Signed domination in Rooted product of a Path with a Cycle graph

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Abstract

A two valued function f defined on the vertices of a graph $G = (V, E)$, $f: V \to \{-1, 1\}$ is a signed dominating function if the sum of its function values over any closed neighborhood of every vertex is at least 1, $i.e. \forall v \in V, f(N[v]) \geq 1$. The weight of a signed dominating function is $w(f) = \sum f(v)$ over all vertices $v \in V$. The signed domination number of a graph G $\gamma_s(G)$ is the minimum weight of a signed dominating function f on G. In this paper we present some results on signed dominating function of rooted product graph of a path with a cycle graph.

Keywords

Rooted product graph, Dominating function , Signed dominating function

I. INTRODUCTION

We use [1] for terminology and notation which are not defined here. For any vertex v ∈ V , the open neighborhood of v in G is N(v) = {u ∈ G | uv ∈ E(G)}, and N[v] = N(v) ∪ {v} denotes its closed neighborhood. Domination theory plays a very important role in graph theory. It has a wide range of applications to various fields like communication, social science, Engineering and others. In recent years attention is given to another more interesting topic, dominating functions. Dunbar[9] and others [5,7,12,20] introduced the concept of signed dominating function, where the vertices of a graph are assigned the values +1 or -1. This variation of dominating function has applications in networks of positive and negative spins of electrons, social networks of people or organizations etc. A two valued function f defined on the vertices of a graph $G = (V, E)$, $f: V \to \{-1, 1\}$ is a signed dominating function if the sum of its function values over any closed neighborhood of every vertex is at least 1, $i.e. \forall v \in V, sf(N[v]) \geq 1$. The signed domination number $\gamma_S(G)$ of a graph G is defined to be the minimum weight of signed dominating function of G.

In 1978, Godsil and Mckay [8] introduced a new product on two graphs G_1 *and* G_2, called rooted product denoted by $G_1 \odot G_2$. The rooted product graphs are used in internet systems for connecting internet from one system to other. The Rooted product of a path P_m with a cycle C_n is a graph obtained by taking one copy of a m-vertex graph P_m and m-copies of C_n and then joining the ith vertex of P_m to all verticies of ith copy of C_n.This graph is denoted by $P_m \odot C_n$. Rashmi S B [15, 16] has studied dominating function and total dominating function of rooted product graph $P_m \odot C_n$. In this paper we present some results on signed dominating function of rooted product graph of a path with a cycle graph $P_m \odot C_n$.

II. MAIN RESULTS

Lemma 2.1: Signed domination number of cycle graph is

$$\gamma_s(C_n) = k + r \text{ for } n = 3k + r, 0 \leq r \leq 2$$

Proof *:* Consider cycle graph C_n where every vertex is of degree 2. In the cycle graph if vertex v is such that $f(v) = 1$ then one of its adjacent vertex can be assigned -1 and the other should be +1. So that $f[N(v)] \geq 1$. Next, if a vertex v is such that $f(v) = -1$ then it is necessary that both of its adjacent vertices are assigned with +1. So that $f(N[v]) \geq 1$. Hence, it is clear that for a dominating function in a cycle graph the assignment for vertices should follow the sequence $1, 1, -1, 1, 1, -1, \dots \dots \dots ..$

From this it is clear that weight of the cycle is $1 + 1 - 1 + 1 + 1 - 1, \dots \dots \dots ..$ f(C_n) = k+r if n = 3k+r .

Case(i) : If $n = 3k$ then from each of consecutive three vertex set $1 + 1 - 1 = 1$ will be contributed to the total weight of C_n As there are k such three vertex sets for $n = 3k$ total weight is $1 + 1 + 1 + \cdots \ldots \ldots . . + k\ times = k$ As there can't be more -1 assigned than a sequence specified above this is the minimum weight giving $\boldsymbol{\gamma_s(C_{3k}) = k}$

Case(ii) If $n = 3k + 1$ then for the first 3k vertices from case (i) assignment will be $1, 1, -1,1,1, -1, \ldots \ldots \ldots$, As the 3k th vertex is assigned -1, its neighbor, the last vertex $3k + 1th$ vertex, should be assigned +1. The total weight will be k+1 giving $\gamma_s(C_{3k+1}) = k + 1$

Case (iii): If $n = 3k + 2$ then as in case (ii), the $3k\ th$ vertex is assigned to -1, the last two vertices assigning 1 and -1 it does not satisfy the condition $f(N[v]) \geq 1$ for the (3k+1)th vertex v. So that last two vertices should be assigned to +1 only, giving $\gamma_s(C_{3k+2}) = k + 2$

