Proof: If G_d is a trivial graph, then $\tau(G_d)$ is a two element chain. If G_d is not a trivial graph, then G_d contains at least one edge. Therefore $\tau(G_d)$ contains a subposet as shown in Fig4.7.

Theorem 4.11: A lattice shown in Fig 4.9 cannot be realized as a lattice of trail sets [for $n \geq 3$]

Proof: For $n \geq 3$, G_d will contain a subgraph P_3. Therefore $\tau(G_d)$ will contain a subposet as shown in Fig 4.8. Thus lattice shown in Fig 4.9 cannot be realized as a lattice of trail sets [for $n \geq 3$]
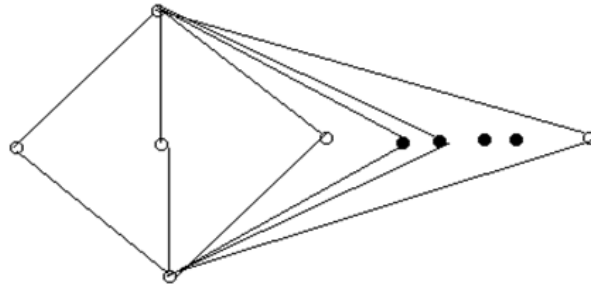
Fig 4.9

Remark4.12: Lattices shown in Fig 4.10, Fig 4.11 cannot be realizable as lattice of trail sets as they are not lower semi modular.
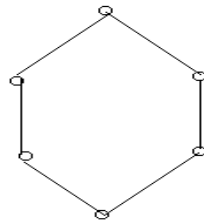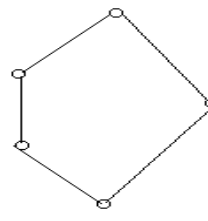

Fig 4.10

Fig 4.11

Theorem4.13: Let T be any vertex of G_d. Degree of T = k in G_d if and only if $\{T\}$ is covered by k elements in $\tau(G_d)$

Proof: If degree of T = k, then there are k edges incident on T . Let $T_1, T_2, \dots T_k$
be the adjacent vertices of T. Then $\{T, T_1\}$, $\{T, T_2\}, \dots \{T, T_k\}$ are the elements which cover $\{T\}$.
Conversely, Let $\{T\}$ be covered by k elements, say $\{T, T_1\}$, $\{T, T_2\}, \dots \{T, T_k\}$ .Therefore vertex T is incident with k edges. Hence degree of T = k

Theorem4.14: $\tau(G_d)$ is complemented if and only if either G_d is a single vertex(trivial graph) or G_d is P_2
Proof: Let $\tau(G_d)$ be complemented. If G_d is not a single vertex or P_2, then G_d must contain P_3 as a subgraph. Let $T_i T_j T_k$ be the subgraph of G_d. Then $\{T_j\}$ is not complemented. And hence $\tau(G_d)$ is not complemented, a contradiction. Conversely, if G_d is a trivial graph, then $\tau(G_d)$ is a two element chain. If G_d is P_2, then $\tau(G_d)$ will be as shown in Fig4.7. Clearly both are complemented.

Theorem4.15: $l(\tau(G_d)) = |V(G_D)|$
Proof: Let V(G_d) = $\{T_1, T_2, \dots T_n\}$. Then the longest chain in T(G_d) is
$\emptyset \prec \{T_1\} \prec \{T_1, T_2\} \prec \cdots \prec \{T_1, T_2, \dots T_{n-1}\} \prec \{T_1, T_2, \dots T_n\}$. Which is of length n.

Theorem4.16: If G_d is P_n , then $|\tau(G_d)| = \frac{n(n+1)}{2} + 1$
Proof: Let G_d be a path with n vertices. Then there are n trail sets with single element, n-1 trail sets with two elements and so on continuing like this finally one trail set with n elements. Including empty set,
$|\tau(G_d)| = n + (n-1) + (n-2) \dots + 1 + 1 = \frac{n(n+1)}{2} + 1$

Theorem4.17: Following statements are equivalent for any derived graph G_d.
1. G_d is P_1 or P_2.
2. $\tau(G_d)$ is distributive.
3. $\tau(G_d)$ is modular.
4. $\tau(G_d)$ satisfies lower covering condition.

