On Relationship of Multiplication Groups and Isostrophic Quasigroups

Benard osoba[#1], oyebo t. Y[*2]
[1]Lecturer, Department of Physical science, Bells university of Technology, Ota. Nigeria.
Senior Lecturer, Department of Mathematics, Lagos State University, Ojo. Nigeria

Abstract

This study presented a kind of characterization of multiplication group of a quasi group $(\boldsymbol{Q},\ \circ)$ *and of a loop* $(\boldsymbol{Q},\ \cdot)$ *that are isostrophic, that is some parastrophes of quasigroup* $(\boldsymbol{Q},\ \circ)$ *with loops* $(\boldsymbol{Q},\cdot)$*. In particular, the middle multiplication groups of a quasi group* $(\boldsymbol{Q},\ \cdot)$ *and of loops* $(\boldsymbol{Q},\ \circ)$ *that are isostrophes* $(\boldsymbol{Q},\ \circ)$ *were studied. Relationship of middle multiplication groups of a quasi group* $(\boldsymbol{Q},\ \cdot)$ *to right(left) multiplication group of a loop* $(\boldsymbol{Q},\ \circ)$ *isostrophes were show to be coincided and their multiplication groups were show to be normal subgroups, using the concept of middle translation*

Keywords— *Quasi group, loop, Isostrophes, Multiplication groups, right (left, middle translation)*

I. INTRODUCTION

A non -empty set '$\boldsymbol{Q}$' with binary operation $'\boldsymbol{A}'$ is called a groupoid $(\boldsymbol{Q},\ \boldsymbol{A})$. Let $(\boldsymbol{Q},\ \boldsymbol{A})$ be a groupoid and $\boldsymbol{a}$ be fixed element in $\boldsymbol{Q}$ then the translation maps $\boldsymbol{L_a}$ and $\boldsymbol{R_a}$ is defined by $\boldsymbol{xL_a = a\cdot x}$ and $\boldsymbol{xR_a = x\cdot a}$ for all $\boldsymbol{a}\in\boldsymbol{Q}$. A groupoid $(\boldsymbol{Q},\ \boldsymbol{A})$ is called quasigroup $(\boldsymbol{Q},\ \cdot)$ if the maps $\boldsymbol{L(a)}: \boldsymbol{G}\to\boldsymbol{G}$ and $\boldsymbol{R(a)}: \boldsymbol{G}\to\boldsymbol{G}$ are bijections for all $\boldsymbol{a}\in\boldsymbol{Q}$ and iftheequations$a{\cdot}x{=}b$and $y\cdot a = b$ have respectively unique solutions $x = a\backslash b$ and $y = b/a$ for all $a,\ b\in Q$.The equations$a{\cdot}x{=}b$and $y\cdot a = b$ have respectively unique solutions $x = a\backslash b$ and $y = b/a$ for all $a, b\in Q$.

Aquasigroup $(\boldsymbol{Q},\ \cdot)$iscalledaloopif$a{\cdot}1{=}a{=}\ 1{\cdot}a$,forall$ainQ$.The group generated by these mappings are called multiplication groups Mlp $(Q,\cdot)$. We donate these groups generated by left, right and middle translations of a quasigroup $(\boldsymbol{Q},\cdot)$ by$\boldsymbol{LM}(\boldsymbol{Q},\ \cdot)$, $\boldsymbol{RM}(\boldsymbol{Q},\ \cdot)$ and $\boldsymbol{PM}(\boldsymbol{Q},\ \cdot)$ respectively [4].

Definition 1.2: A binary groupoid $(\boldsymbol{A},\boldsymbol{Q})$ with a binary operation $'\boldsymbol{A}'$ such that the equality $\boldsymbol{A(x_1,x_2) = x_3}$ knowledge of any two elements of $\boldsymbol{x_1,x_2,x_3}$ uniquely specifics the remaining one is called binary quasigroup [4]

II. PRELIMINARIES

Lemma 2.1: If a quasigroup $(\boldsymbol{Q},\cdot)$ is a group isotope, ie. $(\boldsymbol{Q},\cdot)\sim(\boldsymbol{Q},+)$, where $(\boldsymbol{Q},+)$is a group, then any parastrophe of this quasigroup also is a group isotope [5]

Lemma 2.2: Parastrophic image of a loop is a loop, either an unipotent right loop [4].