Combining these three cases we get the signed domination number,

$$\gamma_s(C_{3k+r}) = k + r, \quad \text{for } n = 3k + r, \ \ 0 \leq r \leq 2.$$

Theorem 2.2: The Signed domination number of rooted product graph $P_m \odot C_n$ is

$$\gamma_s(P_m \odot C_n) = \begin{cases} m(k + r) & for\ n = 3k + r, r = 0,1 \\ mk + 4 & for\ n = 3k + 2 \end{cases}$$

Proof : Consider the rooted product graph $P_m \odot C_n$. Let us define a two valued function on the vertex set of $P_m \odot C_n$ $f: V \to \{-1, 1\}$, first considering possible choices for the assignment of +1 or -1 to root vertices on the path.

Choice (i) : **A**ll the path vertices are assigned to -1 i.e. ,$f_1(v) = -1\ \forall\ v \in V(P_m)$
In the rooted product graph $P_m \odot C_n$, first choose the root vertex of C_n with P_m and assign -1 to each root vertex. From Lemma 1, in the cycle graph C_n, no two adjacent vertices can have value assigned -1 because it does not satisfy the condition $f_1(N[u]) \geq 1$, so assign adjacent vertices of root vertex to +1. Now for the root vertex v, $f_1(N[v]) = -3 + 2 = -1 \ngeq 1$
Hence such assignment of -1 to all the root vertices on path is not valid for f_1 to be a dominating function.

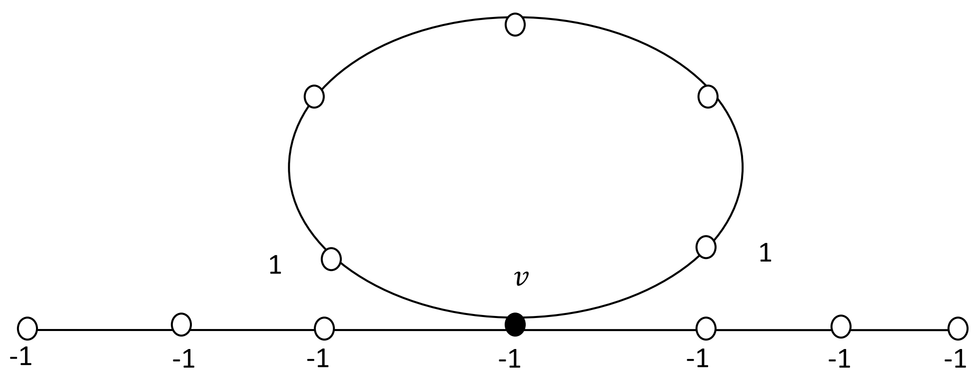

Figure 1 : Assignment of +1, -1 in cycle graph with all root vertices assigned -1

Choice (ii) : All the Path vertices are assigned to +1**, so** $f_2(v) = +1\ \forall\ v \in V(P_m)$
In the rooted product graph $P_m \odot C_n$, first choose the root vertex of C_n with P_m and assign +1 to each root vertex. In the cycle graph C_n adjacent verticies of root vertex can be assigned +1 or -1, so that from lemma 1, for dominating function in rooted product graph the assignment for vertices should follow the sequence $1, 1, -1,1,1, -1, \ldots \ldots \ldots$ in cycle graph part.
As each of these m cycles attached at the root vertex are independent and each path vertex is a vertex of cycle graph as a root, so, total weight of $P_m \odot C_n$ = Sum of weights of m cycles C_n

