Quasi-Boundedness of s-Paratopological Groups

Rafaqat Noreen
Department of Mathematics, COMSATS University Islamabad,
Chack Shahzad, Islamabad 45550, Pakistan.

Abstract

In this paper, our focus is to define and study the boundednessfor s-paratopological groups.Premeager property for s-paratopological groups is discussed. It is proved that every open subgroup of a quasi-bounded, premeager s-paratopological group is premeager. For bounded homomorphisms on s-paratopological group, new notions nb_q*-quasi bounded and* $b_q b_q$ *-quasi bounded homomorphisms are introduced and discussed.*

Keywords - *s- paratopological group, quasi-bounded s- paratopological group, ω-quasi-bounded s-paratopological group, Premeager,* nb_q*-quasi boundedhomomorphism,* $b_q b_q$*- quasi boundedhomomorphism.*

2010hematics Subject Classification: 54A05,54B05, 54C08,54H11

I. INTRODUCTION

By a paratopological group G we mean a group $(G,*)$ endowed with a topology τ making the group operation continuous, or equivalently, for each $x, y \in G$ and for each open neighbourhood W containing $x * y$, there exist open neighbourhoods U containing x and V containing y such that $U * V \subset W$. If in addition, the operation of taking inverses is continuous, then the paratopological group $(G,*,\tau)$ is a topological group. Sorgenfrey line £ is the well-known example for a paratopological group which fails to be a topological group. Paratopological groups have been studied extensively by celebrated mathematicians like; A.V. Arhangel'skii, M. Tkachenkov, T. Banakh, C. Liu, and A.V. Ravsky [1], [2],[13],[17].

Khan et al. in [10] introduced four classes of paratopologized groups which are s- (S-, irresolute-, Irr-) paratopological groups. Each paratopologized group is defined in such a way that the topology τ is endowed upon a group $(G,*)$ such that the group operation satisfies certain condition which is either weaker or stronger than continuity.

K.H. Azar [3] defined the bounded topological groups. However, it was called pseudobounded instead of bounded in [13], since the boundedness has other meaning in topological algebra. The relevant concept of boundedness for irresolute paratopological groups was introduced in [16]. In this paper, we shall define the quasi-bounded and ω-quasi-bounded s-paratopological groups.We shall define and discuss premeager property for s-paratopological groups.

Bounded homomorphisms and their algebraic and topological algebraic structure are of interest for their own right and also for their applications in other area of mathematics. Therefore, it will be of interest to consider different types of bounded homomorphisms on s- paratopological groups. So, in this paper we shall define the new notions nb_q-quasi bounded and $b_q b_q$-quasi bounded of bounded homomorphisms on s-paratopological groups.

II. PRELIMINARIES

Semi-open sets in topological spaces were defined by Levine [12] in 1963. The notion received lot of attraction by many topologists and consequently many results of topological spaces were generalized, since the inception of semi-open sets and semi-continuity. A subset A of a topological space X is said to be semi-open if there exists an open set U in X such that $U \subset A \subset Cl(U)$, or equivalently if $A \subset Cl(Int(A))$. $SO(X)$ denotes the collection of all semi-open sets in X and $SO(X,x)$ in the collection of all semi-open sets containing x. $sInt(A)$ represents the semi interior of A, which is union of all semi-open sets contained in A. The complement of a semi-open set is said to be semi-closed, the semi closure of $A \subset X$, denoted by $sCl(A)$, is the intersection of all semi-closed subsets of X containing A[6, 7]. Let us mention that$x \in sCl(A)$ if and only if for any semi-open set Ucontaining x, $U \cap A \neq \emptyset$.