Proof: (1) $\Rightarrow$ (2): If G_d is P_1, then $\tau(G_d)$ is a two element chain. If G_d is P_2 then $\tau(G_d)$ is of the form as shown in Fig 4.7. Both are distributive.

$(2) \Rightarrow (3) \Rightarrow (4)$ is true for all lattices.

To prove $(4) \Rightarrow (1)$: Let $\tau(G_d)$ satisfy lower covering condition. If G_d is not P_1 or P_2, then G_d contains P_3 , say $T_iT_jT_k$. Clearly $\emptyset = \{T_i\} \wedge \{T_k\} \prec \{T_i\}$. But $\{T_k\} \prec \{T_j, T_k\} < \{T_i\} \vee \{T_k\}$. Contrdiction to $\tau(G_d)$ satisfies lower covering condition. Hence G_d must be P_1 or P_2.

***Remark4.18*:** $\tau(G_d)$ is dually atomic. Dual atom $D_i = V(G_d) - T_i$,where T_i is a pendant vertex of G_d. Thus the number of dual atoms of $\tau(G_d)$ = the number of pendant vertices of G_d.

Remark4.19: For any graph G, $\tau(G)$ satisfies Jordan dedekind chain condition since it is lower semi modular.

V. ON THE LATTICE OF TRAIL SETS OF A CONNECTED DIRECTED GRAPH

A (directed) walk in a digraph is an alternating sequence of vertices and edges $v_0,e_1,v_1\ldots e_n,v_n$ in which each edge e_i is $v_{i-1}v_i$. A closed walk has the same first and last vertices, and a spanning walk contains all the vertices. A path is a walk in which all vertices are distinct, and a closed path is called a cycle. If there is a path from u to v , then v is said to be reachable from u. Each walk is directed from the first vertex v_0 to the last vertex v_n. A **semiwalk** is an alternating sequence $v_0,e_1,v_1\ldots e_n,v_n$ of vertices and edges in which each edge e_i is $v_{i-1}v_i$ or v_iv_{i-1}. Similarly a semipath, semicycle are also defined.

A digraph is strongly connected , if every two vertices are mutually reachable, it is unilaterally connected , if for any two points at least one is reachable from the other, and it is weakly connected if every two points are joined by a semipath. A digraph is disconnected if it is not even weak. For any digraph G, the undirected graph obtained by removing the directions is called it's underlyng graph.

Let G be a connected directed graph. Let E (G) be the edge set of G. A subset A of E (G) is said to be Trail set of G if it satisfies the following conditions,

i)   For every $e_i \in A$, all edges which lie on the trail connecting end vertices of $e_i \in A$. Also if there is any circuit from any end vertex of e_i to itself , then all the edges of that circuit belong to A.

ii)   For any trail $e_i, e_{j,\ldots} e_k, \in A$ all the edges connecting end vertices of this trail belong to A and also all edges which connect end vertices of a trail contained in the above trail belong to A.

Let $\tau(G)$ be the set of all trail sets of G together with the empty set. Define a binary relation $\leq$ on $\tau(G)$ by, for A, B $\in \tau(G)$, A $\leq$ B if and only if A $\subseteq$ B. Then clearly $\leq$ is a partial order on $\tau(G)$. Moreover $<\tau(G), \subseteq>$ forms a lattice where for A, B $\in$ $\tau(G)$,   A $\wedge$ B = A$\cap$B and A $\vee$ B =$<$A$\cup$B$>$ is the smallest convex edge set containing A$\cup$B.

For example, the lattice given in Fig 5. 2 represents the lattice $< \tau(G), \subseteq>$of the connected digraph G given in Fig 5. 1.

Fig 5.1

Fig 5.2
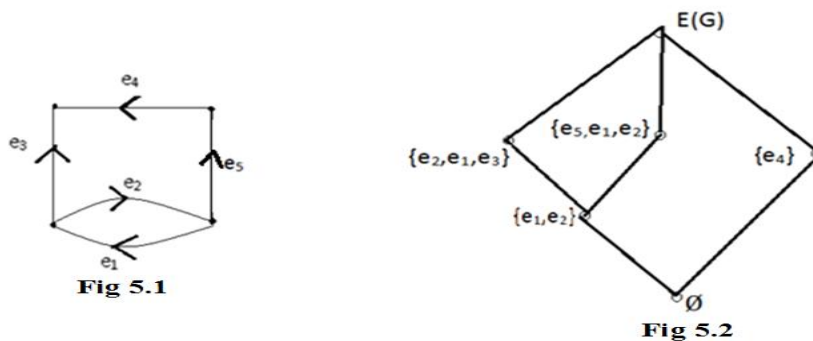
Here after we use $\tau(G)$ to denote $< \tau(G), \subseteq>$ where G is a connected directed graph.

