Maximum Distance in Graphs

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Abstract

For two vertices u and v of a graph G, the usual distance $d(u,v)$, is the length of the shortest path between u and v. In this paper we introduce Maximum distance $M-$distance by considering the length of the shortest path, the sum of the degrees of all vertices in the path in addition the total number of vertices in the path. We also define the Maximum radius, Maximum diameter, Maximum eccentricity and Maximum center of G.

Keywords

M-distance, Maximum radius, Maximum diameter.

I. INTRODUCTION

In this paper we introduce the maximum distance of a graph G denoted as M-distance and defined as $d^M(u,v) = min\{l^M(P)\}$.Where$l^M(P) = d(u,v) + \sum_{x\in V(P)} degx + \sum_{w\in P(G)}|w|$.

Consider the following graph figure-1:

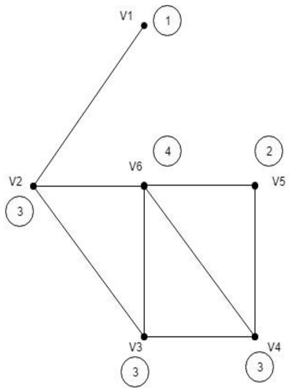

figure-1

$l^M(P_1)_{v_1v_2} \Rightarrow 2+1+1+3 = 7$

$l^M(P_2)_{v_1v_3} \Rightarrow 3+2+1+3+3 = 12$

$l^M(P_3)_{v_1v_4} \Rightarrow 4+3+1+3+3+3 = 17$ $[v_1, v_2, v_3, v_4]$

$l^M(P_4)_{v_1v_4} \Rightarrow 4+3+1+3+4+3 = 18$ $[v_1, v_2, v_6, v_4]$

$l^M(P_5)_{v_1v_5} \Rightarrow 4+3+1+3+4+2 = 17$ $[v_1, v_2, v_4, v_5]$

$l^M(P_6)_{v_1v_6} \Rightarrow 3+2+1+3+4 = 13$ $[v_1, v_2, v_6]$

Remark: Observe that for any two vertices u, v of G we have, $d(u,v) \le d^M(u,v)$. The equality holds good if and only if u, v are identical.

***Theorem 1**: If G is a connected graph then the $M-$ distance is a metric on the set of vertices of G.*

Proof:

Let G be a connected graph. Then it is clear by definition, that $d^M(u,v) \geq 0$ and $d^M(u,v) = 0 \Leftrightarrow u = v$. We also have $d^M(u,v) = d^M(u,v)$. Thus it remains to show that d^M satisfies the triangle inequality.

Let $u, v, w \in V(G)$. Let P and Q be $u - v$ and $w - v$ paths in G respectively such that $d^M(u,v) = l^M(P)$ and $d^M(u,v) = l^M(Q)$. Let $R = P \cup Q$ be the $u - v$ path obtained by joining P and Q at W.

$$d^M(u,w) + d^M(w,v)$$

$$= \left(l^M(P) + \sum_{x \in V(P)} \deg x\right) + \sum_{w \in P(G)} |w| + \left(l^M(Q) + \sum_{y \in V(Q)} \deg y\right) + \sum_{w \in P(G)} |w|$$

$$= l^M(P \cup Q) + \sum_{x \in V(P)} \deg x + \sum_{y \in V(Q)} \deg y + \sum_{w \in P(G)} |w|$$

$$= l^M(P \cup Q) + \sum_{x \in V(P \cup Q)} \deg \mathrm{x} + \deg y + \sum_{w \in P \cup Q(G)} |w|$$

$$\geq d^M(u,v) + \deg w + \sum_{w \in P \cup Q(G)} |w| \geq d^M(u,v)$$

Thus the triangle inequality holds and hence d^M is a metric on the vertex set $V(G)$.

Next we have a consequence of the above proof.

Corollary 1.1: For any three vertices u, v, w of a graph G, we have

$$d^M(u,v) \leq d^M(u,w) + d^M(w,v) - (degw + 1)$$

Preposition 1.2:

In a connected graph G, two distinct vertices u, v are adjacent $\Leftrightarrow d^M(u,v) = \deg(u) + \deg(v) + 3.$

Proof:

If $u, v \in V(G)$ are adjacent then $d(u,v) = 1$ and hence $d^M(u,v) = d(u,v) + \deg u + \deg v + \sum_{P=uv} V(P) = 1 + \deg u + \deg v + 2 = \deg u + \deg v + 3.$

Conversely, if $d^M(u,v) = \deg u + \deg v + 3,$ then by definition of $M -$ distance we get $d(u,v) + \sum_{w \in V(P)} degw + \sum_{P=uv} V(P).$

Hence $d(u,v) = 1, \sum_w \deg w = 0 + \sum_{P=uv} V(P) = 2.$ This implies u and v are adjacent.

