Maximum Solution of Interval Valued Fuzzy Relation Equation using Solution Operators

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Abstract

In this paper we define interval valued fuzzy relation equation and solution operators. Also, we find the maximal solution of interval valued fuzzy relation equation with respect to unknowns R and Q. Examples are given to illustrate its effectiveness.

Keywords

Fuzzy composition, Fuzzy matrices, Fuzzy relational equation and Interval valued fuzzy matrices.

I. INTRODUCTION

A fuzzy matrix is a matrix over a fuzzy algebra and the algebraic operations on fuzzy matrices are max-min operations, which are different from that of the standard operations on real and complex matrices. Thomason has introduced the concept of fuzzy matrices [17]. After that a lot of works have been done on fuzzy matrices and its variants [10, 13, 14]. It is well known that the membership valued completely depends on the decision makers, its habit, mentality etc. So, sometimes it happens that the membership valued cannot be measured as a point but, it can be measured appropriately as an interval. The elements of Fuzzy matrix are the subintervals of the unit interval [0,1], then the fuzzy matrix is known as interval valued fuzzy matrix (IVFM). The concept of IVFMS as a generalization of fuzzy matrix was introduced and developed by shyamal and pal [16], by extending the max-min operations on fuzzy algebra F=[0,1], for elements $a, b \in F$ determined as $a+b = \max\{a, b\}$ and $a \,.\, b = \min\{a,b\}$ and the standard order ≥ of real numbers over F. A study on fuzzy relational equation was introduced by sanchez [15] and later developed by many researchers. A new algorithm is proposed to solve the fuzzy relation equation P o Q = R with max-min composition and max-product composition [5]. In [9], consistency of the interval valued fuzzy relational equations and the complete set of solutions of xA = b where A is an interval valued fuzzy matrix and b is an interval valued vector is determined and equivalent condition for the existence of maximum solution is obtained. In [11] modified algorithm is proposed to solve the interval valued fuzzy relational equation as a generalization of that of fuzzy relational equations. In section 2, we present the basic definitions and example required results on interval valued fuzzy relational equation. In section 3, we introduce interval valued fuzzy relation equations and solution operators. In section 4, we discuss the methods of finding the maximal solution of interval valued fuzzy relational equation with respect to unknowns R and Q.

II. PRELIMINARIES

1) ***Fuzzy Relation*:** Let X and Y be two non empty sets. A fuzzy relation R between X and Y is a fuzzy subset of X × Y where $\mu_R : X \times Y \rightarrow [0, 1]$

2) ***Interval Valued Fuzzy Relation*:** Let X and Y be two non empty interval valued fuzzy sets. An interval valued fuzzy relation R between X and Y is defined as an interval valued fuzzy subset of X ×Y associated with an interval-valued membership function $\mu_R(x,y) = [\mu_{R_L}(x, y), \mu_{R_U}(x, y)]$ for all (x, y) in X × Y where the values of $\mu_{R_L}(x, y)$ and $\mu_{R_U}(x, y)$ denote the left and right end points of $\mu_R(x, y)$ with $0 \le \mu_{R_L}(x, y) \le \mu_{R_U}(x, y) \le 1$

3) ***Interval valued Inverse fuzzy Relation :*** A fuzzy relation $R^{-1} \subseteq Y \times X$ is called the inverse of the interval valued fuzzy relation $R \subseteq X \times Y$ which is defined as $R^{-1}(y, x) = R(x, y)\ \forall x \in X$ and $\forall y \in Y$ for all pairs $(y, x) \in Y \times X$ and $\mu_{R^{-1}}(y, x) = \mu_R(x, y)$ where $\mu_R = [\mu_{R_L}, \mu_{R_U}]$, $\mu_{R^{-1}}$ is the membership function of R^{-1}. R^{-1} is also defined as $R^{-1} = R^t$ and
$(R^{-1})^{-1} = R$.

A. Interval Valued Fuzzy Relation Composition Rules

1) ***Max-Min composition***: Consider two interval valued fuzzy relations R_1 and R_2

$R_1(x, y) \subseteq X \times Y$ and $R_2(y, z) \subseteq Y \times Z$

The max-min composition of R_1 and R_2 is given as follows:

$$R_1\ o\ R_2 = \left\{(x, z), \max_{y \in Y} (\min(\mu_{R_1}(x, y), \mu_{R_2}(y, z))\right\}$$

$$[R_{1L}, R_{1U}]\ o\ [R_{2L}, R_{2U}] = \left\{(x, z), \max_{y \in Y} \left(\min\left([\mu_{R_{1L}}(x, y), \mu_{R_{1U}}(x, y)], [\mu_{R_{2L}}(y, z), \mu_{R_{2U}}(y, z)]\right)\right)\right\}$$

2) ***Example of Max-Min Composition:*** Consider X = $\{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and $z = \{z_1, z_2\}$

Let $R_1 \subseteq X \times Y$ and $R_2 \subseteq Y \times Z$ be two interval valued fuzzy relations

$$R_1 = \begin{array}{c|ccc|} & y_1 & y_2 & y_3 \\ x_1 & [.2, .3] & [.4, .8] & [.6, .9] \\ x_2 & [.7, .9] & [.2, .4] & [.3, .6] \\ x_3 & [.5, .7] & [.6, .8] & [.6, .1] \end{array} \text{ and}$$

$$R_2 = \begin{array}{c|cc|} & z_1 & z_2 \\ y_1 & [.5, .8] & [0, .4] \\ y_2 & [.3, .5] & [.6, .8] \\ y_3 & [7, .9] & [.2, .5] \end{array}$$

Now we compute $R_1\ o\ R_2$ by using max-min composition

$$\mu_{R_1 o R_2}(x_1, z_1) = \max\left\{\min\left([.2, .3], [.5, .8]\right), \min\left([.4, .8], [.3, .5]\right), \min\left([.6, .9], [.7, .9]\right)\right\} = [.6, .9]$$

$$\mu_{R_1 o R_2}(x_2, z_1) = \max\left\{\min\left([.7, .9], [.5, .8]\right), \min\left([.2, .4], [.3, .5]\right), \min\left([.3, .6], [.7, .9]\right)\right\} = [.5, .8]$$

$$\mu_{R_1 o R_2}(x_3, z_1) = \max\left\{\min\left([.5, .7], [.5, .8]\right), \min\left([.6, .8], [.3, .5]\right), \min\left([.6, .1], [.7, .9]\right)\right\} = [.6, .9]$$

$$\mu_{R_1 o R_2}(x_1, z_2) = \max\left\{\min\left([.2, .3], [0, .4]\right), \min\left([.4, .8], [.6, .8]\right), \min\left([.6, .9], [.2, .5]\right)\right\} = [.4, .8]$$

$$\mu_{R_1 o R_2}(x_3, z_2) = \max\{\min([.7, .9], [0, .4]), \min([.6, .8], [.6, .8]), \min([.6, .1], [.2, .5])\} = [.6, .8]$$

$$\mu_{R_1 o R_2}(x_2, z_2) = \max\left\{\min\left([.7, .9], [.0, .4]\right), \min\left([.2, .4], [.6, .8]\right), \min\left([.3, .6], [.2, .5]\right)\right\} = [.2, .5]$$

By using the max-min composition we have the following result.

$$T = R_1 \, o \, R_2 = \begin{array}{c c} & \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} [.6,.9] & [.4,.8] \\ [.5,.8] & [.2,.5] \\ [.6,.9] & [.6,.8] \end{bmatrix} \end{array}$$

III. INTERVAL VALUED FUZZY RELATION EQUATION and SOLUTION OPERATORS

1) *Interval Valued Fuzzy Relation Equation:* Let $R = \left[r_{ij} \right]_{m \times n} = \left[r_{ijL}, r_{ijU} \right]_{m \times n}$ and $Q = \left[q_{jk} \right]_{m \times n} = \left[q_{jkL}, q_{jkU} \right]$ be two interval valued Fuzzy matrix, then interval valued Fuzzy Relation Equations (IVFRE) have the form

$$T = R o Q \qquad (1)$$

where R, Q and T are the interval valued Fuzzy relations and o is the max-min composition.

