Common Fixed Point Theorems in Non-Archimedean Menger PM Space

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Abstract

In this paper we proved the existence of the common fixed point theorems in Non-Archimedean Menger PM-space by using the R-weakly commuting mappings, reciprocal continuity are established in this paper. The presented results extend some known existence results from the literature.

Keywords

Fixed point, Non Archimedean Menger PM-space, R-weakly commuting mappings, reciprocally continuous.

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I. INTRODUCTION

In 1942, K. Menger introduced the probabilistic metric space in which the metric whose value is non-negative real number is the probabilistic metric. The generalization of probabilistic metric spaces leads to the investigation of physical quantities and probabilistic functions. Istratescu and Crivat [7] had defined the non-Archimedean Probability metric space and explained basic topological fundamentals of non-Archimedean Probability metric space in [7]. Istratescu et al. showed the existence of fixed point of contractive maps in non-Archimedean PM-space in [7], [8] which was the generalization of the existing. In this paper we find the common fixed point of R-weakly commuting mappings, in N. A. Menger PM-space.

II. PRELIMINARIES

First we need the following definitions and results that will be used subsequently.

Definition 2.1.

Let X be any non-empty set and the set of all left continuous distribution functions be denoted as D. An ordered pair (X, G) is defined to be the non-Archimedean probabilistic metric space (N.A.PM-space), if G is a mapping from $X \times X \to D$ satisfies the following conditions

(i) G(x, y ; t) = 1 for all t > 0 if and only if $x = y$
(ii) G(x , y ; t) = G(y , x ; t)
(iii) G(x , y ; 0) = 0
(iv) $If\ G(x,y,t_1) = G(y,z,t_2) = 1, then\ G(x,z,\max\{t_1,t_2\}) = 1.$

Definition 2.2.

A t-norm is a function $\delta: [0,1] \times [0,1] \to [0,1]$ which is associative, commutative, non-decreasing in each coordinate and $\delta(b,1) = b\ \ for\ all\ b \in [0,1]$.

Definition 2.3.

A N.A.Menger PM-space is said to be in ordered triplet$(X, F\ \delta)$, where δ is a t-norm and (X, F), is a N.A.PM-space fulfilling the following condition,

$F(x,z,\max\{t_1,t_2\}) \geq \delta(F(x,y,t_1),F(y,z,t_2))$ for all $x, y, z \in X,\ \ t_1, t_2\ \geq 0.$

Definition 2.4.

A sequence $\{x_n\}$ in N. A. Menger PM-space$(X, F\ \delta)$ converges to x if and only if for each $\varepsilon>0, \lambda>0$ there exists $M(\varepsilon,\lambda)$ such that $\varphi(F(x_n,x,\varepsilon)) < \varphi(1-\lambda) for\ all\ n,\ n > M.$

Definition 2.5.
A sequence $\{x_n\}$ in N.A.Menger PM-space is said to be Cauchy sequence iff for each $\varepsilon>0, \lambda>0$ there exists an integer $M(\varepsilon,\lambda)$ such that $\varphi\left(F(x_n, x_{n+p}, \varepsilon)\right) < \varphi(1-\lambda) for\ all\ n, \quad n \geq M\ and\ p \geq 1.$

Definition 2.6.
Two maps S and T of a N. A. Menger PM-space $(X, F\ \delta)$ into itself are said to be R weakly commuting of type A_s if for $x \in X$ and R > 0

$$\varphi\left(F\left(STx, TTx,\ t\right)\right) \leq \varphi\left(F\left(Sx, Tx, \frac{t}{R}\right)\right)$$

Definition 2.7.
Two maps S and T of a N. A. Menger PM-space $(X, F\ \delta)$ into itself are said to be R weakly commuting of type A_T if for $x \in X$ and R > 0

$$\varphi\left(F\left(STx, TTx,\ t\right)\right) \leq \varphi\left(F\left(Sx, Tx, \frac{t}{R}\right)\right)$$

Lemma 2.8.
If a function $\psi: [0,\infty) \to [0,\infty)$ satisfies the condition (Φ) then we get

i) For all t > 0, $\lim_{n\to\infty} \psi^n(t) = 0$, where $\psi^n(t)$ is the n[th] iteration of $\psi(t)$
ii) If $\{t_n\}$ is a non-decreasing sequence of real numbers and $t_{n+1} \leq \psi(t_n), n = 1,2, \ldots$ then $\lim_{n\to\infty} t_n = 0.$ In particular, if $t \leq \psi(t)$, $\forall\ t \geq 0$ then t = 0.