Let $(\boldsymbol{Q},\ \cdot)$ guasigroup. We donate the following translations

$RM(\mathrm{Q},\cdot) = \langle x\mathrm{R_a}\ |\mathrm{a}\in\mathrm{Q}\rangle = (x.a\ \ |\mathrm{x}\in\mathrm{Q})$

$LM(\mathrm{Q},\cdot) = \langle x\mathrm{L_a}\ |\mathrm{a}\in\mathrm{Q}\rangle = (a\cdot x\ \ |\mathrm{x}\in\mathrm{Q})$

$PM(\mathrm{Q},\cdot) = \langle x\mathrm{P_a}\mathrm{s}\ |\mathrm{a}\in\mathrm{Q}\rangle = (x\cdot s = a\ \ |\mathrm{x,s}\in\mathrm{Q})$ where L_a, R_a and P_a are permutations of the set Q.

Definition 2.3: Isostrophy of a quasigroup is the operation of parastrophy of the quasigroup and its isotopic image.

Definition 2.4: Quasigroups$(\boldsymbol{Q},\ \cdot)$ and $(\boldsymbol{Q},\ *)$are said to isotopic if there exist triple (α,β,γ) such that $\alpha x.\beta y = \gamma(x*y)$ for all $x, y\in Q$.

Definition 2.5:Let$(Q,\ \cdot)$be a groupoid (quasigroup, loop) and α,β, and γ be three bijections that map Q onto Q. The triple $\sigma = (\alpha,\beta,\gamma)$ is called an antotopism of $(Q,\cdot)$if and only if $\alpha x\cdot\beta y = \gamma(x\cdot y)$ for all $x, y\in Q$. If $\alpha = \beta = \gamma$, then σ is called the autotopism of $(Q,\cdot)$, this triple form a group called the autotopism group of $(Q,\ \cdot)$.

III. MAIN RESULTS

Lemma 3.1: Let $(Q\circ)$ be a quasigroup and the isostroph $(Q,\cdot)$ is a loop such that $x\cdot y=\beta(y)\circ\alpha(x)$. If α is antomorphism of $(Q,\cdot)$ then the following equalities hold:

(i) $P_z^{(\circ)}=\langle\beta^{-1}P_z^{(\cdot)-1}\alpha\mid z\in Q\rangle$

(ii) $PM(Q\circ)\rhd PM(Q,\cdot)$.

Proof: (i) Let $x\cdot y=\beta(y)\circ\alpha(x)=z$ for all $x,y\in Q$ and for any fixed element $z\in Q$.

Consider $x\cdot y=z\Rightarrow x\backslash^{(\cdot)}z=y\Longrightarrow P_z^{(\cdot)}x=y$ (1)

Consider $\beta(y)\circ\alpha(x)=z\Longrightarrow z/^{(\circ)}\alpha(x)=\beta(y)\Longrightarrow\beta(y)=P_z^{(\circ)-1}\alpha(x)$ (2)

Using equalities (1) and (2) we have $P_z^{(\circ)-1}\alpha=\beta P_z^{(\cdot)}\Leftrightarrow P_z^{(\circ)-1}=\beta P_z^{(\cdot)}\alpha^{-1}\Leftrightarrow P_z^{(\circ)}=\beta^{-1}P_z^{(\cdot)-1}\alpha$ (3)

Here, setting $\alpha=\beta$, (3) become $P_z^{(\circ)}=\alpha^{-1}P_z^{(\cdot)-1}\alpha\Leftrightarrow\alpha^{-1}P_z^{(\circ)-1}\alpha=P_z^{(\cdot)}$ (4)

(ii) There exist identity element $e\in Q$ such that$z=e\cdot z=\beta(z)\circ\alpha(e)$ for any fixed element $z\in Q$. This follow the last equality $R_{\alpha(e)}^{(\circ)}\beta(z)=z\Longrightarrow\beta=R_{\alpha(e)}^{(\circ)-1}$. Hence, $\alpha=R_{\alpha(e)}^{(\circ)-1}\in(Q,\circ)$

Next, let α be an automorphism of $(Q,\cdot)$, then $\alpha(x\cdot y)=\alpha(x)\cdot\alpha(y)$ for all $x,y\in Q$.