Every open (closed) set is semi-open (semi-closed). It is known that the union of any collection of semi-open sets is again a semi-open set, while the intersection of two semi-open sets need not be semi-open. The intersection of an open set and a semi-open set is semi-open. If $A \subset X$ and $B \subset Y$ are semi-open in spaces $X$and$Y$, then $A \times B$ is semi-open in the product space $X \times Y$. Basic properties of semi-open sets are given in [12], and of semi-closed sets and the semi closure in [6, 7]. A set $U \subset X$ is a semi neighbourhood of a point $x \in X$ if there exists $A \in SO(X)$ such that$x \in A \subset U$. A set $A \subset X$ is semi-open in X if and only if A is a semi neighbourhood of each of its points. If a semi neighbourhood U of a point x is a semi-open set we say that U is a semi-open neighbourhood of x.

Definition 2.1. [12] Let X and Y be topological spaces. A mapping $f: X \to Y$ is semi continuous if for each open set Vin Y, $f^{\leftarrow}(V) \in SO(X)$.

Clearly, continuity implies semi continuity; the converse needs not be true. Notice that a mapping $f: X \to Y$ is semi continuous if and only if for each $x \in X$ and each neighbourhood V of $f(x)$ there is a semi open neighbourhood U of x with $f(U) \subset V$. Let $X = Y = [0,1]$. Let $f: X \to Y$ be defined as follows: $f(x) = 1\ if\ 0 \le x \le (1/2)$ and $f(x) = 0$, if $(1/2) < x \le 1$. Then f is semi-continuous but not continuous.

In [9], Kempisty defined quasi continuous mappings: a mapping $f: X \to Y$ is said to be quasi continuous at a point $x \in X$ if for each neighbourhood U of x and each neighbourhood W of $f(x)$ there is a nonempty open set $V \subset U$ such that $f(V) \subset W$. f is quasi continuous if it is quasi-continuous at each point (see also[14]). Neubrunnová in [15] proved that semi continuity and quasi continuity coincide.

Definition 2.2. **"**A mapping $f: X \to Y$ between topological spaces $X$and Y is called:

1) ***semi-open***[4], if for every open set A of X, the set $f(A)$ is semi-open in Y;
2) ***quasi-open***[13] if we have $Int(f(U)) \neq \emptyset$for each non-empty open subset U of X."

Definition 2.3.[5] A s-topological group is a group $(G,*)$ with a topology τ such that for each $x, y \in G$ and for each neighbourhood W of$x * y^{-1}$, there are semi-open neighbourhoods U of x and V of y, such that $U * V^{-1} \subset W$.

Definition 2.4.[10] An s-paratopological group is a group $(G,*)$ with a topology τ such that for each $x, y \in G$ and for each open neighbourhood W containing $x * y$, there exist semi-open neighbourhoods U of x and V of y, such that $U * V \subset W$.

Remark 2.5.Every paratopological group is s-paratopological group but converse is not true in general. see([10], Example).

Definition 2.6.[3]Let G be an paratopological group and $A \subseteq G$. We say that A is an pseudobounded subset of G, if for every neighborhood U of the identity element e of G, we have $A \subseteq U^{n}$. If G is an pseudobounded subset of G, then we say that G is pseudobounded.

Definition 2.7. [13] Let G be a paratopological group and $A \subset G$. We say that A is an ω-pseudobounded subset of G, if for every neighborhood U of the identity element e of G, we have $A \subseteq \cup U^{n}$. If G is an ω-pseudobounded subset of G, then we say that G is ω-pseudobounded.

Definition 2.8.A set U in a topological space is called nowhere dense, if$Int(Cl(U)) = \varphi$.

Definition 2.9. [13] Let G be a paratopological group. G is called premeager if, for any its nowhere dense subset A of G, we have $A^{n} \neq G$ for each $n \in \mathbb{N}$.

Lemma 2.10. [10] Every open subgroup H of an s-paratopological group$G$ is also an s-paratopological group (called s-paratopological subgroup of G).

