Semi-Quotients of Paratopologized Groups

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Abstract
Continuing the study of paratopologized groups, our focus in this paper is to investigate the semi-quotient mappings for the paratopologized groups. Semi-quotient mappings are stronger than semi-continuous mappings. Various results on semi-quotients of paratopologized groups are proved. It is proved that $(G/R,*,s\tau_Q)$ *is an irresolute paratopological group and* $(G/R,s\tau_Q)$ *is regular semi-quotient space. Semi-isomorphisms of irresolute paratopological groups are discussed.Semi connectedness of irresolute paratopological groups*$(G/R,*,s\tau_Q)$ *is also investigated.It is shown that If* R *and* $(G/R,*,s\tau_Q)$ *are semi connected, then* G *is semi connected.*

2010 Mathematics Subject Classification: Primary 54H11,22A05,54C08,54H99.

Keywords - *Semi open set, semi closed set, semi homeomorphism, s-paratopological group, irresolute paratopological group, semi connected space, semi component, semi quotient mapping.*

I. INTRODUCTION

Quotient spaces appeared in the work of Moore in 1925 and Alexandroff in 1927.The general concepts of quotient spaces and quotient maps were introduced in 1932 by R.W.Baer and F.Levi.Rafaqat et al. in [17] introduced the new class of mapping called semi-quotient mappings which are stronger than semi-continuous mappings. With reference to the already existing definition and properties of semi-quotient mappings and spaces, it is natural, to find out whether this fact is valid for the paratopologized groups or not. In this paper, we shall conduct the investigation on semi-quotients of paratopologizedgroups. The result of the study will provide a deeper understanding as well as extension knowledge for the concept of semi-quotient mapping and spaces.

By a paratopological group G we mean a group $(G,*)$ endowed with a topology τ making the group operation continuous, or equivalently, for each $x,y\in G$ and for each open neighbourhood W containing $x*y$, there exist open neighbourhoods U containing$x$ and V containing y such that U∗V⊂W. If in addition, the operation of taking inverses is continuous, then the paratopological group $(G,*,\tau)$ is a topological group. Sorgenfrey line £ is the well-known example for a paratopological group which fails to be a topological group.Paratopological groups have been studied extensively by celebrated mathematicians like; A.V.Arhangel'skii, M. Tkachenkov, T. Banakh, C. Liu, and A.V. Ravsky[2], [3], [20], [29].

Bohn and Lee introduced an analogue concept of s-topological group [5]. It was further studied in [6], [17], [15]. Four classes of paratopologized groups which are s- (S-, irresolute-, Irr-) paratopological groups were introduced by Moizet al.in [16].Each paratopologized group is defined in such a way that the topology τ is endowed upon a group $(G,*)$ such that the group operation satisfies certain condition which is either weaker or stronger than continuity. In this paper, It is proved that for some classes of irresolute-paratopological groups $(G,*,\tau)$ the semi-quotient space G/H is regular. Semi-isomorphisms of irresolute paratopological groups are also discussed.

The notion of connectedness is a basic, useful and fundamental notion in topological spaces. Das [11] in 1974 defined the weaker form of connectedness and called it semi connected space. Many mathematicians studied semi connected spaces rigorously. In this paper, we shall investigate some important results related to semi connectedness and semi components in semi quotient irresolute paratopological groups.

II. PRELIMINARIES

To proceed further, we introduce some notions and definitions. Throughout this paper X and Y are always topological spaces on which no separation axioms are assumed. For a subset A of a space X the symbols $Int(A)$ and $Cl(A)$ are used to denote the interior of A and the closure of A. If $f:X\to Y$ is a mapping between topological spaces

X and Y and B is a subset of Y, then $f^{\leftarrow}(B)$ denotes the pre image of B. Our other topological notation and terminology are standard as in [12]. If $(G,*)$ is a group, then e denotes its identity element, and for a given $x \in G$, $l_x: G \to G$, $y \mapsto x * y$, and $r_x: G \to G$, $y \mapsto y * x$, denote the left and the right translation by x, respectively.The operation $*$ we call the multiplication mapping $m: G \times G \to G$, and the inverse operation $x \mapsto x^{-1}$ is denoted by i.