Let $\alpha(x)\cdot\alpha(y)=\alpha(x\cdot y)=z$ for any fixed element $z\in Q$.

Consider this equality $\alpha(x\cdot y)=z\Longrightarrow x\cdot y=\alpha^{-1}(z)\Longrightarrow x\backslash^{(\cdot)}\alpha^{-1}z=y\Longrightarrow P_{\alpha^{-1}(z)}^{(\cdot)}x=y$ (5)

Also consider this equality$\alpha(x)\cdot\alpha(y)=z\Longrightarrow\alpha(x)\backslash z=\alpha(y)\Longrightarrow P_z^{(\cdot)}\alpha(x)=\alpha(y)$ (6)

using (5) and (6), we have $\Longrightarrow P_z^{(\cdot)}\alpha=\alpha P_{\alpha^{-1}(z)}^{(\cdot)}\Longrightarrow P_z^{(\cdot)}=\alpha P_{\alpha^{-1}(z)}^{(\cdot)}\alpha^{-1}\Leftrightarrow P_z^{(\cdot)-1}=\alpha^{-1}P_{\alpha^{-1}(z)}^{(\cdot)-1}\alpha$ (7)

Now, for every fixed element $z\in Q$, using equalities(4) and (7), we want to show that for every $P_z^{(\cdot)}\in PM(Q,\cdot)$ and every

$P_z^{(\circ)}\in PM(Q,\circ)$ we have

$P_z^{(\circ)}P_z^{(\cdot)}P_z^{(\circ)-1}\in PM(Q,\cdot)$: that is$P_z^{(\circ)}P_z^{(\cdot)}P_z^{(\circ)-1}=\alpha^{-1}P_z^{(\cdot)-1}\alpha P_z^{(\cdot)}\alpha P_z^{(\cdot)}\alpha^{-1}=P_{\alpha(z)}^{(\cdot)-1}P_z^{(\cdot)}P_{\alpha(z)}^{(\cdot)}\in PM(Q,\cdot)$.

Also using (4) and (7), we want to show that for every $P_z^{(\cdot)}\in PM(Q,\cdot)$ and every $P_z^{(\circ)}\in PM(Q,\circ)$ we have $P_z^{(\circ)-1}P_z^{(\cdot)}P_z^{(\circ)}\in PM(Q,\cdot)$ which gives

$P_z^{(\circ)-1}P_z^{(\cdot)}P_z^{(\circ)}=\alpha P_z^{(\cdot)}\alpha^{-1}P_z^{(\cdot)}\alpha^{-1}P_z^{(\cdot)-1}\alpha=P_{\alpha(z)}^{(\cdot)}P_z^{(\cdot)}P_{\alpha(z)}^{(\cdot)-1}\in PM(Q,\cdot)$.

Let also consider, $P_z^{(\circ)-1}P_z^{(\cdot)-1}P_z^{(\circ)}$ that is

$P_z^{(\circ)-1}P_z^{(\cdot)-1}P_z^{(\circ)}=\left(P_z^{(\circ)}P_z^{(\cdot)}P_z^{(\circ)-1}\right)^{-1}=\left(P_{\alpha(z)}^{(\cdot)-1}P_z^{(\cdot)}P_{\alpha(z)}^{(\cdot)}\right)^{-1}\in PM(Q,\cdot)$