Remark 5. 1: The smallest element of $\tau(G)$ is empty set and the largest element is E(G).

Remark 5. 2: $\tau(G)$ is atomic where atoms are as follows
(i)sets containing edges of a maximal strongly connected subgraphs of G.
(ii)singleton sets {e_i} if there is no directed trail connecting end vertices of e_i and also there is no directed circuit from any end vertex of e_i to itself.

Theorem 5. 3: $\tau(G)$ is a two element chain if and only if either G is a digraph containing single edge or G is a strongly connected digraph.

Proof: Let $\tau(G)$ be a two element chain, then $\emptyset$ and E(G) are the only trail sets in G such that $\emptyset \prec E(G)$. Therefore E(G) itself is an atom of $\tau(G)$. Hence G is a digraph containing single edge or G is a strongly connected digraph.

Conversely, let G be a strongly connected digraph. Then every pair of vertices are connected by two directed trails. Therefore $\emptyset$ and E(G) are the only trail sets in G such that $\emptyset \prec E(G)$.

Corollary 5.4 : If G is as given in theorem 5.3, then $|\tau(G)| = 2$, $l(\tau(G)) = 1$ and $\tau(G)$ is a Boolean algebra.

Theorem 5. 5: $\tau(G)$ is a three element chain if and only if $G = S \cup A$ where S is the maximal strongly connected subgraph of G and A is a nonempty set of edges whose initial(terminal) vertices belong to S and they share a common terminal(initial) vertex.

Proof: Let $\tau(G)$ be a three element chain of the form $\emptyset \prec A_1 \prec E(G)$. Where A_1 is either singleton set, say $A_1 = \{e_i\}$ or set of edges of a maximal strongly connected subgraph of G. If $A_1 = \{e_i\}$ is an atom and $A_1 \prec E(G)$, then $\tau(G)$ contains a subposet as shown in Fig 5.4 Therefore A_1 must be a set of edges of a maximal strongly connected subgraph of G. Also $G = A_1 \cup A$ where A is a nonempty set of edges whose initial(terminal) vertices belong to A_1 and they share a common terminal(initial) vertex. We take $A_1 = S$.

Conversely ,let $G = S \cup A$ where S is the maximal strongly connected subgraph of G and A is a nonempty set of edges whose initial(terminal) vertices belong to S and they share a common terminal(initial) vertex.Then $\emptyset$, E(S) and E(G) are the only trail sets in G such that $\emptyset \prec E(S) \prec E(G)$.

Corollary 5.6 : If G is as given in theorem 5.5, then $|\tau(G)| = 3$, $l(\tau(G)) = 2$ and $\tau(G)$ is a Boolean algebra.

Theorem 5. 7: Depending on the nature of underlying graph , $\tau(G)$ will be as follows.
Figure 5.4,5.6,5.8,5.10,5.12 represent lattices $\tau(G)$ corresponding to underlying graphs Figures 5.3,5.5,5.7,5.9,5.11 respectively.
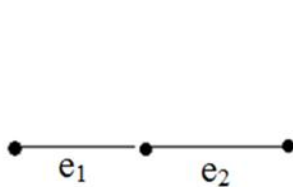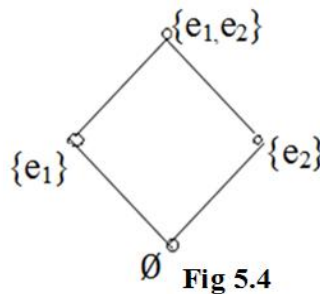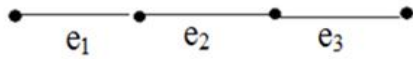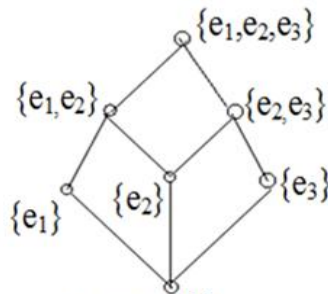