N. Levine [19] defined semi open sets in topological spaces in 1963.Since then many mathematicians explored different concepts and generalized them by using semi open sets (see [1], [10], [25], [27]). A subset A of a topological space X is said to be semi open if there exists an open set U in X such that $U \subset A \subset Cl(U)$, or equivalently if $A \subset Cl(Int(A))$. $SO(X)$denotes the collection of all semi open sets in X, whereas $SO(X,x)$ represents the collection of all semi open sets in X containing x. The complement of a semi open set is said to be semi closed; the semi closure of $A \subset X$, denoted by $sCl(A)$, is the intersection of all semi closed subsets of X containing A[8],[9]. Let us mention that $x \in sCl(A)$ if and only if for any semi-open set U containing x, $U \cap A \neq \emptyset$.

A set $U \subset X$ is a semi neighbourhood of a point $x \in X$ if there exists $A \in SO(X)$ such that $x \in A \subset U$. A set $A \subset X$ is semi open in X if and only if A is a semi neighbourhood of each of its points. If a semi neighbourhood U of a point x is a semi open set we say that U is a semi open neighbourhood of x. Clearly, every open (closed) set is semi open (semi closed). It is known that the union of any collection of semi open sets is again a semi open set, while the intersection of two semi open sets need not be semi open. The intersection of an open set and a semi open set is semi open. If $A \subset X$ and $B \subset Y$ are semi open in spaces $X$and Y, then $A \times B$ is semi open in the product space $X \times Y$. Basic properties of semi open sets are given in [19], and of semi closed sets and the semi closure in [8],[9].

Definition 2.1"A mapping $f: X \to Y$ between topological spaces $X$and Y is called:

1) ***semi open***[4], if for every open set A of X, the set $f(A)$ is semi open in Y;
2) ***pre semi open*** [10], if for every semi open set A of X, the set $f(A)$ is semi open in Y;
3) ***irresolute*** [10], if for every semi open set B in Y, the set $f^{\leftarrow}(B)$ is semi open in X;
4) ***semi homeomorphism*** [10], if it is bijective, pre semi open and irresolute;
5) ***S-homeomorphism***[6], if f is bijective, semi continuous and pre semi open;
6) ***semi continuous***[19], if for each open set V in Y, $f^{\leftarrow}(V) \in SO(X)$".

Kempisty defined quasi continuous mappings: a mapping $f: X \to Y$ is said to be quasi continuous in [14], at a point $x \in X$ if for each neighbourhood U of x and each neighbourhood W of $f(x)$ there is a nonempty open set $V \subset U$ such that $f(V) \subset W$; f is quasi continuous if it is quasi continuous at each point (see also [21]). Neubrunnová in [22] proved that semi continuity and quasi continuity coincide.

Definition 2.2 A topological space (X,τ), is said to be semi compact [7],[13],[30] , if every semi open cover of X has a finite subcover.

Definition 2.3 Two non-null subsets A, B of a topological space (X,τ) are said to be semi separated [11], if and only if $A \cap sCl(B) = sCl(A) \cap B = \emptyset$, where $\emptyset$ denotes the null set.

Definition 2.4[11] In topological space (X,τ), a set which cannot be expressed as the union of two semi separated sets is said to be a semi connected set. The topological space (X,τ) is said to be semi connected if and only if X is semi connected.

Definition 2.5 A subspace (Y,τ_Y) of a topological space (X,τ_X) is semi connected [11], if it is semi connected in the subspace topology. i.e if there do not exist disjoint semi open sets U and V of Y, such that $Y = U \cup V$.

Definition 2.6Let(X,τ) be a topological space and $x \in X$. The semi component [31] of x, denoted by $SC(x)$, is the union of all semi connected subsets of X containing x. Further if $E \subset X$ and if $x \in E$, then the union of all semi connected sets containing x and contained in E is called the semi component of E corresponding to x.