and

$P_z^{(\circ)}P_z^{(\cdot)-1}P_z^{(\circ)-1}=\left(P_z^{(\circ)-1}P_z^{(\cdot)}P_z^{(\circ)}\right)^{-1}=\left(P_{\alpha(z)}^{(\cdot)}P_z^{(\cdot)}P_{\alpha(z)}^{(\cdot)-1}\right)^{-1}\in PM(Q,\cdot)$. Here, we have obtained that $\Phi P^{(\cdot)}\Phi^{-1},\Phi^{-1}P^{(\cdot)}\Phi,\Phi P^{(\cdot)-1}\Phi^{-1},\Phi^{-1}P^{(\cdot)-1}\Phi\in PM(Q,\cdot)$ for each $\Phi\in PM(Q,\circ)$. Hence,$PM(Q\circ)\rhd PM(Q,\cdot)$■

Corollary 3.2 Let $(Q\circ)$ be a quasigroup and $(Q,\cdot)$ be a loop such that isostroph $(Q,\cdot)$ is given as $x\cdot y=\beta(y)\circ\alpha(x)$. where $\alpha,\beta\in S_Q$. If α is antomorphism of $(Q,\cdot)$ then$PM(Q\circ)=PM(Q,\cdot)$

Proof: using equality (3) in proposition (3.1), $P_z^{(\circ)}=\langle\beta^{-1}P_z^{(\cdot)-1}\alpha\mid\alpha,\beta\in S_3\rangle\in PM(Q,\cdot)$, for any fixed $z\in Q$, this imply that $PM(Q\circ)\supseteq PM(Q,\cdot)$ and using (4) $P_z^{(\cdot)}=\langle\alpha^{-1}P_z^{(\circ)-1}\alpha\mid\alpha,\beta\in S_Q\rangle\in PM(Q,\circ)$, this imply that $PM(Q\circ)\subseteq PM(Q,\cdot)$. Hence $PM(Q\circ)=PM(Q,\cdot)$■

Proposition 3.3 Let $(Q\circ)$ be a quasigroup and $(Q,\cdot)$ be a loop such that isostroph $(Q,\cdot)$ is given as $x\cdot y=\alpha(x)/\beta(y)$. where $\alpha,\beta\in S_Q$, If α is antomorphism of $(Q,\cdot)$ then the following hold:

(i) $L_z^{(\circ)} = \langle \beta^{-1} P_z^{(\cdot)-1} \alpha \mid z \in Q \ \rangle$

(ii) $LM(Q \circ) \rhd PM(Q, \cdot)$

Proof: (i) Let $x \cdot y = \alpha(x)/\beta(y) = z$ for all $x, y \in Q$ and for any fixed element $\in Q$.

Consider $x \cdot y = z \Rightarrow x \backslash^{(\cdot)} z = y \ \Longrightarrow \quad P_z^{(\cdot)} x = y$ (8)

Consider the equality $\alpha(x)/\beta(y) = z \Longrightarrow \beta(y) = z \backslash^{(\circ)} \alpha(x) \Longrightarrow L_z^{(\circ)-1} \alpha(x) = \quad \beta(y)$ (9)

Using equalities (8) and (9), we have

$L_z^{(\circ)-1} \alpha(x) = \beta P_z^{(\cdot)} x \Longrightarrow L_z^{(\circ)-1} \alpha = \beta P_z^{(\cdot)} \ \Longrightarrow \ L_z^{(\circ)-1} = \beta P_z^{(\cdot)} \alpha^{-1} \Leftrightarrow L_z^{(\circ)} = \beta^{-1} P_z^{(\cdot)-1} \alpha \ \ \Leftrightarrow \beta^{-1} L_z^{(\circ)-1} \alpha = P_z^{(\cdot)}$ (10)

(ii) There exist identity element $e \in Q$ such that$z = e \cdot z = \alpha(e)/\beta(z)$ for any fixed element $z \in Q$.

This follow $z = \alpha(e)/\beta(z) \Longrightarrow P_{\alpha(e)}^{(\circ)-1} \beta(z) \Longrightarrow \qquad \beta = P_{\alpha(e)}^{(\circ)} \in (Q, \circ)$ (11)

Here, $\alpha = \beta$ as we set e=1, that is $\beta = P_{\alpha(e)}^{(\circ)} 1 \Longrightarrow \ \beta = 1 \backslash \alpha(1) \Longrightarrow \beta = \alpha(1)$.