II. M –Eccentricity, M – Radius, M –Diameter and M –Denter

The M –Eccentricity of any vertex $v, e^M(v)$ is defined as the maximum distance (M –distance) from v to any other vertex

i.e. $e^M(v) = max\{d^M(u,v): u \in V(G)\}.$

Any vertex u for which $d^M(u,v) = e^M(v)$ is called M –Eccentric vertex of v. Further, a vertex u is said to be M –eccentric vertex of G. If it is the M –eccentric vertex of some vertex.

The maximum radius M −radius denoted by $r^M(G)$ is the minimum M −eccentricity among all vertices of G. i.e $r^M(G) = min\{e^M(v); v \in V(G)\}$. Similarly, the M − diameter, $d^M(G)$, is the maximum M −eccentricity among all vertices of G.

The maximum centre M −centre of G, $c^D(G)$ is the subgraph induced by the set of all vertices of minimum M −eccentricity. A graph is called M −self centered if $c^M(G) = G$ (or) equivalently $r^M(G) = d^M(G)$. Similarly, the set of all vertices of maximum M −eccentricity is the M −periphery of G.

Remark: Observe that since the d^M is a metric, we can check easily $r^M(G) \le d^M(G) \le 2r^M(G)$. The lower bound is clear from the definition and the upper bound follows from the triangular inequality.

Theorem 2.1:

If u, v are two adjacent vertices of a connected graph G, with $e^M(u) \ge e^M(v)$, then $e^M(u) - e^M(v) \le \deg(u) + 3$.

Proof:

Let w be a vertex of G such that $d^M(u,w) = e^M(u)$.

Then $e^M(u) = d^M(u,w) \le d^M(u,v) + d^M(u,w) - \deg(v) - 3 \le d^M(u,w) + e^M(v) - \deg(v) - 3$ (by corollary 1.1).

Further since u, v are adjacent, by proposition (1.2) we get $e^M(u) - e^M(v) \le d^M(u,v) - \deg(v) - 3$ $= \deg(u) + 3$

Proposition 2.2

For complete graphs, K_n, on n vertices, $n \ge 3$ we have $r^M(K_n) = d^M(K_n) = 2n + 1$.

Proposition 2.3

For path graphs P_n on n vertices $n \ge 3$, we have $r^M(P_n) = \begin{cases} 2n \text{ if } n \text{ is odd} \\ 2n + 2 \text{ if } n \text{ is even} \end{cases}$ and $d^M(P_n) = 4n - 3$.

Proposition 2.4

For cycle graphs, C_n, with $n \ge 3$ vertices, $r^M(C_n) = d^M(C_n) = 4n + 2$

Proposition 2.5

For star graph $r^M(S_n) = d^M(S_n) - 2 = n + 3$

Proposition 2.6

For a tree graph T, two distinct vertices u, v are adjacent if and only if $d^M(u,v) = \deg(u) + \deg(v) + 3$.

Proof: Let u, v are adjacent. This implies that $d(u,v) = 1$.

By definition, $d^M(u,v) = |V(P)| + d(u,v) + \deg(u) + \deg(v) = 2 + 1 + \deg(u) + \deg(v) = 3 + \deg(u) + \deg(v)$.

Conversely, If $d^M(u,v) = \deg(u) + \deg(v) + 3$.

This implies that $d(u,v) = 1$ and $|V(P)| = 2$. Hence, u, v are adjacent.

Here we discussed $l^M(P)$ for v_1 with all other vertices in the graph. The path v_1 to v_4 having two different M-distance with the same shortest $d(u,v)$ distance. But we consider $l^M(P)$ with minimum value (17).

Theorem 2.7

For every connected graph $u, v \in G, d^M(u,v) > d(u,v)$.

Proof: In a connected graph $G, d(u,v)$ is the shortest path between two vertices. By the definition of M −distance $d^M(u,v) + \sum_{u \in P} \deg(u) + \sum_{v \in P} |n(v)|$. Then the result is true.

II. APPLICATION OF M-DISTANCE

Graph theory is significantly applied in the field of computer networking to structure a minimum distance path and or an optimal route. Today, due to the overwhelming growth of wireless technology and availability of portable devices at very low price, a new type of networking known as the Ad hoc network has emerged.

Wireless multi-hop networks, in various forms, e.g. wireless sensor networks, underwater sensor networks, vehicular networks, mesh networks and UAV (Unmanned Aerial Vehicle) formations, and under various names, e.g. ad-hoc networks, hybrid networks, delay tolerant networks and intermittently connected networks, are being increasingly used in military and civilian applications. Graph theory, particularly a recently developed branch of graph theory, i.e. random geometric graphs, is well suited to studying these problems. These include but not limited to: cooperative communications; opportunistic routing; geographic routing; statistical characterization (e.g. connectivity, capacity and delay) of multi-hop wireless networks; geometric constraints among connected nodes and their use in autonomous parameter estimation without manual calibration.

The M-distance theorem discussed in this paper is more appropriate to overcome some of the main challenges that are high in ad hoc networks. Our next study will be oriented in addressing one such issue with M-distance theorem.

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