Lemma 2.9.
Let $\{y_n\}$ be a sequence in X such that $\lim_{n\to\infty} F(y_n, y_{n+1}; t) = 1$ for each $t > 0$. If the sequence $\{y_n\}$is not a Cauchy sequence in X, then there exist $\varepsilon_0 > 0$, $t_0 > 0$, and two sequences $\{m_i\}\ and\ \{n_i\}$ of positive integers such that $m_i > n_i + 1$ and $n_i \to \infty$ as $i \to \infty$. $F(y_{m_i}, y_{n_i}; t_0) < 1-\varepsilon_0$ and $F(y_{m_i-1}, y_{n_i}; t_0) \geq 1-\varepsilon_0$, $i = 1, 2, \ldots\ldots$

III.MAIN RESULTS

Theorem 3.1.
Let A, B, S and T be the self-maps of N.A.Menger PM-space (X, F, δ) satisfying
(i) $A(X) \subset T(X)$, $B(X) \subset S(X)$
(ii) One of $A(X), B(X), T(X)$ or $S(X)$ is complete
(iii) The pairs (A,S) and (B,T) are R-weakly commuting of type A_s and A_T respectively
(iv) $\varphi\left(F(Ax, By; t)\right) \leq \max\begin{Bmatrix}\varphi(\psi(F(Sx, Ax, t))), \varphi(\psi(F(Ty, By, t))), \\ \varphi(\psi(F(Sx, Ty, t))), \varphi(\psi(F(Ty, Ax, t)))\end{Bmatrix}$
for all $x, y \in X\ and\ t > 0,$ where $\psi: [0,1] \to [0,1]$ is some continuous function such that $\psi(t) < t\ and\ \psi(1) = 1.$ .Then A, B, S and T have a unique common fixed point in X.
Proof:
Let $x_0 \in X$ be an arbitrary point. As $A(X) \subset T(X)$ and $B(X) \subset S(X)$
there exist $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$, $Bx_1 = Sx_2$. Inductively we can construct
Sequences $\{y_n\}\ and\ \{x_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1}\ ; y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}\ \text{for } n = 0,1\ldots.. \tag{1.1}$$

Now, Using (iv) with $x = x_{2n}$, $y = x_{2n+1}$, we get

$$\varphi(F(y_{2n},y_{2n+1},t) = \varphi(F(Ax_{2n},Bx_{2n+1},t)$$
$$\leq \max\begin{Bmatrix}\varphi(\psi(F(Sx_{2n},Ax_{2n},t))),\varphi(\psi(F(Tx_{2n+1},Bx_{2n+1},t))),\\ \varphi(\psi(F(Sx_{2n},Tx_{2n+1},t))),\varphi(\psi(F(Tx_{2n+1},Ax_{2n},t)))\end{Bmatrix}$$
$$\leq \max\begin{Bmatrix}\varphi(\psi(F(y_{2n-1},y_{2n},t))),\varphi(\psi(F(y_{2n},y_{2n+1},t))),\\ \varphi(\psi(F(y_{2n-1},y_{2n},t))),\varphi(\psi(F(y_{2n},y_{2n},t)))\end{Bmatrix}$$
$$\leq \max\{\varphi(\psi(F(y_{2n-1},y_{2n},t))),\varphi(\psi(F(y_{2n},y_{2n+1},t)))\}$$

If $\varphi(\psi(F(y_{2n},y_{2n+1},t))) \geq, \varphi(\psi(F(y_{2n-1},y_{2n},t)))$ then

$\varphi(F(y_{2n},y_{2n+1},t) \leq \varphi(\psi(F(y_{2n},y_{2n+1},t))) < \varphi\big(F(y_{2n},y_{2n+1},t)\big)$ *a contraction.*

Therefore $\varphi(F(y_{2n},y_{2n+1},t) \leq \varphi\big(\psi\big(F(y_{2n-1},y_{2n},t)\big)\big)$ for all t > 0.

