Mahgoub Transform Method for Solving Linear Fractional Differential Equations

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Abstract
In this paper, Mahgoub Transform Method has been introduced for solving linear Fractional Differential Equations (FDEs) with constant coefficients. The fractional derivatives are described in the Caputo sense. Some fundamental properties of Mahgoub Transform necessary in solving FDE are derived. The efficiency of this method has been demonstrated using examples.

Keywords - *Mahgoub transform, Fractional Differential Equations, Mittag-Leffler.*

I. INTRODUCTION

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. It has been used in various areas such as signal processing, image processing, control engineering, bioengineering, polymer networks, mechanics, and viscoelasticity [1].

Traditional and new integral transform methods have been applied to find the analytical solution of FDE. Some of them are Laplace, Mellin, Fourier, Sumudu, Natural, Elzaki and Kamal [2-8]. Many researchers have shown their interest in finding the numerical solution to both linear and nonlinear FDEs [9-11]. AbdelrahimMahgoub [12] introduced Mahgoub transform method to solve the ordinary differential equations. In this paper, we have introduced the Mahgoub transform method for finding the exact solution of FDEs with Caputo derivatives.

This paper has been organized as follows: In Section 2, basic definitions related to fractional calculus are given. In Section 3, the Mahgoub transform of fractional integrals and derivatives have been discussed. In Section 4, examples of FDE have been provided to illustrate the efficiency of this method.

II. PRELIMINARIES AND NOTATIONS

In this section, fundamental definitions and properties of fractional calculus referred in this paper are introduced.

Definition 1: The Riemann Liouville fractional integral $I^\alpha f(x)$ of order $\alpha \in R, \alpha > 0$ of function $f(x) \in C_\mu, \mu \geq -1$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma\alpha}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \qquad t > 0$$

$$I^0 f(t) = f(t)$$
(1)

Definition 2:The Caputo fractional derivative of order $\alpha \in R, \alpha > 0$ is given by

$$ {}^cD^\alpha \; f(t) = I^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1} f^m(\tau)d\tau, \; t > 0 \qquad (2)$$

where $m-1 \leq \alpha \leq m, m\epsilon\mathbb{N}^+$ and $\Gamma(.)$ denotes the Gamma function.

Definition 3: The Mittag-Leffler function of one parameter α is denoted by $E_\alpha(z)$ and is defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \; , \alpha > 0, Re(\alpha) > 0, z\epsilon C \qquad (3)$$

The Mittag-Leffler function with two parameters αandβ is denoted by $E_{\alpha,\beta}(z)$ and is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \alpha, \beta > 0, Re(\alpha), Re(\beta) > 0 \qquad (4)$$

where, C is the set of complex numbers. For $\beta = 1$, we get, $E_{\alpha,1}(z) = E_\alpha(z)$ which is the direct generalization of exponential series.

III. MAHGOUB TRANSFORM OF FRACTIONAL INTEGRALS AND DERIVATIVES

Mahgoub transform is defined on the set of continuous functions and exponential order. We consider functions in the set A defined by

$$A = \left\{ f(t) : |f(t)| < Pe^{\frac{|t|}{\epsilon_i}} if t \in (-1)^i \times [0, \infty), i = 1,2;\ \epsilon_i > 0 \right\} \quad (5)$$

where ϵ_1, ϵ_2 may be finite or infinite and the constant P must be finite.

Let $f \in A$, thenMahgoub transform is defined as

$$M[f(t)] = H(u) = u \int_0^\infty f(t) e^{-ut} dt,\ t \geq 0, \epsilon_1 \leq u \leq \epsilon_2 \quad (6)$$

Mahgoub transform of simple functions are given below:

$$(i) \quad M[1] = 1 \quad (7)$$

$$(ii) \quad M[t] = \frac{1}{u} \quad (8)$$

$$(iii) \quad M[t^2] = \frac{2}{u^2} \quad (9)$$

$$(iv) \quad M[t^n] = \frac{n!}{u^n} = \frac{\Gamma(n+1)}{u^n} \quad (10)$$

Mahgoub Transform for derivatives are:

$$\text{(i)} \ M[f'(t)] = uH(u) - uf(0) \quad (11)$$

$$(ii) M[f''(t)] = u^2 H(u) - u^2 f(0) - uf'(0) \quad (12)$$

$$(iii) M[f^n(t)] = u^n H(u) - \sum_{k=0}^{n-1} u^{n-k} f^k(0) \quad (13)$$

A. Convolution theorem

Let $F(u)$ and $G(u)$ denote the Mahgoub transform of $f(t)$ and $g(t)$ respectively.

Then

$$M[f(t) * g(t)] = \frac{1}{u} F(u) G(u) \quad (14)$$

B. Inverse Mahgoub transform

If $M[f(t)] = H(u)$, then$f(t)$ is called the inverse Mahgoub transform of $H(u)$. In symbol,

$$f(t) = M^{-1}[H(u)] \quad (15)$$

where M^{-1} is called the inverse Mahgoub transform operator.

Some fundamental properties of Mahgoub Transform necessary in solving FDE are given in the following theorems.

Theorem 1:

If $H(u)$ is Mahgoubtransformof$f(t)$, then Mahgoub transform ofRiemann-Liouville fractional integral is

$$M[I^\alpha f(t)] = u^{-\alpha} H(u) \quad (16)$$

for $m - 1 < \alpha \leq m, m \epsilon \mathbb{N}$.

Proof:

Consider the Riemann-Liouville fractional integral

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau$$

$$= \frac{1}{\Gamma(\alpha)} [t^{\alpha-1} * f(t)] \quad (17)$$

Applying the Mahgoub transform to Eqn. (17) and using (14), we get

$$M[I^\alpha f(t)] = u^{-\alpha} H(u)$$

This completes the proof.