2) *Inverse Interval Valued Fuzzy Relation Equation:*
Consider IVFRE of the form $T = R o Q$, the inverse interval valued Fuzzy relation is defined as
$T^{-1} = (R o Q)^{-1}$
As $(R o Q)^{-1} = Q^{-1} o R^{-1}$ then inverse interval valued fuzzy relation is $T^{-1} = Q^{-1} o R^{-1}$ (2)

3) *Example of Interval Valued Fuzzy Relation Equations:*
Let X = $\{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$ and $z = \{z_1, z_2\}$

Consider two interval valued fuzzy relations $R \subseteq X \times Y$ and $T \subseteq X \times Z$ with membership degrees $\mu_R\,(x, y)$ and $\mu_T\,(x, z)$ respectively.

We have to compute a interval valued fuzzy relation Q by using the IVFRE of the form $T = R \, o \, Q$.

Taking R^{-1} on both sides we have $R^{-1} o \, T = Q$, then $Q = R^{-1} o \, T$,where $R^{-1} \subseteq Y \times X$ is the inverse relation $R \subseteq X \times Y$ with $\mu_{R^{-1}}(y, x) = \mu_R(x, y)$

$$\text{Let} \quad T = \begin{array}{c c} & \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} [.3,.6] & [.4,.7] \\ [0,.3] & [.7,.9] \end{bmatrix} \end{array}$$

$$R = \begin{array}{c c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} [.1,.4] & [.2,.7] & [.5,.9] \\ [.3,.7] & [.5,.7] & [0,.8] \end{bmatrix} \end{array}$$

$$R^{-1} = \begin{array}{c c} & \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} & \begin{bmatrix} [.1,.4] & [.3,.7] \\ [.2,.7] & [.5,.7] \\ [.5,.9] & [0,.8] \end{bmatrix} \end{array}$$

where $R^{-1} \in Y \times X$

Now we compute Q by using the operation o.

Here $\mu_Q(y, z) = \max_{x \in X}\{(\min(\mu_{R^{-1}}(y, x), \mu_T(x, z)\}$

$$\mu_Q(y_1, z_1) = \max\left\{\min\left([.1, .4], [.3, .6]\right), \min\left([.3, .7], [0, .3]\right)\right\} = \max\left\{[.1, .4], [.0, .3]\right\} = [.1, .4]$$

$$\mu_Q(y_1, z_2) = \max\left\{\min\left([.1, .4], [.4, .7]\right), \min\left([.3, .7], [.7, .9]\right)\right\} = \max\left\{[.1, .4], [.3, .7]\right\} = [.3, .7]$$

$$\mu_Q(y_2, z_1) = \max\left\{\min\left([.2, .7], [.3, .6]\right), \min\left([.5, .7], [0, .3]\right)\right\} = \max\left\{[.2, .6], [0, .3]\right\} = [.2, .6]$$

$$\mu_Q(y_2, z_2) = \max\left\{\min\left([.2, .7], [.4, .7]\right), \min\left([.5, .7], [.7, .9]\right)\right\} = \max\left\{[.2, .7], [.5, .7]\right\} = [.5, .7]$$

$$\mu_Q(y_3, z_1) = \max\left\{\min\left([.5, .9], [.3, .6]\right), \min\left([0, .8], [0, .3]\right)\right\} = \max\left\{[.3, .6], [0, .3]\right\} = [.3, .6]$$

$$\mu_Q(y_3, z_2) = \max\left\{\min\left([.5, .9], [.4, .7]\right), \min\left([0, .8], [.7, .9]\right)\right\} = \max\left\{[.4, .7], [0, .8]\right\} = [.4, .8]$$

Therefore Q = $R^{-1} \, o \, T$ = $\begin{array}{c c} & \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} & \begin{bmatrix} [.1, .4] & [.3, .7] \\ [.2, .6] & [.5, .7] \\ [.3, .6] & [.4, .8] \end{bmatrix} \end{array}$, $Q \in Y \times Z$

A. α - Operator

Let $A = \left[a_{ij}\right]_{m \times n}$ and $B = \left[b_{ij}\right]_{m \times n}$ be two interval valued fuzzy matrix, then α operator is defined as

A α B = $\left(\left[a_{ij}\right] \alpha \left[b_{ij}\right]\right)_{m \times n}$

$$= \begin{cases} [1, 1] & \text{if } \left[a_{ijL}, a_{ijU}\right] \le \left[b_{ijL}, b_{ijU}\right] \\ \left[b_{ijL}, b_{ijU}\right] & \text{if } \left[a_{ijL}, a_{ijU}\right] > \left[b_{ijL}, b_{ijU}\right] \end{cases} \qquad (3)$$

It is also called Sanchez Operator.

1) *Remark*: If $\left[a_{ijL}\right] \alpha \left[b_{ijL}\right] = 1$ and $\left[a_{ijU}\right] \alpha \left[b_{ijU}\right] = b_{ijU}$

then $\left[a_{ijL}, b_{ijL}\right] \alpha \left[a_{ijU}, b_{ijU}\right] = \left[b_{ijU}, 1\right]$

2) *Example of α - Operator*: Let A = $\begin{bmatrix} [.2, .5] & [.6, .9] \\ [.3, .4] & [.7, .8] \end{bmatrix}$ and B = $\begin{bmatrix} [.5, .9] & [.6, .7] \\ [.3, .6] & [.2, .5] \end{bmatrix}$ then A α B =

$\begin{bmatrix} [1, 1] & [.7, 1] \\ [1, 1] & [.2, .5] \end{bmatrix}$

3)Properties of α - Operator:

- If $\left[b_{ijL}, b_{ijU}\right] = 0$, then $\left[a_{ijL}, a_{ijU}\right] \alpha \left[b_{ijL}, b_{ijU}\right]$ will be given as

$$A\,\alpha\,0 = \left\{ \begin{array}{ll} [1,\,1] & \text{if } [a_{ijL}, a_{ijU}] = [b_{ijL}, b_{ijU}] \\ [0,\,0] & \text{if } [a_{ijL}, a_{ijU}] > [b_{ijL}, b_{ijU}] \end{array} \right\}$$

