Generation Formula for Integer Solutions to Special Elliptic Paraboloids

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Abstract

Knowing a solution of ternary quadratic diophantine equation representing elliptic paraboloid, a general formula for generating sequence of solutions based on the given solution is illustrated.

Keywords

ternary quadratic, generation of solutions, elliptic paraboloid.

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I. INTRODUCTION

The subject of diophantine equations in number theory has attracted many mathematicians since antiquity. It is well-known that a diophantine equation is a polynomial equation in two or more unknowns with integer coefficients for which integer solutions are required. An integer solution is a solution such that all the unknowns in the equation take integer values. An extension of ordinary integers into complex numbers is the gaussian integers. A gaussian integer is a complex number whose real and imaginary parts are both integers. It is quite obvious that diophantine equations are rich in variety and there are methods available to obtain solutions either in real integers or in gaussian integers.

A natural question that arises now is, whether a general formula for generating sequence of solutions based on the given solution can be obtained? In this context, one may refer [1-7]. The main thrust of this communication is to show that the answer to the above question is in the affirmative in the case of the following ternary quadratic diophantine equations, each representing a elliptic paraboloid.

II. METHOD OF ANALYSIS

Illustration: 1

The ternary quadratic diophantine equation under consideration is

$$16x^2 + 9y^2 = 4z \tag{1}$$

Let (x_0, y_0, z_0) be any solution of (1).

The solution may be in real integers or in gausssian integers or in irrational numbers.

Let (x_1, y_1, z_1) be the second solution of (1), where

$$x_1 = h_0 - x_0 \ , \ y_1 = h_0 - y_0 \ , \ z_1 = z_0 + 6h_0^2 \tag{2}$$

in which h_0 is an unknown to be determined.

Substitution of (2) in (1) gives

$$h_0 = 32x_0 + 18y_0 \tag{3}$$

Using (3) in (2), the second solution is given by

$$x_1 = 31x_0 + 18y_0 \ , \ y_1 = 32x_0 + 17y_0 \tag{4}$$

$$z_1 = z_0 + 6(32x_0 + 18y_0)^2 \tag{5}$$

Let (x_2, y_2, z_2) be the third solution of (1), where

$$x_2 = h_1 - x_1 \ , \ y_2 = h_1 - y_1 \ , \ z_2 = z_1 + 6h_1^2$$

in which h_1 is an unknown to be determined.

The repetition of the above process leads to

$$h_1 = 7^2 h_0 \ , \ x_2 = 1537x_0 + 864y_0 \ , \ y_2 = 1536x_0 + 865y_0 \tag{6}$$

$$z_2 = z_0 + 6(32x_0 + 18y_0)^2 (1 + 49^2) \tag{7}$$

Now, assume the n^{th} solution of (1) to be (x_n, y_n, z_n) where

$$x_n = h_{n-1} - x_{n-1}\ ,\ y_n = h_{n-1} - y_{n-1}\ ,\ z_n = z_{n-1} + 6h_{n-1}^2$$

From (5) and (6), one can observe that

$$z_n = z_0 + 6(32x_0 + 18y_0)^2\left(1 + X + X^2 + ... + X^{n-1}\right)\ ,\ X = 49^2$$

$$= z_0 + \frac{1}{400}\left(49^{2n} - 1\right)(32x_0 + 18y_0)^2 \qquad (8)$$

To obtain the values of x_n, y_n, we proceed as follows:

The solution (4) is written in the matrix form as

$$(x_1, y_1)^t = M(x_0, y_0)^t$$

where t is the transpose and

$$M = \begin{bmatrix} 31 & 18 \\ 32 & 17 \end{bmatrix}$$

In general, $(x_n, y_n)^t = M^n(x_0, y_0)^t$

To find M^n, consider the characteristic equation

$$|M - \lambda I| = 0 \qquad (9)$$

where I is the unit matrix of order 2 and λ is the eigen value of M

Solving (9), the eigen values of M are given by $\lambda_1 = -1$, $\lambda_2 = 49$

It is well-known that

$$M^n = \frac{\lambda_1^n}{\lambda_1 - \lambda_2}(M - \lambda_2 I) + \frac{\lambda_2^n}{\lambda_2 - \lambda_1}(M - \lambda_1 I)$$

$$= \frac{1}{25}\begin{bmatrix} 9(-1)^n + 16(49)^n & 9\left(49^n - (-1)^n\right) \\ 16\left(49^n - (-1)^n\right) & 16(-1)^n + 9\left(49^n\right) \end{bmatrix}$$