Fig 5.3

Fig 5.4

Fig 5.5

Fig 5.6 Ø
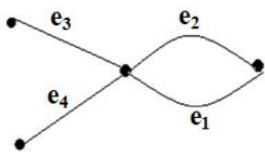
Fig 5.7
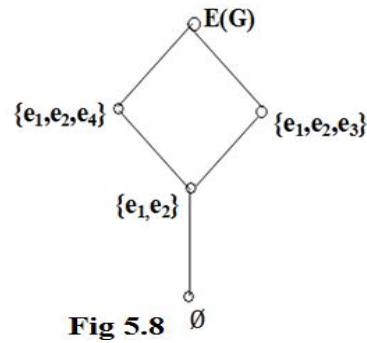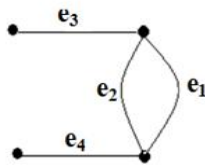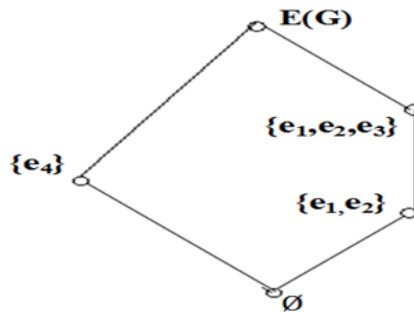

Fig 5.8 Ø
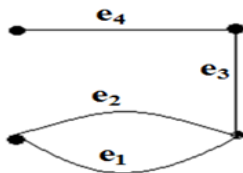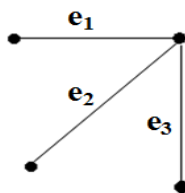

Fig 5.9

Fig 5.10
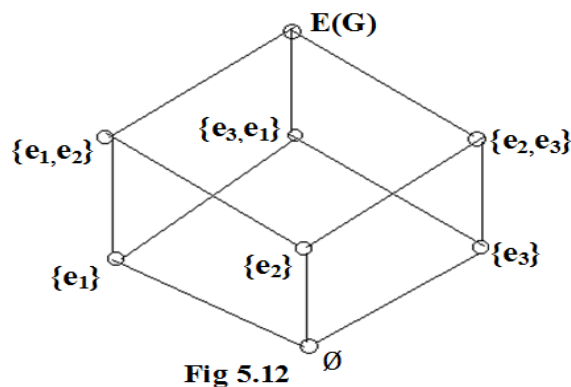

Fig 5.11

Fig 5.12 Ø

Theorem 5.8: **I**f underlying graph of digraph G is of the form as shown in Fig 5.13, then $\tau(G)$ is a Boolean algebra.
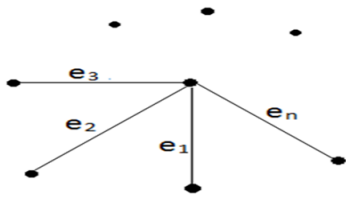

Fig 5.13

Proof: If G is of the form as shown in Fig 5.13 , then $\tau(G)$ is a lattice for A, B $\in \tau(G)$,
A $\wedge$ B = A$\cap$B and A $\vee$ B = A$\cup$B. Clearly $\tau(G)$ is a complemented distributive lattice. Hence $\tau(G)$ is a Boolean algebra.

Theorem 5.9: If G is a directed path with n edges, then

(i) $\qquad |\tau(G)| = \frac{n(n+1)}{2} + 1$

(ii) $\qquad l(\tau(G)) = n$.

Proof: (i) If G is a directed path with n edges, then there are n trail sets with single edge, $n-1$ trail sets with two edges, $n-2$ trail sets with three edges and so on, finally one trail set with n edges. Including empty set,

$$|\tau(G)| = n + (n-1) + (n-2) + \cdots + 1 + 1 \quad = \frac{n(n+1)}{2} + 1$$

(ii) Let $e_1 e_2 \ldots\ldots e_n$ be the directed path.
Then $\emptyset \prec \{e_1\} \prec \{e_1, e_2\} \prec \cdots \prec \{e_1, e_2, \ldots e_n\}$ is the maximum chain.
Hence $l(\tau(G)) = n$