Case (a) $n = 3k + r\ , r = 0,1$

From lemma 1, Weight of each cycle C_n = k+r , total weight of $P_m \odot C_n$

$$\boldsymbol{w(f_2) = (k + r) + (k + r) + - - - - - m\ times} = m(k + r)$$

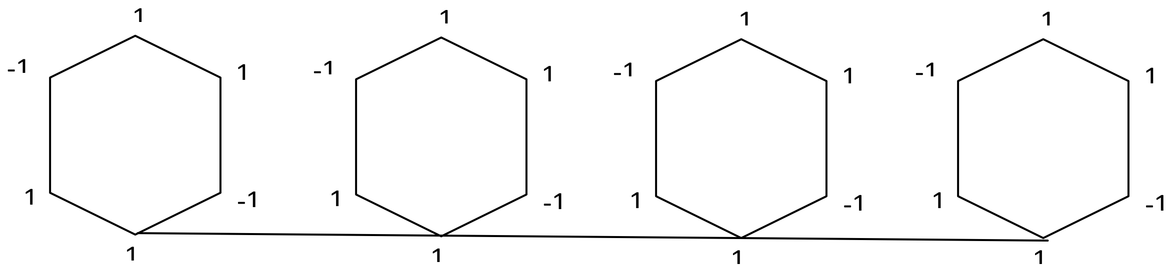

Figure 2 : Signed domination function for $P_4 \odot C_6$

Case (b) $n = 3k + 2$

As for the cycle C_n attached to the first and last vertex in the path graph as root, the weight of the cycle is $(\boldsymbol{k} + \boldsymbol{r})$ for $f_2(N[v]) \geq 1$, but for internal path vertex as root, the weight of the cycle can reduce because adjacent path vertices are mapped to +1, so cycle vertices can be mapped to -1 & -1.Therefore for internal root vertex cycle C_{3k+2} there will be one extra -1 assigned than the end vertex rooted cycle. Weight of internal (m-2) root vertices cycle is k where as for the two end vertices of path as root vertex of cycle C_{3k+2} weight is $k + 2$.

$$\begin{aligned} \boldsymbol{w}(f_2) &= (k+2) + (m-2)k + (k+2) \\ &= (m-2)k + 2(k+2) \\ &= mk - 2k + 2k + 4 \\ &= mk + 4 \end{aligned}$$

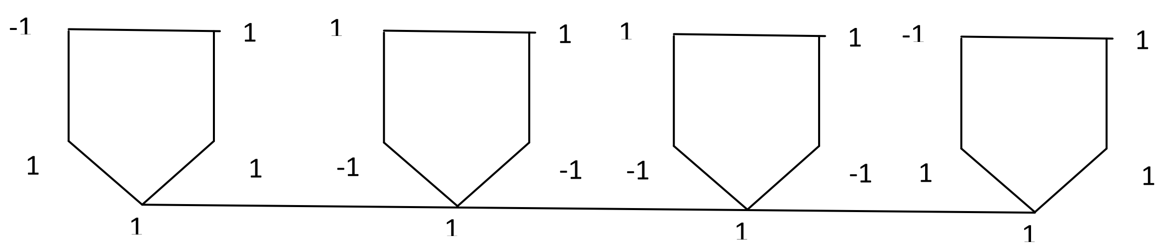

Figure 3 : Signed domination function for $P_4 \odot C_5$

Choice (iii): **I**f all the path vertices are assigned alternately to +1 and -1

In the rooted product graph $P_m \odot C_n$, choose the root vertex of C_n with P_m. In the path P_m all the root vertices are assigned alternately +1 and -1. If the first vertex of path P_m is +1 the adjacency of two vertices in cycle is +1 because the adjacency of path vertex is -1 for $f_3(N[v]) \geq 1$. Again the assignment of vertices in cycle it follows sequence +1,+1,-1,+1,+1,-1,………..

Total weight of each of the m cycles is $k + r$.

$w(f_3) = m(k + r) \qquad n = 3k + r , 0 \leq r \leq 2.$

Choice (iv): **I**f all the path vertices are assigned to +1 +1 -1 …..

In the rooted product graph $P_m \odot C_n$, choose the root vertex of C_n with P_m. In the path P_m all the root vertices are assigned sequence +1 +1 -1. If the first vertex of path P_m is +1 the adjacency of two vertices in cycle is +1 because one of the adjacency of path vertex is -1 for $f_4(N[v]) \geq 1$. Again the assignment of vertices in cycle follows sequence +1,+1,-1,+1,+1,-1,……….. with properly following the root vertex value in the cycle. Total weight of each of the m cycles is $k + r$, $giving$

$$w(f_3) = m(k + r) \qquad n = 3k + r , 0 \leq r \leq 2.$$

Any other choice of assignment of +1, -1 to root vertices will follow similar argument. All the three possible choices of signed dominating functions $f_2,\ f_{3,}\ f_4$ have same weight for n = 3k and 3k+1, but f_2 has minimum weight for n=3k+2. Hence the function with minimum weight is f_2. Therefore the signed domination number of rooted product graph $P_m \odot C_n$ is w(f_2), $giving$

$$\gamma_S(P_m \odot C_n) = mk \qquad if\ n = 3k ,$$
$$= mk + m \qquad if\ n = 3k + 1$$
$$= mk + 4 \qquad if\ n = 3k + 2$$

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