Thus, we have

$$\left.\begin{aligned} x_n &= \frac{1}{25}\left[\left(9(-1)^n + 16(49)^n\right)x_0 + 9\left(49^n - (-1)^n\right)y_0\right] \\ y_n &= \frac{1}{25}\left[\left(16\left(49^n - (-1)^n\right)\right)x_0 + \left(16(-1)^n + 9\left(49^n\right)\right)y_0\right] \end{aligned}\right\} \qquad (10)$$

Thus (8), (10) represents the generation formula for the given elliptic paraboloid in terms of its given solution.

Illustration: 2

The ternary quadratic diophantine equation under consideration is

$$3x^2 + 4y^2 = 17z \qquad (1)$$

Let (x_0, y_0, z_0) be any solution of (1).

The solution may be in real integers or in gausssian integers or in irrational numbers.

Let (x_1, y_1, z_1) be the second solution of (1), where

$$x_1 = 2h_0 + x_0\ ,\ y_1 = h_0 + y_0\ ,\ z_1 = z_0 + h_0^2 \qquad (2)$$

in which h_0 is an unknown to be determined.

Substitution of (2) in (1) gives

$$h_0 = 12x_0 + 8y_0 \qquad (3)$$

Using (3) in (2), the second solution is given by

$$x_1 = 25x_0 + 16y_0\ ,\ y_1 = 12x_0 + 9y_0 \qquad (4)$$

$$z_1 = z_0 + (12x_0 + 8y_0)^2 \qquad (5)$$

Let (x_2, y_2, z_2) be the third solution of (1), where

$$x_2 = 2h_1 + x_1\ ,\ y_2 = h_1 + y_1\ ,\ z_2 = z_1 + h_1^2$$

in which h_1 is an unknown to be determined.

The repetition of the above process leads to

$$h_1 = 33h_0 \ , \ x_2 = 817x_0 + 544y_0 \ , \ y_2 = 408x_0 + 273y_0 \qquad (6)$$

$$z_2 = z_0 + (12x_0 + 8y_0)^2 (1 + 33^2) \qquad (7)$$

Now, assume the n^{th} solution of (1) to be (x_n, y_n, z_n) where

$$x_n = 2h_{n-1} + x_{n-1} \ , \ y_n = h_{n-1} + y_{n-1} \ , \ z_n = z_{n-1} + h_{n-1}^2$$

From (5) and (6), one can observe that

$$z_n = z_0 + (12x_0 + 8y_0)^2 (1 + X + X^2 + ... + X^{n-1}) \ , \ X = 33^2$$

$$= z_0 + \frac{1}{1088}(33^{2n} - 1)(12x_0 + 8y_0)^2 \qquad (8)$$

To obtain the values of x_n, y_n, we proceed as follows:

The solution (4) is written in the matrix form as

$$(x_1, y_1)^t = M(x_0, y_0)^t$$

where t is the transpose and

$$M = \begin{bmatrix} 25 & 16 \\ 12 & 9 \end{bmatrix}$$

In general, $(x_n, y_n)^t = M^n (x_0, y_0)^t$

To find M^n, consider the characteristic equation

$$|M - \lambda I| = 0 \qquad (9)$$

where I is the unit matrix of order 2 and λ is the eigen value of M

Solving (9), the eigen values of M are given by $\lambda_1 = 1$, $\lambda_2 = 33$

It is well-known that

$$M^n = \frac{\lambda_1^n}{\lambda_1 - \lambda_2}(M - \lambda_2 I) + \frac{\lambda_2^n}{\lambda_2 - \lambda_1}(M - \lambda_1 I)$$

$$= \frac{1}{8}\begin{bmatrix} 2 + 6(33)^n & 4\left((33)^n - 1\right) \\ 3\left((33)^n - 1\right) & 6 + 2(33)^n \end{bmatrix}$$

Thus, we have

$$\left.\begin{aligned} x_n &= \frac{1}{8}\left[\left(2 + 6(33)^n\right)x_0 + 4\left((33)^n - 1\right)y_0\right] \\ y_n &= \frac{1}{8}\left[\left(3(33^n - 1)\right)x_0 + \left(6 + 2(33)^n\right)y_0\right] \end{aligned}\right\} \qquad (10)$$

Thus (8), (10) represents the generation formula for the given elliptic paraboloid in terms of its given solution.