- If

$$A = [a_{ij}]_{m\times n} = [a_{ijL},\ a_{ijU}] = [0,\ 0]$$

and $B = [b_{ij}]_{m\times n} = [b_{ijL}, b_{ijU}]$ then

$$A\ \alpha\ B = [0, 0]\ \alpha\ [b_{ijL},\ b_{ijU}] = [1, 1]$$

- If $A = [a_{ij}]_{m\times n} = [a_{ijL},\ a_{ijU}] = [0,\ 0]$ and $B = [b_{ij}]_{m\times n} = [b_{ijL},\ b_{ijU}] = [1,\ 1]$ then

$$A\ \alpha\ B = [a_{ijL},\ a_{ijU}]\ \alpha\ [1,\ 1] = [1,\ 1]$$

- If $A = [a_{ij}]_{m\times n} = [a_{ijL},\ a_{ijU}] = [1,\ 1]$ and $B = [b_{ij}]_{m\times n} = [b_{ijL},\ b_{ijU}]$ then

$$A\ \alpha\ B = [1, 1]\ \alpha\ [b_{ijL}, b_{ijU}] = [b_{ijL}, b_{ijU}]$$

- α - Operator is not commutative

$$[a_{ijL},\ a_{ijU}]\ \alpha\ [b_{ijL},\ b_{ijU}] \neq [b_{ijL},\ b_{ijU}]\ \alpha\ [a_{ijL},\ a_{ijU}]$$

Let $A = \begin{bmatrix} [.2, .5] & [.6, .9] \\ [.3, .4] & [.7, .8] \end{bmatrix}$ and

$B = \begin{bmatrix} [.5, .9] & [.6, .7] \\ [.3, .6] & [.2, .5] \end{bmatrix}$

$$A\ \alpha\ B = \begin{bmatrix} [1,\ 1] & [.7, .1 \\ [1,\ 1] & [.2,\ .5] \end{bmatrix} \quad (4)$$

$$B\ \alpha\ A = \begin{bmatrix} [.2, .5] & [1, 1] \\ [1, 1] & [1, 1] \end{bmatrix} \quad (5)$$

From equation (4) & (5) $A\,\alpha\,B \neq B\,\alpha\,A$

Therefore α - Operator is not commutative.

- α - Operator is not associative

$$[a_{ijL}, a_{ijU}]\ \alpha\ \left([b_{ijL}, b_{ijU}]\ \alpha\ [c_{ijL}, c_{ijU}]\right) \neq \left([a_{ijL}, a_{ijU}]\ \alpha\ [b_{ijL}, b_{ijU}]\right)\ \alpha\ [c_{ijL}, c_{ijU}]$$

Let $A = \begin{bmatrix} [.2, .6] & [.4, .7] \\ [.5, .7] & [.3, .8] \end{bmatrix}$, $B = \begin{bmatrix} [.1, .4] & [.3, .7] \\ [.5, .6] & [.6, .8] \end{bmatrix}$ and $C = \begin{bmatrix} [.3, .7] & [.4, .8] \\ [.2, .6] & [.7, .9] \end{bmatrix}$

$$\text{Consider } A\alpha(B\alpha C) = \begin{bmatrix} [.2, .6] & [.4, .7] \\ [.5, .7] & [.3, .8] \end{bmatrix}\ \alpha\ \begin{bmatrix} [1, 1] & [1, 1] \\ [.2,\ 1] & [1, 1] \end{bmatrix} = \begin{bmatrix} [1, 1] & [1, 1] \\ [.2, 1] & [1, 1] \end{bmatrix} \quad (6)$$

Consider $(A\alpha B)\alpha C = \begin{bmatrix} [.1,.4] & [.3,.1] \\ [.6,1] & [1,1] \end{bmatrix} \alpha \begin{bmatrix} [.3,.7] & [.4,.8] \\ [.2,.6] & [.7,.9] \end{bmatrix} = \begin{bmatrix} [1,1] & [.8,1] \\ [.2,.6] & [.7,.9] \end{bmatrix}$ (7)

From equation (6) & (7) $A\alpha(B\alpha C) \neq (A\alpha B)\alpha C$.

B. γ- Operator

Let $A = \left[a_{ij}\right]_{m\times n}$ and $B = \left[b_{ij}\right]_{m\times n}$ be two Interval valued Fuzzy matrix, then **γ** operator is defined as

$$A\gamma B = \begin{cases} [1,1] & \text{if } \left[a_{ijL}, a_{ijU}\right] = \left[b_{ijL}, b_{ijU}\right] \\ [0,0] & \text{if } \left[a_{ijL}, a_{ijU}\right] \neq \left[b_{ijL}, b_{ijU}\right] \end{cases}$$

γ- Operator is also known as equality operator.

1) ***Remark :*** If $\left[a_{ijL}\right] \gamma \left[b_{ijL}\right] = 1$ and $\left[a_{ijU}\right] \gamma \left[b_{ijU}\right] = 0$ then $\left[a_{ijL}, b_{ijL}\right] \gamma \left[a_{ijU}, b_{ijU}\right] = [0, 1]$

2) ***Example of γ- Operator:*** Let A = $\begin{bmatrix} [.2,.5] & [.6,.9] \\ [.3,.4] & [.7,.8] \end{bmatrix}$ and B = $\begin{bmatrix} [.5,.9] & [.6,.7] \\ [.3,.6] & [.2,.5] \end{bmatrix}$ then A **γ** B =

$$\begin{bmatrix} [0,0] & [0,1] \\ [0,1] & [0,0] \end{bmatrix}$$

3) ***Properties of γ- Operator:***

- γ- Operator holds the commutative property
- γ- Operator does not holds the associative property.

C. σ- Operator

Let $A = \left[a_{ij}\right]_{m\times n}$ and $B = \left[b_{ij}\right]_{m\times n}$ be two Interval valued Fuzzy matrix, then σ operator is defined as

$$A\ \sigma\ B = \left(\left[a_{ij}\right] \sigma \left[b_{ij}\right]\right)_{m\times n}$$

$$= \begin{cases} [0,0] & \text{if} \left[a_{ijL}, a_{ijU}\right] < \left[b_{ijL}, b_{ijU}\right] \\ \left[b_{ijL}, b_{ijU}\right] & \text{if} \left[a_{ijL}, a_{ijU}\right] \geq \left[b_{ijL}, b_{ijU}\right] \end{cases}$$

1) Remark: If $\left[a_{ijL}\right] \sigma \left[b_{ijL}\right] = b_{ijL}$ *and* $\left[a_{ijU}\right] \sigma \left[b_{ijU}\right] = 0$, then $\left[a_{ijL}, b_{ijL}\right] \sigma \left[a_{ijU}, b_{ijU}\right] = \left[0, b_{ijL}\right]$

2) ***Properties of σ- Operator***:

- σ - Operator does not holds the commutative property
- σ - Operator does not satisfies the associative property

D. ε- Operator

Let $A = \left[a_{ij}\right]_{m\times n}$ and $B = \left[b_{ij}\right]_{m\times n}$ be two Interval valued Fuzzy matrix, then ε operator is defined as

$$A\ \varepsilon\ B = \left(\left[A_{ij} \right] \varepsilon \left[B_{ij} \right] \right)_{m\times n}$$

$$= \begin{cases} [0,\ 0] & if \left[a_{ijL},\ a_{ijU} \right] \geq \left[b_{ijL},\ b_{ijU} \right] \\ \left[b_{ijL},\ b_{ijU} \right] & if \left[a_{ijL},\ a_{ijU} \right] < \left[b_{ijL},\ b_{ijU} \right] \end{cases}$$

1) ***Remark:*** If $\left[a_{ijU} \right] \varepsilon \left[b_{ijU} \right] = 0 \quad and \quad \left[a_{ijL} \right] \varepsilon \left[b_{ijL} \right] = b_{ijL}$, then $\left[a_{ijL}, b_{ijU} \right] \sigma \left[b_{ijL}, b_{ijU} \right] = \left[0, b_{ijL} \right]$

2) *Properties of ε - Operator*:

- ε - Operator does not hold the commutative property
- ε- Operator does not hold the associative property.