Using equality(11), then (10) become$L_z^{(\circ)-1} = P_{\alpha(e)}^{(\circ)} P_z^{(\cdot)} P_{\alpha(e)}^{(\circ)-1} \Leftrightarrow L_z^{(\circ)} = \quad P_{\alpha(e)}^{(\circ)-1} P_z^{(\cdot)-1} P_{\alpha(e)}^{(\circ)}$ (12)

Next, let α be an automorphism of $(Q, \cdot)$, then$\alpha(x) \cdot \alpha(y) = \alpha(x \cdot y)$ for all $x, y \in Q$.

Let $\alpha(x) \cdot \alpha(y) = \alpha(x \cdot y) = z$ for any fixed element $z \in Q$.

Consider this equality $\alpha(x \cdot y) = z \Longrightarrow x \cdot y = \alpha^{-1} z \Longrightarrow x \backslash^{(\cdot)} \alpha^{-1} z = y \Longrightarrow \ \ P_{\alpha^{-1}(z)}^{(\cdot)} x = y$ (13)

Also consider the equality$\alpha(x) \cdot \alpha(y) = z \Longrightarrow \ \alpha(x) \backslash z = \alpha(y) \Longrightarrow P_z^{(\cdot)} \alpha(x) = \alpha(y)$ (14)
using equalities(13) and (14), we have

$P_z^{(\cdot)} \alpha = \alpha P_{\alpha^{-1}(z)}^{(\cdot)} \ \Longrightarrow P_z^{(\cdot)} = \alpha P_{\alpha^{-1}(z)}^{(\cdot)} \alpha^{-1} \Leftrightarrow P_z^{(\cdot)-1} = \alpha^{-1} P_{\alpha^{-1}(z)}^{(\cdot)-1} \alpha$ (15) here,
using (11), equality (15) become$P_z^{(\cdot)} = P_{\alpha(e)}^{(\circ)} P_{\alpha^{-1}(z)}^{(\cdot)} P_{\alpha(e)}^{(\circ)-1} \Leftrightarrow P_z^{(\cdot)-1} = P_{\alpha(e)}^{(\circ)-1} P_{\alpha^{-1}(z)}^{(\cdot)-1} P_{\alpha(e)}^{(\circ)}$ (16)

Now, for every fixed element $z \in Q$, using equalities(12) and (16), we want to show that for every $P_z^{(\cdot)} \in PM(Q, \cdot)$ and every $L_z^{(\circ)} \in LM(Q, \circ)$ we have $L_z^{(\circ)} P_z^{(\cdot)} L_z^{(\circ)-1} \in PM(Q, \cdot)$, that is

$L_z^{(\circ)} P_z^{(\cdot)} L_z^{(\circ)-1} = P_{\alpha(e)}^{(\circ)-1} P_z^{(\cdot)-1} P_{\alpha(e)}^{(\circ)} P_z^{(\cdot)} P_{\alpha(e)}^{(\circ)} P_z^{(\cdot)} P_{\alpha(e)}^{(\circ)-1} = P_{\alpha(z)}^{(\cdot)-1} P_{(z)}^{(\cdot)} P_{\alpha(z)}^{(\cdot)} \in PM(Q,\cdot),$

and also using (12) and (16), we want to show that for every $P_z^{(\cdot)} \in PM(Q, \cdot)$ and every $L_z^{(\circ)} \in LM(Q, \circ)$ we have$L_z^{(\circ)-1} P_z^{(\cdot)} L_z^{(\circ)} \in PM(Q, \cdot)$, that is

$$L_z^{(\circ)-1} P_z^{(\cdot)} L_z^{(\circ)} = P_{\alpha(e)}^{(\circ)} P_z^{(\cdot)} P_{\alpha(e)}^{(\circ)-1} P_z^{(\cdot)} P_{\alpha(e)}^{(\circ)-1} P_z^{(\cdot)-1} P_{\alpha(e)}^{(\circ)} = P_{\alpha(z)}^{(\cdot)} P_{(z)}^{(\cdot)} P_{\alpha(z)}^{(\cdot)-1} \in PM(Q, \cdot)$$