Hence $\varphi(F(y_n,y_{n+1},t) \leq \varphi\big(\psi\big(F(y_{n-1},y_n,t)\big)\big)$ for all t > 0 and $n = 1,2,\ldots$.

Therefore, by lemma 2.8, $\lim_{n\to\infty}\varphi\big(F(y_n,y_{n+1},t)\big) = 0$ *for all* $t > 0$.

Before preceding the proof of the theorem, we have to prove a claim.

Claim:

Let A, B, S and T : X → X be the maps satisfying (i) , (ii) and the sequence $\{y_n\}$defined by (1.1) such that $\lim_{n\to\infty}\varphi\big(F(y_n,y_{n+1},t)\big) = 0$ is a Cauchy sequence in X.

Proof:

Since $\varphi\in\Omega$ it follows that $\lim_{n\to\infty}F(y_n,y_{n+1},t) = 1$ *for each* $t > 0$ if and only if

$\lim\limits_{n\to\infty}\varphi\big(F(y_n,y_{n+1},t)\big) = 0$ *for each* $t > 0$.

By Lemma 2.9, if $\{y_n\}$ is not a Cauchy sequence in X, then there exists $\varepsilon_0 > 0$, $t_0 > 0$ and two sequences $\{m_i\}$ and $\{n_i\}$ of positive integers such that

(a) $m_i > n_i + 1$ and $n_i \to \infty$ as $i \to \infty$

(b) $\varphi(F(y_{m_i},y_{n_i};t_0)) > \varphi(1-\varepsilon_0)$ and $\varphi(F(y_{m_i-1},y_{n_i};t_0)) \leq \varphi(1-\varepsilon_0)$; $i = 1,2,..$,

Since $\varphi(t) = 1 - t$. Thus, we have

$$\varphi(1-\varepsilon_0) < \varphi(F(y_{m_i},y_{n_i};t_0))$$
$$\leq \varphi(F(y_{m_i},y_{m_i-1};t_0)) + \varphi(F(y_{m_i-1},y_{n_i};t_0))$$

$$\leq \varphi(F(y_{m_i},y_{m_i-1};t_0)) + \varphi(1-\varepsilon_0). \tag{1.2}$$

As $i \to \infty$ in (1.2) we get , $\lim\limits_{n\to\infty}\varphi(F(y_{m_i},y_{n_i};t_0)) = \varphi(1-\varepsilon_0)$ (1.3)

On the other hand, we have

$$\varphi(1-\varepsilon_0) < \varphi(F(y_{m_i},y_{n_i},a;t_0))$$
$$\leq \varphi(F(y_{n_i},y_{n_i+1};t_0)) + \varphi(F(y_{m_i},y_{n_i+1};t_0)) \tag{1.4}$$

Now, consider $\varphi(F(y_{m_i},y_{n_i+1};t_0))$ in (1.4) and assume that both m_i *and* n_i are even. Then by lemma (1.2), we have

$$\varphi(F(y_m,y_{n+1},t) = \varphi(F(Ax_m,Bx_{n+1},t)$$
$$\leq \max\begin{Bmatrix}\varphi(\psi(F(Sx_m,Ax_m,t))),\varphi(\psi(F(Tx_{n+1},Bx_{n+1},t))),\\ \varphi(\psi(F(Sx_m,Tx_{n+1},t))),\varphi(\psi(F(Tx_{n+1},Ax_m,t)))\end{Bmatrix}$$
$$\leq \max\begin{Bmatrix}\varphi(\psi(F(y_m,y_n,t))),\varphi(\psi(F(y_n,y_{n+1},t))),\\ \varphi(\psi(F(y_{m-1},y_n,t))),\varphi(\psi(F(y_{m-1},y_m,t)))\end{Bmatrix} \tag{1.5}$$