Theorem 5.10: Let G be a digraph containing Euler trail. An element A $\in \tau(G)$ is doubly irreducible if and only if $A = \{e_i\}$ where e_i is a pendant edge without containing circuits at its end vertex OR A = S , where S is a set of edges of a maximal strongly connected subgraph of G such that A is adjacent to at most one vertex of G.
Proof: Let G be a digraph containing Euler trail. Let A $\in \tau(G)$ be doubly irreducible. Then $A = \{e_i\}$ or A = S. Otherwise A will be join reducible.
If $A = \{e_i\}$ where e_i is a pendant edge containing circuits at its end vertex, then $A = \{e_i\}$ can not be a trail set. If $A = \{e_i\}$ is not a pendant edge, then either e_i belongs to a directed circuit or there exist a directed path $e_j e_i e_k$. If e_i belongs to a directed circuit, then $A = \{e_i\}$ can not be a trail set. If there exist a directed path $e_j e_i e_k$, then $\{e_i\} = <\{e_j, e_i\}> \wedge <\{e_k, e_i\}>$,a contradiction. Let A = S, where S is a set of edges of a maximal strongly connected sub graph of G. If A is adjacent to two or more vertices of G say e_j, e_k be the corresponding edges , then $A = <A \cup \{e_j\}> \wedge <A \cup \{e_k\}>$, a contradiction.

Conversely, if $A = \{e_i\}$ where e_i is a pendant edge without containing circuits at its end vertex OR A = S , where S is a set of edges of a maximal strongly connected subgraph of G such that A is adjacent to at most one vertex of G, then A is an atom of $\tau(G)$. Therefore A is join irreducible. Let $A = \{e_i\}$ where e_i is a pendant edge without containing circuits at its end vertex. If A is meet reducible say $A = B \wedge C = B \cap C$ for some $B, C \in \tau(G)$ such that $A \neq B, A \neq C$ .Then $e_i \in B \cap C$. Consider $e_j \in B$, $e_k \in C$ where $e_j \neq e_i$, $e_k \neq e_i$. Let $e_i f_1 f_2 \ldots e_j$ be the shortest path connecting e_i, e_j in B. Also let $e_i g_1 g_2 \ldots e_k$ be the shortest path connecting e_i, e_k in C. If $f_1 = g_1$, then $f_1 \in B \cap C$ a contrdiction to $B \cap C = \{e_i\}$. Also if $f_1 \neq g_1$,then e_i contains circuit at its end vertex since G contains an euler trail a contradiction. If A = S and $A = B \wedge C = B \cap C$ for some $B, C \in \tau(G)$ such that $A \neq B, A \neq C$, then A is adjacent to atleast two vertices. Hence a contradiction.

Theorem 5.11 Let G be a directed graph containing Euler trail. Then $\tau(G)$ is complemented if and only if G is any of the following three forms.

i)   G contains a spanning circuit.
ii) G is a directed path containing atmost two edges.
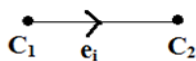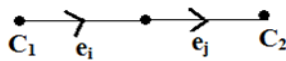**iii)** G is of the form as shown in Fig 5.14, Fig 5.15 where C_1 and C_2 are directed circuits.
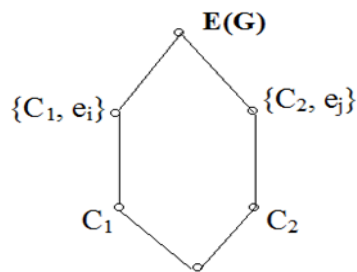

Fig 5.14

Fig 5.15

Fig 5.16

Proof : Let $\tau(G)$ be complemented. If G is not of the form as given in the theorem, then either G contains a directed path containing three edges $e_j e_i e_k$ OR $G = S \cup A$ where S is the maximal strongly connected subgraph of G and A is a nonempty set of edges whose initial(terminal) vertices belong to S and they share a common terminal(initial) vertex. If G contains a directed path containing three edges $e_j e_i e_k$, then $\{e_i\}$ is not complemented. If $G = S \cup A$, then $\tau(G)$ is a three element chain. Hence $\tau(G)$ is not complemented.

Conversely, if G contains a spanning circuit or G is a directed path containing one edge, then $\tau(G)$ is a two element chain. If G is a directed path with two edges or of the form as shown in Fig 5.14 where C_1 and C_2 are directed circuits, then $\tau(G)$ is of the form as shown in Fig 5.4 . If G is of the form as shown in Fig 5.15, then $\tau(G)$ is of the form as shown in Fig 5.16. In all these cases $\tau(G)$ is complemented. Hence the result.

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