E. Composition of α - Operator

Consider two interval valued fuzzy relations $R \subseteq X \times Y$ and $Q \subseteq Y \times Z$. Relationship between these two interval valued fuzzy relations when using α composition is defined as $R\ \alpha\ Q \subseteq X \times Z$ with membership function is defined as

$$\mu_{R\alpha Q}(x,\ z) = \max_{y\varepsilon Y} \left\{ \mu_R(x,\ y)\ \alpha\ \mu_Q(y,\ z) \right\} \qquad \forall x \in X,\ y \in Y\ and\ z \in Z$$

1) *Example of α Composition*: Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and $Z = \{z_1, z_2, z_3\}$

Consider two interval valued fuzzy relations $R \subseteq X \times Y$ and $Q \subseteq Y \times Z$ which are given below respectively. We are to compute $T \subseteq X \times Z$ using α composition.

$$R = \begin{array}{c c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} [.1, .4] & [.3, .6] & [.2, .5] \\ [.2, .7] & [.4, .7] & [.3, .8] \\ [0, .3] & [.2, .1] & [.6, .9] \end{bmatrix} \end{array}$$

$$Q = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} & \begin{bmatrix} [.3, .7] & [.2, .8] & [.6, .7] \\ [.2, .4] & [.3, .7] & [.5, .8] \\ [.4, .7] & [.5, .6] & [.2, .9] \end{bmatrix} \end{array}$$

$$T(x,\ z) = R(x,\ y)\ @\ Q(y,\ z) \qquad \forall x \in X, y \in Y\ and\ z \in Z$$

$$T(x,\ z) = R(x,\ y) @ Q(y,\ z) = \begin{bmatrix} [.1, .4] & [.3, .6] & [.2, .5] \\ [.2, .7] & [.4, .7] & [.3, .8] \\ [0, .3] & [.2, .1] & [.6, .9] \end{bmatrix} \alpha \begin{bmatrix} [.3, .7] & [.2, .8] & [.6, .7] \\ [.2, .4] & [.3, .7] & [.5, .8] \\ [.4, .7] & [.5, .6] & [.2, .9] \end{bmatrix}$$

$$\mu_{R\alpha Q}(x_1,\ z_1) = \min_{y \in Y} \left\{ [.1,\ .4]\ \alpha\ [.3,\ .7], [.3,\ .6]\ \alpha\ [.2,\ .4], [.2,\ .5]\ \alpha\ [.4,\ .7] \right\}$$

$$= \min \{ [1,\ 1], [-2,\ .4], [1,\ 1] \} = [.2,\ .4]$$

$$\mu_{R\alpha Q}(x_1,\ z_2) = \min_{y \in Y} \left\{ [.1, .4]\ \alpha\ [.2, .8], [.3, .6]\ \alpha\ [.3, .7], [.2, .5]\ \alpha\ [.5, .6] \right\}$$

$$= \min \left\{ [1, 1], [1, 1], [1, 1] \right\} = [1, 1]$$

$$\mu_{R\alpha Q}(x_1, z_3) = \min_{y\in Y}\left\{[.1,\ .4]\ \alpha\ [.6,\ .7], [.3,\ .6]\ \alpha\ [.5,\ .8], [.2,\ .5]\ \alpha\ [.2,\ .9]\right\}$$
$$= \min\left\{[1, 1], [1, 1], [1, 1]\right\} = [1,1]$$

$$\mu_{R\alpha Q}(x_2, z_1) = \min_{y\in Y}\left\{[.2,\ .7]\ \alpha\ [.3,\ .7], [.4,\ .7]\ \alpha\ [.2,\ .4], [.3,\ .8]\ \alpha\ [.4,\ .7]\right\}$$
$$= \min\left\{[1, 1], [.2,.4], [.7, 1]\right\} = [.2,\ .4]$$

$$\mu_{R\alpha Q}(x_2, z_2) = \min_{y\in Y}\left\{[.2,.7]\ \alpha\ [.2,\ .8], [.4,\ .7]\ \alpha\ [.3,\ .7], [.3,\ .8]\ \alpha\ [.5,\ .6]\right\}$$
$$= \min\left\{[1, 1], [.3,\ 1], [.6, 1]\right\} = [.3, 1]$$

$$\mu_{R\alpha Q}(x_2, z_3) = \min_{y\in Y}\left\{[.2,\ .7]\ \alpha\ [.6,\ .7], [.4,\ .7]\ \alpha\ [.5,\ .8], [.3,\ .8]\ \alpha\ [.2,\ .9]\right\}$$
$$= \min\left\{[1,1], [1,1], [.2,1]\right\} = [.2, 1]$$

$$\mu_{R\alpha Q}(x_3, z_1) = \min_{y\in Y}\left\{[0,\ .3]\ \alpha\ [.3,\ .7], [.2,\ 1]\ \alpha\ [.2,\ .4], [.6,\ .9]\ \alpha\ [.4,\ .7]\right\}$$
$$= \min\left\{[1, 1], [.4, 1], [.4,\ .7]\right\} = [.4, .7]$$

$$\mu_{R\alpha Q}(x_3, z_2) = \min_{y\in Y}\left\{[0,\ .3]\ \alpha\ [.2,\ .8], [.2, 1]\ \alpha\ [.3,\ .7], [.6,\ .9]\ \alpha\ [.5,\ .6]\right\}$$
$$= \min\left\{[1,1], [.7,1], [.5,\ .6]\right\} = [.5,.6]$$

$$\mu_{R\alpha Q}(x_3, z_3) = \min_{y\in Y}\left\{[0,\ .3]\ \alpha\ [.6,\ .7], [.2,\ 1]\ \alpha\ [.5,\ .8]\ [.6,\ .9]\ \alpha\ [.2,\ .9]\right\}$$
$$= \min\left\{[1, 1], [.8, 1], [.2,\ .9]\right\} = [.2,\ .9]$$

$$\text{Therefore T (x,z)} = \begin{array}{c|ccc} & z_1 & z_2 & z_3 \\ \hline x_1 & [.2,.4] & [1, 1] & [1, 1] \\ x_2 & [.2,.4] & [.3,\ 1] & [.2, 1] \\ x_3 & [.4,.7] & [.5, .6] & [.2, .9] \end{array}$$

F. Composition of γ - Operator

Consider two interval valued fuzzy relations $R \subseteq X \times Y$ and $Q \subseteq Y \times Z$. Relationship between these two interval valued fuzzy relations when using γ composition is defined as
$R\ \gamma\ Q \subseteq X \times Z$ with the membership function is defined as

$$\mu_{R\gamma Q}(x, z) = \min_{y\varepsilon Y}\left\{\mu_R(x, y)\ \gamma\ \mu_Q(y, z)\right\} \quad \forall x \in X, y \in Y \ \ and \ \ z \in Z$$

1) *Example of γ Composition*: Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$ and $Z = \{z_1, z_2\}$