Now, consider $\varphi(F(y_{m_i-1},y_{n_i};t_0))$ from (1.5)

$$\varphi(F(y_{m_i-1},y_{n_i};t_0)) \leq \varphi(F(y_{m_i-1},y_{m_i};t_0)) + \varphi(F(y_{m_i},y_{n_i};t_0)). \tag{1.6}$$

Using (1.6) in (1.5) and letting $i\to\infty$.

$\varphi(1-\varepsilon_0) \leq \max\{\ \varphi(\psi(1-\varepsilon_0)),0,\varphi(\psi(1-\varepsilon_0)),0\}$ *i.e*, $\varphi(1-\varepsilon_0) \leq \varphi(\psi(1-\varepsilon_0))$

This is a contradiction. Hence the sequence $\{y_n = Ax_n\}$ defined by the result (1.1) is a Cauchy sequence.

Case I:

T(X) is complete and $\{\boldsymbol{y_n}\}$is a Cauchy sequence in T(X) so $\{\boldsymbol{y_n}\}$ converges to some $z(say)\in T(X)$ and hence the subsequence's $\{Ax_{2n}\},\{Bx_{2n+1}\}$ *and* $\{Tx_{2n+1}\}$ must also converge to $z \in T(X)$. Since $z \in T(X)$, there exists $uz \in T(X)$ such that $z = Tu$.

Step I:
By putting $x = x_{2n}, y = u$ in (iv), we get
$\varphi(F(Ax_{2n}, Bu, t))$
$\leq \max\{\varphi(\psi(F(Sx_{2n}, Ax_{2n}, t))), \varphi(\psi(F(Tu, Bu, t))), \varphi(\psi(F(Sx_{2n}, Tu, t))), \varphi(\psi(F(Tu, Ax_{2n}, t)))\}$

Letting $n \to \infty$, we have $\varphi(F(z, Bu, t)) \leq \psi\left[\max\begin{Bmatrix}\varphi(F(z,z,t)), \varphi(F(z,Bu,a;t)), \\ \varphi(F(z,z,t)), g(F(z,z,t))\end{Bmatrix}\right]$
which implies that $z = Bu = Tu$. As (B,T) is R-weakly commuting of type A_T so $\varphi(F(BTu, TTu, t)) \leq \varphi\left(F\left(Bu, Tu, \frac{t}{R}\right)\right)$, R > 0, which yields $BTu = TTu$ i.e., $Bz = Tz$.

Step II:
By putting $x = x_{2n}, y = z$ in (iv), we have
$\varphi(F(Ax_{2n}, Bz, t))$
$\leq \max\{\varphi(\psi(F(Sx_{2n}, Ax_{2n}, t))), \varphi(\psi(F(Tz, Bz, t))), \varphi(\psi(F(Sx_{2n}, Tz, t))), \varphi(\psi(F(Tz, Ax_{2n}, t)))\}$
Taking. $n \to \infty$. we get $Bz = z = Tz$

Step III:
As $B(X) \subset S(X)$, there exists $v \in X$ such that $z = Bz = Sv$.
By putting $x = v, y = z$ in (iv) , we get
$\varphi(F(Av, Bz, t))$
$\leq \max\{\varphi(\psi(F(Sv, Av, t))), \varphi(\psi(F(Tz, Bz, t))), \varphi(\psi(F(Sv, Tz, t))), \varphi(\psi(F(Tz, Av, t)))\}$
$\varphi(F(Av, z, t)) \leq \max\{\varphi(\psi(F(z, Av, t))), 0, 0, \varphi(\psi(F(z, Av, t)))\}$
Therefore, $Av = z = Sv$. Since (A,S) is R-weakly commuting of type A_s, so
$\varphi(F(ASv, SSv, t)) \leq \varphi\left(F\left(Av, Sv, \frac{t}{R}\right)\right) = 0$, which implies that $ASv = SSv$ i.e., $Az = Sz$.