III. QUASI-BOUNDED AND ω -QUASI-BOUNDED s-PARATOPOLOGICAL GROUPS

In this section we will define quasi-bounded and ω-quasi-bounded s-paratopological groups. K.H. Azar defined the bounded topological groups [3]. However, it was called pseudobounded instead of bounded in [13], since the boundedness has other meaning in topological algebra.

Definition 3.1. Let G be an s-paratoplogical group and $A \subseteq G$. We say that A is quasi-bounded subset of G, if for every open neighbourhood U of e, there is a natural number n, such that $A \subseteq U^{\mathrm{n}}$. If G is itself quasi-bounded, then we say that G is quasi-bounded.

Definition 3.2. Let G be an s-paratopological group and $A \subseteq G$. We say that A is an ω- quasi-bounded subset of G, if for every open neighbourhood U of e, there is a natural number n, such that $A \subseteq \cup U^{\mathrm{n}}$. If G is an ω- quasi-bounded subset of G, then we say that G is ω- quasi-bounded.

Example 3.3. Let $X = \{(x,1): 0 \leq x < 1\}$, and let the topology on X be generated by the base consisting of sets of the form $\{(x,1) \in X: x_0 < x < x_0 + 1/k\} \cup \{(x_0,1)\}$, where $0 \leq x_0 < 1$ and $k \in \mathbb{N}$. There exists a natural structure of an Abelian group on X such that the multiplication $(u,v) \to u \cdot v$ is continuous. For example, if $u = (x,1)$ and $v = (y,1)$ are two points in X, then $u \cdot v = (x+y,1)$, if $x+y < 1$, and $u \cdot v = (x+y-1,1)$ if $x+y \geq 1$, that is, the space X admits a structure of a paratopological group. Therefore it is s-paratoplogical group too. Obviously, X is quasi-bounded s-paratopological group.

The following examples show that s-paratoplogical group may not be quasi-bounded s-paratopological group.

Example 3.4. Let $(\mathbb{R},+)$ be the group of real numbers with usual operation of addition and $\beta = \{(a,b),[1,c);$ where $a < b$ and $a,b,c \in \mathbb{R}\}$ be the basis for topology τ for $\mathbb{R}$.Then by example 3.1 [10, $(\mathbb{R},+,\tau)$ is s-paratopological group. It is easy to see that $(G,+,\tau)$ is non quasi-bounded s-paratopological group.

Example 3.5. It is well known that the Sorgenfrey line ([8], Example 1.2.2) is a first-countable and non-pseudobounded paratopological group, where as a set the Sorgenfrey line is the set of real numbers and its topology is generated by taking as a basis the half open intervals $[a,b), a < b$. The Sorgenfrey line is s-paratopological group by Remark 3.2 [10]. Hence Sorgenfrey line is non quasi-bounded s-paratopological group.

Obviously, quasi-bounded s-paratopological group is ω- quasi-bounded. However, there exists an ω- quasi-bounded s-topological group which is not quasi-bounded, see the following example.

Example 3.6. Let $(\mathbb{R},+)$ be the real line endowed with the Euclidean topology, where ` $+$ ′ is the additive operation. Clearly, the $\mathbb{R}$ with the additive operation is ω- quasi-bounded s-paratopological group. However, the $\mathbb{R}$ with the additive operation is not quasi-bounded s-paratopological group.

Theorem 3.7. Suppose that G is a s- paratopological group, and U any semi-open neighborhood of the neutral element e in G. Then $sCl(M) \subset MU^{-1}$, for each subset M of G.

Proof. Put $F = G \backslash \{gU: g \in G, gU \cap M = \emptyset\}$. Then, clearly, F is a semi-closed subset of G and $M \subset F$. Take any $y \in F$. Then $yU \cap M \neq \emptyset$, that is, $yh = m$, for some $h \in U$ and $m \in M$. Hence, $y = mh^{-1} \in MU^{-1}$. Thus, $\subset MU^{-1}$. Since $M \subset F$, it follows that $sCl(M) \subset MU^{-1}$.