Consider two interval valued fuzzy relations $R \subseteq X \times Y$ and $Q \subseteq Y \times Z$ are given below respectively.
We are to compute $T \subseteq X \times Z$ using γ composition.

$$R = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & [.2, .5] & [.3, .6] & [.6, .8] \\ x_2 & [.4, .7] & [.5, .9] & [.2, .3] \end{array}$$

z_1 z_2

$$Q = \begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array}\begin{bmatrix} [.3, .7] & [.5, .8] \\ [.6, .7] & [.2, .4] \\ [.6, .9] & [.1, .5] \end{bmatrix}$$

$$R \gamma Q = \begin{bmatrix} [.2, .5] & [.3, .6] & [.6, .8] \\ [.4, .7] & [.5, .9] & [.2, .3] \end{bmatrix} \gamma \begin{bmatrix} [.3, .7] & [.5, .8] \\ [.6, .7] & [.2, .4] \\ [.6, .9] & [.1, .5] \end{bmatrix}$$

$$\mu_{R\gamma Q}(x_1, z_1) = \min\left\{ [.2, .5] \gamma [.3, .7], [.3, .6] \gamma [.6, .7], [.6, .8] \gamma [.6, .9] \right\} = [0, 0]$$

$$\mu_{R\gamma Q}(x_1, z_2) = \min\left\{ [.2, .5] \gamma [.5, .8], [.3, .6] \gamma [.2, .4], [.6, .8] \gamma [.1, .5] \right\} = [0, 0]$$

$$\mu_{R\gamma Q}(x_2, z_1) = \min\left\{ [.4, .7] \gamma [.3, .7], [.5, .9] \gamma [.6, .7], [.2, .3] \gamma [.6, .9] \right\} = [0, 0]$$

$$\mu_{R\gamma Q}(x_2, z_2) = \min\left\{ [.4, .7] \gamma [.5, .8], [.5, .9] \gamma [.2, .4], [.2, .3] \gamma [.1, .5] \right\} = [0, 0]$$

$$R \gamma Q = \begin{array}{c} \\ x_1 \\ x_2 \end{array}\begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} [0, 0] & [0, 0] \\ [0, 0] & [0, 0] \end{bmatrix} \end{array}$$

Let us point out some other properties of α-operator as follows:

- min (a, (a @ b)) = min (a, b) $\leq$ b
- a α b $\geq$ b
- a α max (b, c) $\geq$ a @ b
- a @ min (a, b) $\geq$ b
- a @ max (b, c) $\geq$ a @ c

IV. MAXIMUM SOLUTION of INTERVAL VALUED FUZZY RELATION EQUATION

In this section we discuss the methods of finding the maximal solution of interval valued fuzzy relation equation (IVFRE) with respect to unknowns R and Q. The methods for finding the maximal Q and maximal R respectively for IVFRE of the form R o Q=T. Here ∇ denotes the maximal solution.

A. Determination of Maximal Q

For determination of maximal Q, we discuss some results.

1) *Theorem:* If R and Q are two interval valued Fuzzy relations and $R \subseteq X \times Y$ and $Q \subseteq Y \times Z$, then $Q \subseteq R^{-1} @ \left(R o Q \right)$.

where R $= \left[R_L, R_U \right] = \left[r_{ijL}, r_{ijU} \right]$

Q $= \left[Q_L, Q_U \right] = \left[q_{ijL}, q_{ijU} \right]$

o = max-min composition and

@ = composition made by α-operator.

Proof:

Let A $= R^{-1} @ \left(R \; o \; Q \right) \subseteq Y \times Z$

$$= \left[R_L^{-1} @ \left(R_L \; o \; Q_L \right), R_U^{-1} @ \left(R_U \; o \; Q_U \right) \right] \subseteq Y \times Z$$

Consider $\mu_{A_L}(y, z) = \min\limits_{x \in X} \left\{ \mu_{R_L^{-1}}(y, x) \; \alpha \; \mu_{R_L o Q_L}(x, z) \right\}$

$$= \min_{x \in X} \left\{ \mu_{R_L} \left(x,\ y\right) \ \alpha \ \mu_{R_L o Q_L} \left(x,\ z\right) \right\}$$

$$= \min_{x \in X} \left\{ \mu_{R_L} \left(x,\ y\right) \ \alpha \ \left(\max_{t \in Y} \left\{ \min \left[\mu_{R_L} \left(x,\ t\right), \ \mu_{Q_L} \left(t,\ z\right) \right] \right\} \right) \right\}$$

$$= \min_{x \in X} \left\{ \mu_{R_L} (x,\ y) \ \alpha \ \min \left(\left[\mu_{R_L} (x,\ y),\ \mu_{Q_L} (y,\ z) \right], \max_{t \in Y,\, t \neq y} \left\{ \min \left[\mu_{R_L} (x,\ t),\ \mu_{Q_L} (t,\ z) \right] \right\} \right) \right\}$$

Now $\mu_{A_L} \left(y,\ z\right) \geq \min_{x \in X} \left\{ \mu_{R_L} \left(x,\ y\right) \alpha \min \left(\mu_{R_L} \left(x,\ y\right), \ \mu_Q \left(y,\ z\right) \right) \right\}$

We know that a α (min (a, b)) $\geq$ b

$\mu_{A_L} \left(y,\ z\right) \geq \mu_{Q_L} \left(y,\ z\right) \quad \forall\ y \in Y \ and \ z \in Z$

Hence $Q_L \subseteq R_L^{-1}\ @\ \left(R_L\ o\ Q_L\right)$. Similarly $Q_U \subseteq R_U^{-1}\ @\ \left(R_U\ o\ Q_U\right)$. Therefore $Q \subseteq R^{-1}\ @\ \left(R\ o\ Q\right)$.

2) ***Example :*** Let $X = \left\{x_1,\ x_2,\ x_3\right\}$, $Y = \left\{y_1,\ y_2,\ y_3\right\}$ and $Z = \left\{z_1,\ z_2,\ z_3\right\}$

Given R =
$$\begin{array}{c c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} [.2,\ .3] & [.4,\ .5] & [.5,\ .6] \\ [.1,\ .5] & [.3,\ .7] & [.4,\ .6] \\ [.4,\ .6] & [.5,\ .9] & [.6,\ .8] \end{bmatrix} \end{array}$$

$$Q = \begin{array}{c c} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} & \begin{bmatrix} [.3,\ .5] & [.6,\ .7] & [.3,\ .6] \\ [.2,\ .4] & [.3,\ .4] & [.6,\ .7] \\ [.7,\ .9] & [.5,\ .8] & [.9,\ 1] \end{bmatrix} \end{array}$$

$$R \text{ o } Q = \begin{bmatrix} [.5,\ .6] & [.5,\ .6] & [.5,\ .6] \\ [.4,\ .6] & [.4,\ .6] & [.4,\ .7] \\ [.6,\ .8] & [.5,\ .8] & [.6,\ .8] \end{bmatrix}$$

$$R^{-1} @\ RoQ = \begin{bmatrix} [.2,\ .3] & [.1,\ .5] & [.4,\ .6] \\ [.4,\ .5] & [.3,\ .7] & [.5,\ .9] \\ [.5,\ .6] & [.4,\ .6] & [.6,\ .8] \end{bmatrix} @ \begin{bmatrix} [.5,.6] & [.5,\ .6] & [.5,\ .6] \\ [.4,\ .6] & [.4,\ .6] & [.4,\ .7] \\ [.6,\ .8] & [.5,\ .8] & [.6,\ .8] \end{bmatrix}$$