Illustration: 3

The ternary quadratic diophantine equation under consideration is

$$2x^2 + 3y^2 = 11z \qquad (1)$$

Let (x_0, y_0, z_0) be any solution of (1).

The solution may be in real integers or in gausssian integers or in irrational numbers.

Let (x_1, y_1, z_1) be the second solution of (1), where

$$x_1 = h_0 - 3x_0 \ , \ y_1 = 2h_0 - 3y_0 \ , \ z_1 = 9z_0 + h_0^2 \qquad (2)$$

in which h_0 is an unknown to be determined.

Substitution of (2) in (1) gives

$$h_0 = 4x_0 + 12y_0 \qquad (3)$$

Using (3) in (2), the second solution is given by

$$x_1 = x_0 + 12y_0 \ , \ y_1 = 8x_0 + 21y_0 \qquad (4)$$

$$z_1 = 9z_0 + (4x_0 + 12y_0)^2 \qquad (5)$$

Let (x_2, y_2, z_2) be the third solution of (1), where

$$x_2 = h_1 - 3x_1 \ , \ y_2 = 2h_1 - 3y_1 \ , \ z_2 = 9z_1 + h_1^2$$

in which h_1 is an unknown to be determined.

The repetition of the above process leads to

$$h_1 = 5^2 h_0 \; , \; x_2 = 97x_0 + 264y_0 \; , \; y_2 = 176x_0 + 537y_0 \qquad (6)$$

$$z_2 = 9^2 z_0 + 9(4x_0 + 12y_0)^2 + 25^2(4x_0 + 12y_0)^2 \qquad (7)$$

Now, assume the n^{th} solution of (1) to be (x_n, y_n, z_n) where

$$x_n = h_{n-1} - 3x_{n-1} \; , \; y_n = 2h_{n-1} - 3y_{n-1} \; , \; z_n = 9z_{n-1} + h_{n-1}^2$$

From (5) and (6), one can observe that

$$z_n = 9^n z_0 + 9^{n-1}(4x_0 + 12y_0)^2 + 9^{n-2}\, 25^2(4x_0 + 12y_0)^2 + \ldots + 25^{2n-2}(4x_0 + 12y_0)^2 \qquad (8)$$

To obtain the values of x_n, y_n, we proceed as follows:

The solution (4) is written in the matrix form as

$$(x_1, y_1)^t = M(x_0, y_0)^t$$

where t is the transpose and

$$M = \begin{bmatrix} 1 & 12 \\ 8 & 21 \end{bmatrix}$$

In general, $(x_n, y_n)^t = M^n (x_0, y_0)^t$

To find M^n, consider the characteristic equation

$$|M - \lambda I| = 0 \qquad (9)$$

where I is the unit matrix of order 2 and λ is the eigen value of M

Solving (9), the eigen values of M are given by $\lambda_1 = -3$, $\lambda_2 = 25$

It is well-known that

$$M^n = \frac{\lambda_1^n}{\lambda_1 - \lambda_2}(M - \lambda_2 I) + \frac{\lambda_2^n}{\lambda_2 - \lambda_1}(M - \lambda_1 I)$$

$$= \frac{1}{7}\begin{bmatrix} 6(-3)^n + 25^n & 3(25^n - (-3)^n) \\ 2(25^n - (-3)^n) & (-3)^n + 6(25^n) \end{bmatrix}$$

Thus, we have

$$\left.\begin{aligned} x_n &= \frac{1}{7}\left[\left(6(-3)^n + 25^n\right)x_0 + 3\left(25^n - (-3)^n\right)y_0\right] \\ y_n &= \frac{1}{7}\left[2\left(25^n - (-3)^n\right)x_0 + \left((-3)^n + 6\left(25^n\right)\right)y_0\right] \end{aligned}\right\} \qquad (10)$$

Thus (8), (10) represents the generation formula for the given elliptic paraboloid in terms of its given solution.

To conclude, one may attempt for obtaining generation formula for other choices of elliptic paraboloid.

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