Theorem 3.8. If a s-paratopological group $(G,\cdot,\tau)$ contains a quasi-bounded (ω- quasi-bounded) dense subgroup, then $(G,\cdot,\tau)$ is quasi-bounded (ω- quasi-bounded).

Proof. Let H be a quasi-bounded dense subgroup of a s-paratopological $(G,\cdot,\tau)$. Take a semi-open neighbourhood U of the identity e in $(G,\cdot,\tau)$. Since H is a quasi-bounded subset of $(G,\cdot,\tau)$, there exists $n \in \mathbb{N}$ such that $H \subseteq U^{\mathrm{n}}$, equivalently, $H \subseteq U^{-\mathrm{n}}$. Hence, using Theorem 3.7, $G = sCl(H) \subseteq HU^{-1} \subseteq U^{-\mathrm{n}-1}$, hence, $G = U^{\mathrm{n}+1}$. Using a similar argument, we can prove that if H is an ω- quasi-bounded dense subgroup of a s-paratopological group G, then $(G,\cdot,\tau)$ is ω- quasi-bounded.

Theorem 3.9. Let $f: G \to H$ be a continuous homomorphism from the an s-paratopological group G onto the s-paratopological group H. If G is quasi-bounded (ω- quasi-bounded), then H isquasi-bounded (ω- quasi-bounded).

Proof. Suppose that G is ω- quasi-bounded. Take V an open neighbourhood of the identity in H. Put $U = f^{-1}(V)$. Since f is continuous homomorphism, U is an open neighbourhood of the identity in G. By hypothesis, $G = \cup U^{\mathrm{n}}$.We conclude, $H = \cup f(U)^{\mathrm{n}} = \cup (f(U))^{\mathrm{n}} = \cup V^{\mathrm{n}}$, so H is ω- quasi-bounded. The proof of the quasi-bounded case is similar.

Theorem 3.10. Let G and H be s-paratopological groups and suppose that $f : G \to H$ is group isomorphism. If f is continuous and $E \subseteq G$ is quasi-bounded subset of G, then $f(E)$ is quasi-bounded subset of H.

Proof. Let U be an open neighbourhood of $e \in H$. Then $f^{-1}(V)$ is an open neighbourhood of e. Since E is quasi-bounded subset of G, there is a natural number n, such that $E \subseteq (f^{-1}(V))^{n} \subseteq f^{-1}(V^{n})$, because f is group isomorphism. $f(E) \subseteq ff^{-1}(V^{n})$. This implies that $f(E) \subseteq V^{n}$. Thus $f(E)$ is quasi-bounded subset of H.

Theorem 3.11. Let $f: G \to H$ be a semi continuous homomorphism from the s-paratopological group G onto the s- paratopological group H. If G is quasi-bounded (ω- quasi-bounded), then H is quasi-bounded (ω- quasi-bounded).

Proof. Suppose that $(G,\cdot,\tau)$ is ω- quasi-bounded. Take V a open neighbourhood of the identity in H. Put $U = f^{-1}(V)$. Since f is semi continuous homomorphism, U is a semi open neighbourhood of the identity in G. By hypothesis, $G = \cup U^{n}$.We conclude, $H = \cup\ f(U)^{n} = \cup\ (f(U))^{n} = \cup V^{n}$, so H is ω- quasi-bounded. The proof of the quasi-bounded case is similar.

IV. PREMEAGER s-PARATOPLOGICAL GROUPS

In this section, we will define and discuss the premeager property for s-paratopological groups.

Definition 4.1. Let $(G,\cdot,\tau)$ be an s-paratopological group. G is called premeager if, for any its nowhere dense subset A of G, we have $A^{n} \neq G$ for any $n \in \mathbb{N}$.

The Sorgenfrey line X ($X = \mathbb{R}$) does not have the premeager property. In particular, the Euclidean line does not have the premeager property.