Theorem 2:

If $H(u)$ is Mahgoub transform of $f(t)$, then Mahgoub transform of Caputo fractional derivative is

$$M[\,{}^{c}D^{\alpha} f(t)] = u^{\alpha}H(u) - \sum_{k=0}^{m-1} u^{\alpha-k} f^{k}(0) \tag{18}$$

for $m - 1 < \alpha \leq m, m \in \mathbb{N}$.

Proof:

Consider the Caputo fractional derivative

$${}^{c}D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{m}(\tau)d\tau,\ t > 0$$

$$= \frac{1}{\Gamma(m-\alpha)}\left[t^{m-\alpha-1} * f^{(m)}(t)\right] \tag{19}$$

Applying Mahgoub transform to Eqn. (19) and using (14), we get

$$M[\,{}^{c}D^{\alpha} f(t)] = \frac{1}{\Gamma(m-\alpha)} M(t^{m-\alpha-1})M(f^{(m)}(t))$$

$$= u^{\alpha}H(u) - \sum_{k=0}^{m-1} u^{\alpha-k} f^{k}(0)$$

after simplification. This completes the proof.

C. Mahgoub transform of Mittage-Leffler function

The Mahgoub transform of the Mittage-Leffler function is given by the following theorem.

Theorem 3:

$$M^{-1}\left[\frac{u^{\alpha-\beta+1}}{u^{\alpha}-a}\right] = t^{\beta-1}E_{\alpha,\beta}(at^{\alpha}),\quad |u^{\alpha} - a| < 1 \tag{20}$$

Proof:

Using the definition of Mahgoub transform, we have

$$M\left[t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})\right] = u\int_0^{\infty} t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})e^{-ut}dt$$

$$= \sum_{k=0}^{\infty} \frac{a^k}{\Gamma(\alpha k+\beta)} u\int_0^{\infty} t^{\alpha k+\beta-1}e^{-ut}dt$$

$$= \sum_{k=0}^{\infty} \frac{a^k}{\Gamma(\alpha k+\beta)} M\left(t^{\alpha k+\beta-1}\right) = \frac{u^{\alpha-\beta+1}}{u^{\alpha}-a}$$

Then,

$$M^{-1}\left[\frac{u^{\alpha-\beta+1}}{u^{\alpha}-a}\right] = t^{\beta-1}E_{\alpha,\beta}(at^{\alpha}).$$

IV. ILLUSTRATIVE EXAMPLES

Example 1:

Consider the inhomogeneous Bagley-Torvik equation

$$D^2y(t) + {}^{c}D^{\frac{3}{2}}y(t) + y(t) = 1 + t \tag{21}$$

$$y(0) = y'(0) = 1$$

Using Mahgoub Transform to Eqn. (21), we get

$$M(D^2y(t)) + M({}^{c}D^{\frac{3}{2}}y(t)) + M(y(t)) = M(1+t)$$

Using the properties (13) and (18), we get

$$H(u) = \left(1 + \frac{1}{u}\right)$$

By taking the Inverse Mahgoub Transform, we get the exact solution of this problem as

$$y(t) = 1 + t.$$

Example 2:

Consider the inhomogeneous linear equation

$$ {}^{c}D^{\alpha}y(t) + y(t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + t^2 - t, \tag{22}$$

$$\mathrm{y}(0) = 0\,, where\ \ 0 \le \alpha \le 1$$

Applying Mahgoub Transform to Eqn. (22), we get

$$M(\,{}^{c}D^{\alpha}y(t)) + M(y(t)) = M\left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\right) - M\left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right) + M(t^2 - t)$$

Using the properties (13) and (18), we get

$$H(u) = \frac{2}{u^2} - \frac{1}{u}$$

By taking the Inverse Mahgoub Transform, we get the exact solution of this problem as

$$y(t) = t^2 - t.$$

Example 3:

Consider the linear initial value problem

$$ {}^{c}D^{\alpha}y(t) + y(t) = 0, \tag{23}$$

$$y(0) = 1,\ y'(0) = 0, where\ \ 0 \le \alpha \le 2$$

ApplyingMahgoub Transform to Eqn. (23), we get

$$M(\,{}^{c}D^{\alpha}y(t)) + M(y(t)) = 0.$$

Using the properties (13) and (18), we get

$$H(u) = \frac{u^{\alpha}}{(u^{\alpha}+1)}$$

By taking the Inverse Mahgoub Transform, we get the exact solution of this problem as

$$y(t) = E_{\alpha}(-t^{\alpha}).$$

Example 4:

Consider the linear initial value problem

$$ {}^{c}D^{\alpha}y(t) = y(t) + 1,\ y(0) = 0, \tag{24}$$

$$where\ \ 0 \le \alpha \le 1$$

Using Mahgoub Transform to Eqn. (24), we get

$$M(\,{}^{c}D^{\alpha}y(t)) = M\big(y(t)\big) + M(1).$$

Using the properties (13) and (18), we get

$$H(u) = \frac{1}{(u^{\alpha}-1)}$$

By taking the Inverse Mahgoub Transform, we get the exact solution of this problem as

$$y(t) = t^{\alpha}E_{\alpha,\alpha+1}(t^{\alpha})$$

V. CONCLUSION

In this paper, the Mahgoub Transformation method has been successfully applied to obtain an exact solution of linear fractional ordinary differential equations with constant coefficients.Some fundamental properties of Mahgoub Transform necessary in solving FDE are derived.By solving the illustrative examples, it is concluded that the Mahgoub Transform is efficient, reliable and powerful for finding analytic solution of linear fractional differential equations with constant coefficients.

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