Definition 2.7A space G is totally semi disconnected, if the singletons are the only semi connected subsets of$G$. Equivalently, a space G is totally semi disconnected if each one-point subset in G is its only semi connected component. Of course, every discrete space is totally semi disconnected.

Definition 2.8 [25] Let A be a subset of a space X. Then a point $x \in A$ is said to be a semi isolated point of A, if there is a semi open set U such that $U \cap A = \{x\}$.

Definition 2.9[25] A set A is said to be semi discrete, if each point of A is semi isolated.

Definition 2.10 [17] A mapping $f: X \to Y$ is:

1) ***an S-isomorphism*** if it is an algebraic isomorphism and (topologically) an S-homeomorphism;
2) ***a semi-isomorphism*** if it is an algebraic isomorphism and a semi-homeomorphism.

Definition 2.11 A mapping $f: X \to Y$ is irreolute-perfect if it is irresolute, s-closed, surjective, and $f^{\leftarrow}(y)$ is s-compact relative to X, for each y in Y .

Definition 2.12 [17] A mapping $f: X \to Y$ from a space X onto a space Y is said to be semi quotient provided a subset V of Y is open in Y if and only if $f^{\leftarrow}(V)$ is semi open in X.

Definition 2.13 [17] Let X be a topological space and Y a set and $f: X \to Y$ be a mapping, defined by

$$s\tau_Q = \{V \subset Y : f^{\leftarrow}(V) \in SO(X)\}$$

Then the family $s\tau_Q$ is a generalized topology on Y(i.e. $\emptyset \in s\tau_Q$) and the union of any collection of sets in $s\tau_Q$ is again in $s\tau_Q$ generated by f ; it is the semi quotient generalized topology. But $s\tau_Q$ needs not be a topology on Y[32]. It happens if X is an extremally disconnected space, because in this case the intersection of two semi open sets in X is semi open [26].

Lemma 2.14[31] If the topological space (X, τ) is separated by semi open sets C and D and if Y is semi connected subspace of X, then Y lies entirely within either C or D.

Lemma 2.15semi homeomorphic image of a semi connected space is semi connected.

Lemma 2.16 [31] Let A be semi connected subspace of (X, τ). Let B be subspace of G such that $A \subseteq B \subseteq sCl(A)$. Then B is semi connected.

Lemma 2.17[23] If $f: X \to Y$ is a semi-continuous mapping and $X_\circ$ is an open set in X, then the restriction $f\,|_{X_\circ}: X_\circ \to Y$ is semi-continuous.

Lemma 2.18[3] Let $\pi: G \to G/R$ be a canonical projection mapping. Then for any subset $U$of G, $\pi^{\leftarrow}(\pi(U)) = U * R$.

III. PARATOPOLOGIZED GROUPS

In this section we give some information on s-paratopological groups and irresolute paratopological groups which were introduced and studied first in [16].

Definition 3.1 An s-paratopological group [16], is a group $(G,*)$ with a topology τ such that for each $x, y \in G$ and each open neighbourhood$W$containing $x * y$, there exist semi open neighbourhoods U containing x and V containing y, such that $U * V \subset W$.

Definition 3.2 An irresolute-paratopological group [16] is a group $(G,*)$ with a topology τ such that for each $x, y \in G$ and each semi open neighbourhood W containing $x * y$, there exist semi open neighbourhoods U containing x and V containing y, such that $U * V \subset W$.

Definition 3.3 Semi component of an identity element e of an irresolute paratopological group $(G,\cdot,\tau)$ is the largest semi connected subset of G that contains the identity element e of the group G.

Lemma 3.4[16] Let $(G,*,\tau)$ be an irresolute paratopological group. Then left translation $l_x: G \to G$ is semi homeomorphism.

Theorem 3.5 If a mapping $f: X \to Y$ between topological spaces X and Y is irresolute-perfect, then for any compact subset K of Y, the pre image $f^{\leftarrow}(K)$ is an s-compact subset of X.