Step IV:
By putting $x = z, y = z$ in (iv) and assuming $Az \neq Bz$, we get
$\varphi(F(Az, Bz, t))$
$\leq \max\{\varphi(\psi(F(Sz, Az, t))), \varphi(\psi(F(Tz, Bz, t))), \varphi(\psi(F(Sz, Tz, t))), \varphi(\psi(F(Tz, Az, t)))\}$
i.e., $\varphi(F(Az, Bz, t)) \leq \max\{0, 0, \varphi(\psi(F(Az, Bz, t))), \varphi(\psi(F(Bz, Az, t)))\}$
which is a contradiction and we get $Az = Bz$. Combining all the results we get $z = Az = Bz = Sz = Tz$. That is z is a common fixed point of A, B, S and T.

Case II:
S(X) is complete so the sequence $\{y_n\}$ must converge to $z \in S(X)$ and hence the subsequence's $\{Ax_{2n}\}, \{Bx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ must also converge to $z \in S(X)$. As $z \in S(X)$, there exists w$\in X$ such that $z = Sw$.

Step I:
By putting $x = w, y = x_{2n}$ in (iv), we get
$\varphi(F(Aw, Bx_{2n}, t))$
$\leq \max\{\varphi(\psi(F(Sw, Aw, t))), \varphi(\psi(F(Tx_{2n}, Bx_{2n}, t))), \varphi(\psi(F(Sw, Tx_{2n}, t))), \varphi(\psi(F(Tx_{2n}, Aw, t)))\}$
Letting $n \to \infty$, we get $Aw = z = Sw$. Now, (A,S) are R-weakly commuting of type A_s,
so $\varphi(F(ASv, SSv, t)) \leq \varphi\left(F\left(Av, Sv, \frac{t}{R}\right)\right)$, R > 0 , which implies that $ASw = SSw$, i.e $Az = Sz$.

Step II:
Putting $x = z, y = x_{2n}$ in (iv) , we get
$\varphi(F(Az, Bx_{2n}, t))$
$\leq \max\{\varphi(\psi(F(Sz, Az, t))), \varphi(\psi(F(Tx_{2n}, Bx_{2n}, t))), \varphi(\psi(F(Sz, Tx_{2n}, t))), \varphi(\psi(F(Tx_{2n}, Az, t)))\}$

Letting $n \to \infty$, we get $Az = z = Sz$.

Step III:
As $A(X) \subset T(X)$, there exists $u_1 \in X$ such that $z = Az = Tu_1$. By putting $x = x_{2n}$, $y = u_1$ in (iv), we obtain
$\varphi(F(Ax_{2n}, Bu_1, t))$
$\leq \max\{\varphi(\psi(F(Sx_{2n}, Ax_{2n}, t))), \varphi(\psi(F(Tu_1, Bu_1, t))), \varphi(\psi(F(Sx_{2n}, Tu_1, t))), \varphi(\psi(F(Tu_1, Ax_{2n}, t)))\}$
Letting $n \to \infty$, we get $z = Bu_1 = Tu_1$. Now (B,T) is R-weakly commuting of type A_T
so $\varphi(F(BTu_1, TTu_1, t)) \leq \varphi\left(F\left(Bu_1, Tu_1, \frac{t}{R}\right)\right), R > 0$. This implies that $BTu_1 = TBu_1$ i.e., $Bz = Tz$.

Step IV:
By putting $x = z$, $y = z$ in (iv), we get $Az = BZ$.
Hence $z = Az = Sz = Bz = Tz$. That is z is the common fixed point of A, B, S and T.

Case III:
If A(X) or B(X) is complete. As $A(X) \subset T(X)$ and $B(X) \subset S(X)$, the result follows respectively from Case I and Case II. The uniqueness of the fixed point follows directly from (iv) and hence the theorem

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