Proof. Let C be the usual Cantor set in [0,1]. It is well known that C is nowhere dense in X. By ([11], Lemma A1), we have $C + C = [0,2]$, where `$+$'is the usual addition. Let $A = \cup\ (2n + C)$, where $\mathbb{Z}$ is the integer. Then A is nowhere dense in X, but $A + A = X$ since $C + C = [0,2]$.

Theorem 4.2. Let $f : G \to H$ be a quasi- open, continuous homomorphism, where G, H are s- paratopological groups. If G is premeager, then H is also premeager.

Proof. Let A be any nowhere dense subset of H. Suppose that there exists some $n \in \mathbb{N}$, such that $A^{n} = H$. Therefore, $(f^{-1}(A))^{n} = f^{-1}(A^{n}) = f^{-1}(H) = G$. Since G is premeager, the set $f^{-1}(A)$ is a non-nowhere dense subset of X. Hence there is a non-empty open subset U of X such that $U \subset Cl(f^{-1}(A))$. It follows $U \subset Cl(f^{-1}(A)) \subset f^{-1}(Cl(A))$ that $f(U) \subset Cl(A)$. Since f is quasi-open, we have $\phi \neq Int(f(U)) \subset f(U) \subset Cl(A)$, which is a contradiction.

Since open maps are quasi-open maps, so, we have a corollary.

Corollary 4.3. Let $f: G \to H$ be an open and continuous homomorphism map, where G, H are s- paratopological groups. If G is premeager, then H is also premeager.

Theorem 4.4. Let $(G,\cdot,\tau)$ be a quasi-bounded and premeager s-paratopological group. Then every open subgroup of $(G,\cdot,\tau)$ is premeager.

Proof. Let H be an open subgroup of G. Suppose that H is non-premeager. Then there exists a nowhere dense subset A of H and an $n \in \mathbb{N}$, such that $A^{n} = H$. Since G is quasi-bounded, it follows that there is an $m \in N$ such that $H^{m} = G$. Hence $(A^{n})^{m} = H^{m} = G = A^{nm}$. However, the set A is a nowhere dense subset of G, which is a contradiction.

V. nb_q-QUASI BOUNDED AND $b_q b_q$-QUASI BOUNDED HOMOMORPHISM

Bounded homomorphisms and their algebraic and topological algebraic structure are of interest for their own right and also for their applications in other area of mathematics. Therefore, it will be of interest to consider different types of bounded homomorphisms on s- paratopological groups. So, in this section we will define new notions nb_q-quasi bounded and $b_q b_q$-quasi bounded of bounded homomorphisms on s- paratopological group.

Definition 5.1. Let G and H be two s-paratopological groups. A homomorphism $\phi: G \to H$ is said to be

1) ***nb_q-quasi bounded,*** if there exists an open neighborhood U of e_G such that $\phi(U)$ is quasi bounded in H ;
2) ***$b_q b_q$-quasi bounded,*** if for every quasi bounded set $B \subset G$, $\phi(B)$ is quasi bounded in H.

The set of all nb_q-quasi bounded ($b_q b_q$-quasi bounded) homomorphisms from an s-paratopological group G to an s-paratopological group H is denoted by

$$Hom_{nb_q}(G,H)(Hom_{b_q b_q}(G,H)).$$

We write $Hom(G)$ instead of $Hom(G,G)$.

Theorem 5.2. For s-paratopological groups G and H the following holds:

$$Hom_{nb_q}(G,H) \subset Hom_{b_q b_q}(G,H)$$

Proof. Let $\phi : G \to H$ be an nb_q-quasi bounded homomorphism. Then it is $b_q b_q$-quasi bounded. For, suppose $B \subset G$ is a quasi bounded set. Since ϕ is nb_q-quasi bounded there is an open neighborhood U of e_G such that $\phi(U)$ is quasi bounded in H. Quasi Boundedness of B implies $B \subset U^n$ for some natural number n. We prove that $\phi(B)$ is quasi bounded in H. Let V be an open neighborhood of e_H. Quasi boundedness of $\phi(U)$ implies that there is $m \in \mathbb{N}$ such that $\phi(U) \subset V^m$. Then

$$\phi(B) \subset \phi(U^n) = (\phi(U))^n \subset V^{mn}$$

i.e. $\phi(B)$ is quasi bounded in H.