Proof.Let $\{U_i : i \in \Lambda\}$ be a semi open cover of $f^{\leftarrow}(K)$. Then for each $y \in K$ the set $f^{\leftarrow}(y)$ can be covered by finitely many U_i. Let $U(y)$ denote their union. Then $O(y) = Y \setminus f[y \setminus U(y)]$ is an open neighbourhood of y in Y because f is an s-closed mapping. So, $K \subset \cup_{y \in K} O(y)$, and because K was assumed to be compact so, there are finitely many points $y_1, y_2, \cdots, y_n$ in K such that $K \subset \cup_{i=1}^{n} O(y_i)$. It follows that $f^{\leftarrow}(K) \subset \cup_{i=1}^{n} f^{\leftarrow}(O(y_i)) \subset \cup_{i=1}^{n} U(y_i)$, hence $f^{\leftarrow}(K)$ is s-compact in y.

Theorem 3.6For every disjoint s-compact subsets A_1, A_2 of a semi Hausdorff extremely disconnected irresolute paratopological group G,there exists a semi-open neighbourhood$U$ of the identity such that $UA_1 \cap UA_2 = \emptyset$.

Proof.For every points $x \in A_1, y \in A_2$, there exists a semi-open neighbouhood $V(x,y)$ of identity such that $V(x,y)x \cap V(x,y)y = \emptyset$. Let $U(x,y)$ be a semi-open neighbourhood of identity such that $U^2(x,y) \subset V(x,y)$.For every$x \in A_1$, choose a finite family $\mathcal{F}(x)$ such that$A_2 \subset \cup \{U(x,y)y: y \in \mathcal{F}(x)\}$ Put $U(x) = \cap \{U(x,y): y \in \mathcal{F}(x)\}$ There exists a finite family χ such that $A_1 \subset \cup \{U(x)x: x \in \chi\}$.Put $U = \cap \{U(x): x \in \chi\}$.Then

$$UA_1 \cap UA_2 \subset U \cup \{U(x)x: x \in \chi\} \cap UA_2$$

$$\subset \cup \{U^2(x)x: x \in \chi\} \cap UA_2$$

$$= \cup \{U^2(x)x \cap UA_2: x \in \chi\}$$

$$\subset \cup \{U^2(x)x \cap U \cup \{U(x,y)y: y \in \mathcal{F}(x)\}: x \in \chi\}$$

$$\subset \cup \{U^2(x,y)x \cap U^2(x,y)y: x \in \chi, y \in \mathcal{F}(x)\} = \emptyset.$$

Theorem 3.7 Let G be an extremely disconnected irresolute paratopological group, $A \subset G$ be a s- compact subspace, $F \subset G$ be a semi-closed set and $A \cap F = \emptyset$.Then there exists a semi-open neighbouhood $V(x)$ of identity such that $V(x) \cap F = \emptyset$.

Proof.For every point $x \in A$ there exists a semi-open neighbourhood$V(x)$ of identity such that $V(x)x \cap F = \emptyset$. Let $U(x)$ be a semi-open neighbourhood of identity such that $U^2(x) \subset V(x)$. There exists a finite family χ such that $A \subset \cup \{U(x): x \in \chi\}$. Put $U = \cap \{U(x): x \in \chi\}$. Then $UA \cap F \subset U \cup \{U(x): x \in \chi\} \cap F \subset \cup \{U^2(x): x \in \chi\} \cap F = \emptyset$.

Theorem 3.8Let$(G, ., \tau)$ be an irresolute paratopological group and H be a semi connected component of e, then H is semi closed, and an invariant subgroup of G.