$$= \begin{bmatrix} \min\left([1,1] \quad [1,1] \quad [1,1] \right) & \min\left([1,1] \quad [1,1] \quad [1,1] \right) & \min\left([1,1] \quad [1,1] \quad [1,1] \right) \\ \min\left([1,1] \quad [.6,\ 1] \quad [.8,1] \right) & \min\left([1,1] \quad [.6,\ 1] \quad [.8,1] \right) & \min\left([1,1] \quad [1,1] \quad [.8,\ 1] \right) \\ \min\left([1,1] \quad [1,1] \quad [1,1] \right) & \min\left([1,1] \quad [1,1] \quad [.5,1] \right) & \min\left([1,1] \quad [1,1] \quad [1,1] \right) \end{bmatrix}$$

$$= \begin{bmatrix} [1,1] & [1,1] & [1,1] \\ [.6,\ 1] & [.6,\ 1] & [.8,\ 1] \\ [1,1] & [.5,1] & [1,1] \end{bmatrix}$$

Therefore Q $\subseteq$ R^{-1} @ $(R\, o\, Q)$.

3) ***Theorem*:** If R and Q be two interval valued fuzzy relations $R \subseteq X \times Y$ and $Q \subseteq Y \times Z$, then $R o \left(R^{-1} @\ T \right) \subset T$

where R $= \left[R_L, R_U\right] = \left[r_{ijL}, r_{ijU}\right]$

Q $= \left[Q_L, Q_U\right] = \left[q_{ijL}, q_{ijU}\right]$

o = max-min composition

@= composition made by α-operator.

4) *Theorem*: If R and Q be a two interval valued fuzzy relations $R \subseteq X \times Y$ and $Q \subseteq Y \times Z$, then $R \subseteq \left(Q\,@\,(R\,o\,Q)^{-1}\right)^{-1}$

where R $= \left[R_L, R_U\right] = \left[r_{ijL}, r_{ijU}\right]$

Q $= \left[Q_L, Q_U\right] = \left[q_{ijL}, q_{ijU}\right]$

o = max-min composition , @ = composition made by α-operator.

5) *Theorem:* *If* Q and T be the two interval valued fuzzy relations $Q \subseteq Y \times Z$ and $T \subseteq X \times Z$ then $\left(Q\ @\ T^{-1}\right)^{-1} o\ Q \subset T$

where Q $= \left[Q_L, Q_U\right] = \left[q_{ijL}, q_{ijU}\right]$

T $= \left[T_L, T_U\right] = \left[t_{ijL}, t_{ijU}\right]$

o = max-min composition, @ = composition made by α-operator.

6) *Theorem*:

Let $R = \left[R_L, R_U\right] \subseteq X \times Y$ and $T = \left[T_L, T_U\right] \subseteq X \times Z$ be the two interval valued fuzzy relations, the set of fuzzy relations $Q = \left[Q_L, Q_U\right] \in Y \times Z$ such that $\left[R_L, R_U\right] o \left[Q_L, Q_U\right] \subseteq \left[T_L, T_U\right]$ contains a greatest element $\left[R_L^{-1}, R_U^{-1}\right]\ @\ \left[T_L, T_U\right]$.

7) *Theorem*:

Let $R = \left[R_L, R_U\right] \subseteq X \times Y$ and $T = \left[T_L, T_U\right] \subseteq X \times Z$ be the two interval valued fuzzy relations,

$Q = \left[Q_L, Q_U\right] \in Y \times Z$ such that $\left[R_L, R_U\right] o \left[Q_L, Q_U\right] \subseteq \left[T_L, T_U\right]$.

$$S\left[Q_L, Q_U\right] = \left\{\left[Q_L, Q_U\right] \in Y \times Z \,\middle|\, \left[R_L, R_U\right] o \left[Q_L, Q_U\right] = \left[T_L, T_U\right]\right\} \neq \varphi$$

and $\left[R_L^{-1}, R_U^{-1}\right] @ \left[T_L, T_U\right] \in S\left[Q_L, Q_U\right]$, then $\left[R_L^{-1}, R_U^{-1}\right] @ \left[T_L, T_U\right]$ is the greatest element in $S\left[Q_L, Q_U\right]$

Proof:

Let $S\left[Q_L, Q_U\right]^*$

$$= \left\{ \text{Interval valued Fuzzy } \left[Q_L, Q_U\right] \in \left(Y \times Z\right) \,\middle|\, \left[R_L, R_U\right] o \left[Q_L, Q_U\right] \subseteq \left[T_L, T_U\right] \right\}$$ and $S\left[R_L, R_U\right]^* \neq \varphi$

Let $\left[Q_L, Q_U\right] \subseteq S\left(\left[R_L, R_U\right]\right)^* : \left[R_L, R_U\right] o \left[Q_L, Q_U\right] = \left[T_L, T_U\right]$ then, we have

$$\left[R_L^{-1}, R_U^{-1}\right] @ \left(\left[R_L, R_U\right] o \left[Q_L, Q_U\right]\right) \subseteq \left[R_L^{-1}, R_U^{-1}\right] @ \left[T_L, T_U\right]$$

we know that $\left[Q_L, Q_U\right] \subseteq \left[R_L^{-1}, R_U^{-1}\right] @ \left(\left[R_L, R_U\right] o \left[Q_L, Q_U\right]\right)$

then it shows that

$$\left[Q_L, Q_U\right] \subseteq \left[R_L^{-1}, R_U^{-1}\right] @ \left[T_L, T_U\right]$$

By theorem 4.1.5 we have

$$\left[R_L^{-1}, R_U^{-1}\right] @ \left[T_L, T_U\right] \in S\left[Q_L, Q_U\right]$$

Then it shows that $\left[R_L^{-1}, R_U^{-1}\right] @ \left[T_L, T_U\right] \in S\left(\left[Q_L, Q_U\right]\right)^*$

then $\left[R_L^{-1}, R_U^{-1}\right] @ \left[T_L, T_U\right]$ will be the greatest element in $S\left(\left[Q_L, Q_U\right]\right)^*$.

Hence $\left[R_L^{-1}, R_U^{-1}\right] @ \left[T_L, T_U\right]$ be the greatest element in $S\left(\left[Q_L, Q_U\right]\right)^*$.

Therefore $\left[Q_L^{\nabla}, Q_U^{\nabla}\right] = \left[R_L^{-1}, R_U^{-1}\right] @ \left[T_L, T_U\right]$

which is the maximum relation $\left[Q_L, Q_U\right]$ satisfying the equation $\left[R_L, R_U\right] o \left[Q_L, Q_U\right] = \left[T_L, T_U\right]$.

8) *Necessary condition for existence of* Q^{∇} :

The necessary condition for the existence of $Q^{\nabla} = \left[Q_L^{\nabla}, Q_U^{\nabla}\right]$ satisfying the interval valued fuzzy relation equation (1) is

$$\mu_{\left[T_L, T_U\right]}(x, z) \leq \max_{y \in Y} \mu_{\left[R_L, R_U\right]}(x, y) \qquad \forall x \in X \text{ and } z \in Z \qquad (8)$$

9) *Example of determining the maximal* Q^{∇} :

Consider $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and $Z = \{z_1, z_2, z_3\}$.