Let also consider this $L_z^{(\circ)-1} P_z^{(\cdot)-1} L_z^{(\circ)}$:

$L_z^{(\circ)-1} P_z^{(\cdot)-1} L_z^{(\circ)} = \left(L_z^{(\circ)} P_z^{(\cdot)} L_z^{(\circ)-1}\right)^{-1} = \left(P_{\alpha(z)}^{(\cdot)-1} P_{(z)}^{(\cdot)} P_{\alpha(z)}^{(\cdot)}\right)^{-1} \in PM(Q, \cdot)$
and

$L_z^{(\circ)} P_z^{(\cdot)-1} L_z^{(\circ)-1} = \left(L_z^{(\circ)-1} P_z^{(\cdot)} L_z^{(\circ)}\right)^{-1} = \left(P_{\alpha(z)}^{(\cdot)} P_{(z)}^{(\cdot)} P_{\alpha(z)}^{(\cdot)-1}\right)^{-1} \in PM(Q, \cdot)$. We have obtained that $\Phi P^{(\cdot)} \Phi^{-1}, \Phi^{-1} P^{(\cdot)} \Phi, \Phi P^{(\cdot)-1} \Phi^{-1}, \ \Phi^{-1} P^{(\cdot)-1} \Phi, \ \ \in PM(Q, \cdot)$ for each $\Phi \in LM(Q, \circ)$. Hence, $LM(Q \circ) \rhd PM(Q, \cdot)$■

Corollary 3.4: Let $(Q \circ)$ be a quasigroup and $(Q, \cdot)$ be a loop such that isostroph $(Q, \cdot)$ is given as $x \cdot y = \alpha(x)/\beta(y)$. where $\alpha, \beta \in S_Q$, If α is antomorphism of$(Q, \cdot)$ then $LM(Q \circ) = PM(Q, \cdot)$

Proof: using the equality (12) in proposition (3.3), $L_z^{(\circ)} = P_{\alpha(e)}^{(\circ)-1} P_z^{(\cdot)-1} P_{\alpha(e)}^{(\circ)} \in LM(Q, \circ)$, this imply that $LM(Q \circ) \subseteq PM(Q, \cdot)$ with $\beta = P_{\alpha(e)}^{(\circ)} \in (Q, \circ)$. Also, using equality (16) in proposition (3.3), we have $P_z^{(\cdot)} = P_{\alpha(e)}^{(\circ)} L_z^{(\circ)-1} P_{\alpha(e)}^{(\circ)-1} \in PM(Q, \cdot)$, hence $LM(Q \circ) \supseteq PM(Q, \cdot)$ so, $LM(Q \circ) = PM(Q, \cdot)$■

Proposition 3.5. Let $(Q \circ)$ be a quasigroup and $(Q, \cdot)$ be a loop such that isostroph $(Q, \cdot)$ is given as $x \cdot y = \beta(y) \backslash \alpha(x)$. where $\alpha, \beta \in S_Q$. If α is antomorphism of $(Q, \cdot)$ then the following hold:

(i) $R_z^{(\circ)} = \langle \beta^{-1} P_z^{(\cdot)-1} \alpha \mid z \in Q \rangle$

(ii) $RM(Q \circ) \rhd PM(Q, \cdot)$.

Proof: (1) Let $x \cdot y = \beta(y) \backslash \alpha(x) = z$ for all $x, y \in Q$ and for any fixed element $z \in Q$.