Theorem 5.3. Each continuous homomorphism between two s-paratopological groups is $b_q b_q$- quasi bounded.

Proof. Let G and H be s-paratopological groups, $\phi: G \to H$ be a continuous homomorphism, and B a quasi bounded subset of G. Suppose an open neighborhood V of e_H is given. There exists an open neighborhood U of e_G such that $\phi(U) \subset V$. Also, since B is quasi bounded in G, there is an $n \in \mathbb{N}$ with $B \subset U^n$. Thus,

$$\phi(B) \subset \phi(U^n) = (\phi(U))^n \subset V^n$$

i.e. $\phi(B)$ is quasi bounded in H.

VI. CONCLUSION

In this paper, we have discussed the boundedness for s-paratopological groups.Premeager property for s-paratopological groups is investigated. For bounded homomorphisms on s-paratopological group, new notions nb_q- quasi bounded and $b_q b_q$ -quasi bounded homomorphisms are introduced and discussed.The results of the study will provide a deeper understanding as well as extension knowledge for the concept of bounbedness.

VII. ACKNOWLEDGEMENT

Author is thankful to the referee for his careful reading and suggestions for the improvement of this paper.

REFERENCES

[1] A.V. Arhangel'skii, E.A. Reznichenko, Paratopological and semitopological groups versus topological groups, Topology Appl. 151 (2005), 107-119.

[2] A. V. Arhangel'skii, M. Tkachenko,Topological Groups and Related Structures, Atlantis Press and World Sci., 2008.

[3] K. H. Azar, Bounded topological groups, arXiv: 1003.2876.

[4] N. Biswas, On some mapping on topological spaces, Bull. cal. Math. Soc., 61(1969), 127-135.

[5] E. Bohn, J. Lee, Semi-topological groups, Amer. Math. Monthly, 72(1965), 996-998.

[6] S.G. Crossley, S.K. Hildebrand, Semi-closed sets and semi-continuity in topological spaces, Texas J. Sci., 22(1971), 123-126.

[7] S.G. Crossley, S.K. Hildebrand, Semi-topological properties, Fund. Math., 74(1972), 233-254.

[8] R. Engelking, General Topology (revised and completed edition), Heldermann Verlag, Berlin, 1989.

[9] S. Kempisty, Sur les fonctions quasicontinues, Fund. Math., 19(1932), 184-197.

[10] M. Khan and R. Noreen, On Paratopologized Groups, Analele Uni. Oradea Fasc. Math. Tom XXIII 2(2016), 147-156.

[11] A.B. Kharazishvili, Nonmeasurable sets and functions, Mathematical Studies 195, North-Holland, 2004.

[12] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70(1963), 36-41.

[13] F. Lin and S. Lin, pseudobounded or ω-pseudobounded paratopological groups, Filomat, 25(3)(2011), 93-103.

[14] S. Marcus, Sur les fonctions quasicontinues au sens de S. Kempisty, Coll. Math., 8(1961), 47-53.

[15] A. Neubrunnov´a, On certain generalizations of the notion of continuity, Mat. Časopis, 23(1973), 374-380.

[16] R. Noreen and M. Khan , Quasi-boundedness of Irresolute paratopological groups, (http./doi.org/10. 1080/25742558.2018.1458553),cogent Mathematics and Statistics, (2018) 5: 1458553.

[17] O.V. Ravsky, Paratopological groups I, Matem. Studii , 16 (2001), 37-48.