Suppose $\left[R_L, R_U\right] \subseteq X \times Y$ *and* $\left[T_L, T_U\right] \subseteq X \times Z$ are two interval valued fuzzy relations given below respectively.

$$R = \begin{array}{c c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} [.6, .8] & [.3, .4] & [.7, .9] \\ [.2, .3] & [.7, .1] & [.4, .6] \\ [.5, .7] & [.8, .9] & [.3, .6] \end{bmatrix} \end{array}$$

and

$$T = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} [.4, .5] & [.3, .4] & [.4, .5] \\ [.6, .7] & [.3, .6] & [.5, .7] \\ [.6, .7] & [.3, .6] & [.5, .7] \end{bmatrix} \end{array}$$

$Q^{\nabla} = R^{-1} @ T$

$$= \begin{bmatrix} [.6, .8] & [.2, .3] & [.5, .7] \\ [.3, .4] & [.7, .1] & [.8, .9] \\ [.7, .9] & [.4, .6] & [.3, .6] \end{bmatrix} @ \begin{bmatrix} [.4, .5] & [.3, .4] & [.4, .5] \\ [.6, .7] & [.3, .6] & [.5, .7] \\ [.6, .7] & [.3, .6] & [.5, .7] \end{bmatrix}$$

$$= \begin{bmatrix} \min\left([.4,.5] \; [1,1] \; [1,1]\right) & \min\left([.3,.4] \; [1,1] \; [.3,.6]\right) & \min\left([.4,.5] \; [1,1] \; [1,1]\right) \\ \min\left([1,1] \; [.6,.7] \; [.6,.7]\right) & \min\left([1,1] \; [.3,.6] \; [.3,.6]\right) & \min\left([1,1] \; [.5,.7] \; [.5,.7]\right) \\ \min\left([.4,.5] \; [1,1] \; [1,1]\right) & \min\left([.3,.4] \; [.3,1] \; [1,1]\right) & \min\left([.4,.5] \; [1,1] \; [1,1]\right) \end{bmatrix}$$

$$Q^{\nabla} = \begin{bmatrix} [.4, .5] & [.3, .4] & [.4, .5] \\ [.6, .7] & [.3, .6] & [.5, .7] \\ [.4, .5] & [.3, .4] & [.4, .5] \end{bmatrix}$$

$\left[Q_L^{\nabla}, Q_U^{\nabla}\right]$ satisfies IVFRE (1)

i.e., $\left[R_L, R_U\right] \, o \, \left[Q_L^{\nabla}, Q_U^{\nabla}\right] = \left[T_L, T_U\right]$

$$R \, o \, Q^{\nabla} = \begin{bmatrix} [.6, .8] & [.3, .4] & [.7, .9] \\ [.2, .3] & [.7, 1] & [.4, .6] \\ [.5, .7] & [.8, .9] & [.3, .6] \end{bmatrix} o \begin{bmatrix} [.4, .5] & [.3, .4] & [.4, .5] \\ [.6, .7] & [.3, .6] & [.5, .7] \\ [.4, .5] & [.3, .4] & [.4, .5] \end{bmatrix}$$

$$= \begin{bmatrix} [.4, .5] & [.3, .4] & [.4, .5] \\ [.6, .7] & [.3, .4] & [.5, .7] \\ [.6, .7] & [.3, .6] & [.5, .7] \end{bmatrix}$$

$$= \left[T_L, T_U\right]$$

$$= T.$$

10) *Example of determining the maximal* Q^{∇} **:**

Consider $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$ and $Z = \{z_1, z_2\}$

Suppose $\left[R_L, R_U\right] \subseteq X \times Y$ *and* $\left[T_L, T_U\right] \subseteq X \times Z$ are two interval valued fuzzy relations given below respectively.

we are to compute $\left[Q_L^{\nabla}, Q_U^{\nabla}\right]$ when

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & \quad y_2 \end{array} \\ \begin{bmatrix} [.2, .4] & [.6, .8] \\ [.3, .8] & [.7, 1] \\ [.5, .9] & [.2, .3] \end{bmatrix} \end{array}$$

$$T = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & \quad z_2 \end{array} \\ \begin{bmatrix} [.6, .7] & [.3, .4] \\ [.6, .7] & [.3, .5] \\ [.5, .6] & [.2, .5] \end{bmatrix} \end{array}$$

Now $Q^{\nabla} = \begin{bmatrix} [.6, 1] & [.2, .5] \\ [.6, .7] & [.3, .4] \end{bmatrix}$

Q^{∇} also satisfies the IVFRE. i.e., $R \, o \, Q = T$

B. Determination of maximal R

For determination of maximal R, we discuss some results

1) *Theorem*: Let $Q = \left[Q_L,\ Q_U\right] \subseteq Y \times Z$ and $T = \left[T_L, T_U\right] \subseteq X \times Z$ be the two interval valued fuzzy relations, then the set of interval valued fuzzy relations $\left[R_L, R_U\right] \in X \times Y$ such that $\left[R_L,\ R_U\right] o \left[Q_L, Q_U\right] \subseteq \left[T_L, T_U\right]$ contains a greatest element $\left(\left[Q_L, Q_U\right] @ \left[T_L, T_U\right]\right)^{-1}$

2) *Theorem*: Let $Q = \left[Q_L, Q_U\right] \subseteq Y \times Z$ and $T = \left[T_L, T_U\right] \subseteq X \times Z$ be the two interval valued fuzzy relations, $S\left(\left[R_L,\ R_U\right]\right)$ be the set of interval valued fuzzy relations $[R_L, R_U] \in X \times Y$ such that $\left[R_L,\ R_U\right] o \left[Q_L, Q_U\right] = \left[T_L, T_U\right]$ $S\left[R_L, R_U\right] = \left\{ Interval\ valued\ fuzzy \left[R_L, R_U\right] \in X \times Y \middle| \left[R_L, R_U\right] o \left[Q_L, Q_U\right] = \left[T_L, T_U\right] \right\} \neq \varphi$

if and only if $\left(\left[Q_L, Q_U\right] @ \left[T_L^{-1}, T_U^{-1}\right]\right)^{-1} \in S\left(\left[R_L, R_U\right]\right)$ and it is the greatest element in $S\left(\left[R_L, R_U\right]\right)$

Proof:

Let $S\left(\left[R_L, R_U\right]\right)^* = \left\{ Interval\ valued\ fuzzy \left[R_L, R_U\right] \in X \times Y \middle| \left[R_L, R_U\right] o \left[Q_L, Q_U\right] \subseteq \left[T_L, T_U\right] \right\}$

and $S\left(\left[R_L, R_U\right]\right)^* \neq \varphi$

Let $\left[R_L, R_U\right] \subseteq S\left(\left[R_L, R_U\right]\right)^* : \left[R_L, R_U\right] o \left[Q_L, Q_U\right] = \left[T_L, T_U\right]$

then we have $\left(\left[Q_L, Q_U\right] @ \left(\left[R_L, R_U\right] o \left[Q_L, Q_U\right]\right)^{-1}\right)^{-1} \subseteq \left(\left[Q_L, Q_U\right] @ \left[T_L, T_U\right]^{-1}\right)^{-1}$