Consider $x \cdot y = z \Rightarrow x \backslash^{(\cdot)} z = y \Longrightarrow \quad P_z^{(\cdot)} x = y$ (17)

Consider the equality $\beta(y) \backslash \alpha(x) = z \Longrightarrow \beta(y) = \alpha(x) /^{(\circ)} z \Longrightarrow R_z^{(\circ)-1} \alpha(x) = \beta(y)$ (18)
Using equalities (17) and (18), we have

$\beta P_z^{(\cdot)}(x) = R_z^{(\circ)-1} \alpha(x)$,this

follow$\beta P_z^{(\cdot)} = R_z^{(\circ)-1} \alpha \Longrightarrow P_z^{(\cdot)} = \beta^{-1} R_z^{(\circ)-1} \alpha \Longrightarrow \beta P_z^{(\cdot)} \alpha^{-1} = R_z^{(\circ)-1} \Longrightarrow R_z^{(\circ)} = \beta^{-1} P_z^{(\cdot)-1} \alpha$ ` (19)

(ii) Since $(Q, \cdot)$ is loop, there exist an identity element $e \in Q$ such that $z = e \cdot z = \beta(z) \backslash \alpha(e)$ for any fixed element $z \in Q$. This follow form the last equality

$$z = \beta(z) \backslash \alpha(e) \Longrightarrow \alpha(e) /^{\circ} z = \beta(z) \Longrightarrow P_{\alpha(e)}^{(\circ)-1}(z) = \beta(z) \Longrightarrow \beta = P_{\alpha(e)}^{(\circ)-1} \in (Q \circ) \quad (20)$$

Here, $\alpha = \beta$ if$e = 1$ as show above

Next, let α be an automorphism of $(Q, \cdot)$, then $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$ for all $x, y \in Q$.

Let $\alpha(x) \cdot \alpha(y) = \alpha(x \cdot y) = z$ for any fixed element $z \in Q$.

Consider $\alpha(x \cdot y) = z \Longrightarrow x \cdot y = \alpha^{-1}(z) \Longrightarrow x \backslash^{(\cdot)} \alpha^{-1}(z) = y \Longrightarrow P_{\alpha^{-1}(z)}^{(\cdot)} x = y$ (21)

Also consider $\alpha(x) \cdot \alpha(y) = z \Longrightarrow \alpha(x) \backslash z = \alpha(y) \Longrightarrow P_z^{(\cdot)} \alpha(x)\, \alpha(y)$ (22)

Here, using equalities (21) and (22), we have $\Longrightarrow P_z^{(\cdot)} \alpha = \alpha P_{\alpha^{-1}(z)}^{(\cdot)} \Longrightarrow P_z^{(\cdot)} = \alpha P_{\alpha^{-1}(z)}^{(\cdot)} \alpha^{-1} \Leftrightarrow$

$P_z^{(\cdot)-1} = \alpha^{-1} P_{\alpha^{-1}(z)}^{(\cdot)-1} \alpha = P_{\alpha(e)}^{(\circ)} P_{\alpha^{-1}(z)}^{(\cdot)-1} P_{\alpha(e)}^{(\circ)-1} \Leftrightarrow P_z^{(\cdot)} = P_{\alpha(e)}^{(\circ)-1} P_{\alpha^{-1}(z)}^{(\cdot)} P_{\alpha(e)}^{(\circ)}$ (23)

Now, for every fixed element $z \in Q$. Using equalities(19) and (23), we want to show that for every $P_z^{(\cdot)} \in PM(Q, \cdot)$ and every $R_z^{(\circ)} \in RM(Q,\circ)$ we have

$R_z^{(\circ)} P_z^{(\cdot)} R_z^{(\circ)-1} \in PM(Q, \cdot)$, that is

$R_z^{(\circ)} P_z^{(\cdot)} R_z^{(\circ)-1} = P_{\alpha(e)}^{(\circ)} P_z^{(\cdot)-1} P_{\alpha(e)}^{(\circ)-1} P_z^{(\cdot)} P_{\alpha(e)}^{(\circ)-1} P_z^{(\cdot)} P_{\alpha(e)}^{(\circ)} = P_{\alpha(z)}^{(\cdot)-1} P_{(z)}^{(\cdot)} P_{\alpha(z)}^{(\cdot)} \in PM(Q, \cdot)$ and using also(19) and (23), we want to show that for every$P_z^{(\cdot)} \in PM(Q, \cdot)$ and every$R_z^{(\circ)} \in RM(Q, \circ)$, we have $R_z^{(\circ)-1} P_z^{(\cdot)} R_z^{(\circ)} \in PM(Q, \cdot)$. That is