By known lemma, we have

$$\left[R_L, R_U\right] \subseteq \left(\left[Q_L, Q_U\right] @ \left(\left[R_L, R_U\right] o \left[Q_L, Q_U\right]\right)^{-1}\right)^{-1}$$

then it shows that

$$\left[R_L, R_U\right] \subseteq \left(\left[Q_L, Q_U\right] @ \left[T_L, T_U\right]^{-1}\right)^{-1}$$

Now from theorem B . 1) we have

$$\left(\left[Q_L, Q_U\right] @ \left[T_L, T_U\right]^{-1}\right)^{-1} \in S\left(\left[R_L, R_U\right]\right)$$

Then it shows that $\left(\left[Q_L, Q_U\right] @ \left[T_L, T_U\right]^{-1}\right)^{-1} \in S\left(\left[R_L, R_U\right]\right)^*$

then $\left(\left[Q_L, Q_U\right] @ \left[T_L, T_U\right]^{-1}\right)^{-1}$ will be the greatest element in $S\left(\left[R_L, R_U\right]\right)^*$. So

$\left[R_L^{\nabla}, R_U^{\nabla}\right] = \left(\left[Q_L, Q_U\right] @ \left[T_L, T_U\right]^{-1}\right)^{-1}$

is the maximum relation for $\left[R_L, R_U\right]$ satisfying the equation $\left[R_L, R_U\right] o \left[Q_L, Q_U\right] = \left[T_L, T_U\right]$

3) *Necessary condition for Existence of* R^{∇} :

The necessary condition for the existence of $R^{\nabla} = \left[R_L^{\nabla}, R_U^{\nabla}\right]$ satisfying the IVFRE (1) is

$$\mu_{\left[T_L, T_U\right]}\left(x, z\right) \leq \max_{y \in Y} \mu_{\left[Q_L, Q_U\right]}\left(y, z\right) \qquad \forall x \in X \text{ and } z \in Z$$

4) *Example of Determining the maximal* R^{∇} :

Consider $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and $Z = \{z_1, z_2, z_3\}$

Suppose $Q \subseteq Y \times Z$ *and* $T \subseteq X \times Z$ are two interval valued fuzzy relations given below respectively.

$$Q = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} & \begin{bmatrix} [.6, .8] & [.3, .4] & [.8, .9] \\ [.8, 1] & [.2, .3] & [.3, .4] \\ [.2, .4] & [.3, .7] & [.6, .7] \end{bmatrix} \end{array}$$

and $$T = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} [.4, .6] & [.3, .6] & [.5, .6] \\ [.6, .8] & [.3, .7] & [.6, .7] \\ [.2, .5] & [.3, .7] & [.4, .7] \end{bmatrix} \end{array}$$

Note**:** First we check the necessary condition for the existence of $\left[R_L^{\nabla}, R_U^{\nabla}\right]$ using theorem B. 3).

Here given Q and T satisfies the equation

$$\mu_{\left[T_L, T_U\right]}(x, z) \leq \max_{y \in Y} \mu_{\left[Q_L, Q_U\right]}(y, z)$$

Now we compute $\left[R_L^{\nabla}, R_U^{\nabla}\right] = \left(\left[Q_L^{\nabla}, Q_U^{\nabla}\right] @ \left[T_L, T_U\right]^{-1}\right)^{-1}$

$$\left[T_L, T_U\right]^{-1} = \begin{array}{c c} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array} & \begin{bmatrix} [.4, .6] & [.6, .8] & [.2, .5] \\ [.3, .6] & [.3, .7] & [.3, .7] \\ [.5, .6] & [.6, .7] & [.4, .7] \end{bmatrix} \end{array}$$

$$\left[Q_L, Q_U\right] @ \left[T_L, T_U\right]^{-1}$$

$$= \begin{bmatrix} [.6, .8] & [.3, .4] & [.8, .9] \\ [.8, 1] & [.2, .3] & [.3, .4] \\ [.2, .4] & [.3, .7] & [.6, .7] \end{bmatrix} @ \begin{bmatrix} [.4, .6] & [.6, .8] & [.2, .5] \\ [.3, .6] & [.3, .7] & [.3, .7] \\ [.5, .6] & [.6, .7] & [.4, .7] \end{bmatrix}$$

$$= \begin{bmatrix} \min\left([.4,.6] \quad [1,1] \quad [.5,.6]\right) & \min\left([1,1] \quad [1,1] \quad [.6,.7]\right) & \min\left([.2,.5] \quad [1,1] \quad [.4,.7]\right) \\ \min\left([.4,.6] \quad [1,1] \quad [1,1]\right) & \min\left([.6,.8] \quad [1,1] \quad [1,1]\right) & \min\left([.2,.5] \quad [1,1] \quad [1,1]\right) \\ \min\left([1,1] \quad [.6,1] \quad [.5,.6]\right) & \min\left([1,1] \quad [1,1] \quad [1,1]\right) & \min\left([1,1] \quad [1,1] \quad [.4,1]\right) \end{bmatrix}$$

$$= \begin{bmatrix} [.4, .6] & [.6, .7] & [.2, .5] \\ [.4, .6] & [.6, .8] & [.2, .5] \\ [.5, .6] & [1, 1] & [.4, 1] \end{bmatrix}$$

$$\left[R_L^{\nabla}, R_U^{\nabla}\right] = \left(\left[Q_L, Q_U\right] @ \left[T_L, T_U\right]^{-1}\right)^{-1}$$

$$= \begin{array}{c c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} [.4, .6] & [.4, .6] & [.5, .6] \\ [.6, .7] & [.6, .8] & [1, 1] \\ [.2, .5] & [.2, .5] & [.4, 1] \end{bmatrix} \end{array}$$

Note:

Here we check whether $\left[R_L^{\nabla}, R_U^{\nabla}\right]$ satisfies the IVFRE i.e., $\left[R_L, R_U\right] o \left[Q_L, Q_U\right] = \left[T_L, T_U\right]$ or not

$$\left[R_L^{\nabla}, R_U^{\nabla}\right] o \left[Q_L, Q_U\right]$$

$$= \begin{bmatrix} [.4, .6] & [.4, .6] & [.5, .6] \\ [.6, .7] & [.6, .8] & [1, 1] \\ [.2, .5] & [.2, .5] & [.4, 1] \end{bmatrix} o \begin{bmatrix} [.6, .8] & [.3, .4] & [.8, .9] \\ [.8, 1] & [.2, .3] & [.3, .4] \\ [.2, .4] & [.3, .7] & [.6, .7] \end{bmatrix}$$

$$= \begin{bmatrix} [.4, .6] & [.3, .6] & [.5, .6] \\ [.6, .8] & [.3, .7] & [.6, .7] \\ [.2, .5] & [.3, .7] & [.4, .7] \end{bmatrix}$$

$$= \left[T_L, T_U\right]$$

$$= \left[\; T \;\right]$$

Here $\left[R_L^{\nabla}, R_U^{\nabla}\right]$ satisfies the IVFRE $\left[R_L, R_U\right] o \left[Q_L, Q_U\right] = \left[T_L, T_U\right]$

V. CONCLUSION

In this paper, definition of interval valued fuzzy relation equation and some solution operators are introduced. Also we find the maximal solution of interval valued fuzzy relation equation using the composition $RoQ=T$. Numerical examples are given to clarify the developed theory.

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