$R_z^{(\circ)-1} P_z^{(\cdot)} R_z^{(\circ)} = P_{\alpha(e)}^{(\circ)-1} P_z^{(\cdot)} P_{\alpha(e)}^{(\circ)} P_z^{(\cdot)} P_{\alpha(e)}^{(\circ)} P_z^{(\cdot)-1} P_{\alpha(e)}^{(\circ)-1} = P_{\alpha(z)}^{(\cdot)} P_{(z)}^{(\cdot)} P_{\alpha(z)}^{(\cdot)-1} \in PM(Q, \cdot)$ we also have the equality $R_z^{(\circ)-1} P_z^{(\cdot)-1} R_z^{(\circ)} = \left(R_z^{(\circ)} P_z^{(\cdot)} R_z^{(\circ)-1}\right)^{-1} = \left(P_{\alpha(z)}^{(\cdot)-1} P_{(z)}^{(\cdot)} P_{\alpha(z)}^{(\cdot)}\right)^{-1} \in PM(Q, \cdot)$

and

$R_z^{(\circ)} P_z^{(\cdot)-1} R_z^{(\circ)-1} = \left(R_z^{(\circ)-1} P_z^{(\cdot)} R_z^{(\circ)}\right)^{-1} = \left(P_{\alpha(z)}^{(\cdot)} P_{(z)}^{(\cdot)} P_{\alpha(z)}^{(\cdot)-1}\right)^{-1} \in PM(Q, \cdot)$.

Here, we have obtained that $\Phi P^{(\cdot)} \Phi^{-1}, \Phi^{-1} P^{(\cdot)} \Phi, \Phi P^{(\cdot)-1} \Phi^{-1}, \Phi^{-1} P^{(\cdot)-1} \Phi, \in PM(Q, \cdot)$ for each $\Phi \in RM(Q, \circ)$, hence, we have proved that $RM(Q \circ) \rhd PM(Q, \cdot)$■

Corollary 3.6: Let $(Q\circ)$ be a quasigroup and $(Q,\ \cdot)$ be a loop such that isostroph $(Q,\ \cdot)$ is given as $x\cdot y=\beta(y)\backslash\alpha(x)$. where $\alpha,\beta\in S_Q$. If α is antomorphism of$(Q,\ \cdot)$ then$RM(Q\circ)=PM(Q,\ \cdot)$

Proof: using the equality (19) and (23)in proposition (5.5), the proof is simple■

REFERENCES

[1] I. Grecu, (2014), On multiplication groups of isostrophic Quasigroup, Revista Stiintifica a UniversitatiideStatainMoldova, nr. 7(77).
[2] H. O. Pflugfelder,(1990)*Quasigroupsandloops:Introduction.*,SigmaSeriesinPureMath.7Heldermann Verlag,Berlin.
[3] P. Syrbu, and I. Grecu, (2013), On some groups related to middle Bol loops, Revist? a Stiintica a UniversitatiideStatdinMoldova,7(67),1018.
[4] V. A. Shcherbacov, (2011), A-nuclei and A-centers of quasigroup, Institute of mathematics and computer Science Academiy of Science of Moldova Academiei str. 5, Chisinau, MD -2028, Moldova.
[5] F. N. Sokhatskii. On isotopes of groups, I Ukrain. Mat. Zh., 49(10): 1387-1398, 1995. (in Ukrainian)
[6] R. H .Bruck, A Survey of Binary Systems, 1958, Springer-Verlag, Berlin-G¨ottinge-Heidelberg.
[7] R. Capodaglio Di Cocco, On the Isotopism and the Pseudo-Automorphisms of the Loops, Bollettino U.M.I. 7(1993), 199-205.