Proof.By the Lemma 3.4, $l_a: G \to G$ is a semi homeomorphism for every $a \in G$. This implies $l_a(H) = a.H$. Since H is semi connected, therefore by Lemma 2.15, $a.H$ is semi connectd. Similarly $r_{a^{-1}}(a.H) = a.H.a^{-1}$ is semi connected. Since $e \in a.H.a^{-1}$ and H is the biggest semi-connected subset of G containing e, therefore $a.H.a^{-1} \subset H$ and $H \subset a.H.a^{-1}$. This implies $H = a.H.a^{-1}$. By Lemma 2.16, $sCl(H)$ is semi connected. Since H is largest semi connected subset, therefore $sCl(H) \subseteq H$. This implies $sCl(H) = H$. Thus H is semi closed invariant subgroup.

Theorem 3.9Let$(G, ., \tau)$ be an irresolute paratopologal and let C be the semi component of identity in G. Then for all $a \in G$, $aC = Ca$ is the semi component of a.

Proof.The mapping $x \to ax$ is semi homeomorphism of G by Lemma 3.4.$C$is an invariant subgroup of G, by Theorem 3.8. Hence $Ca = aC$ is semi component of a.

Theorem 3.10LetG, H and K be irresolute paratopological groups, $\phi: G \to H$ an irresolute homomorphism, $\psi: G \to K$ an irresolute endomorphisms, such that $ker\,\psi \subset ker\phi$. Assume also that for each semi-open neighbourhood U of e_H, there is a semi-open neighbourhood V of $e_K$with $\psi^{\leftarrow}(V) \subset \phi^{\leftarrow}(U)$.Then there is an irresolute homomorphism $f: K \to H$ such that$\phi = f \circ \psi$.

Theorem 3.11Suppose that G, H and K are irresolute paratopological groups. Let $\phi: G \to H$ be an irresolute homomorphism, $\psi: G \to K$ an irresolute endomorphism such that $ker\,\psi \subset ker\phi$.If ψ is pre-semi-open, then there is an irresolute homomorphism $f: K \to H$ such that$\phi = f \circ \psi$.

Proof.By Theorem 3.10, there exists a homomorphism $f: K \to H$ satisfying $\phi = f \circ \psi$.We prove that f is irresolute. Let V be a semi-open set in H. From $\phi = f \circ \psi$, it follows $f(V) = \psi(\phi^{\leftarrow}(V))$. Since ϕ is irresolute, the set $\phi^{\leftarrow}$(V) is semi-open in G, and pre-semi-openness of ψ implies that $\psi(\phi^{\leftarrow}(V))$ is semi-open, i.e. $f^{\leftarrow}(V)$ is semi-open in K. This means that f is irresolute.

IV. SEMI-QUOTIENTS OF PARATOPOLOGIZED GROUPS

With reference to the already existing definition and properties of semi-quotient spaces and semi-quotient maps, it is natural, to find out whether this fact is valid for the paratopologized groups or not. In this section , we shall conduct the investigation on semi-quotients of paratopologized groups.The result of the study will provide a deeper understanding as well as extension knowledge for the concept of semi-quotient mapping and spaces.

Now, we apply the construction of $s\tau_Q$ to paratopologized groups and establish some properties of their semi quotients.

If G is a paratopological group and R a closed subgroup of G, we can look at the collection G/R of left cosets of R in G (or the collection $R\backslash G$ of right cosets of R in G), and endow G/R (or $G\backslash R$) with the semi quotient structure induced by the natural projection $\pi: G \to G/R$. Recall that G/R is not a group under coset multiplication unless R is a normal subgroup of G.

Theorem 4.1 Let $(G,*,\tau)$ be an extremally disconnected s-(irresolute)paratopological group, R its invariant subgroup.Thenπ: $(G,*,\tau) \to (G/R,*,s\tau_Q)$is s-open.

Proof.Let $V \subset G$ be semi open. By definition of $s\tau_Q$, $\pi(V) \in s\tau_Q$ if and only if $\pi^{\leftarrow}(\pi(V)) \subset G$ is semi open, i.e. $V * R$ is semi open in G. But $V * R$ is semi open in G because $V \in SO(G)$ and $(G,*,\tau)$ is an irresolute paratopological group. So, $\pi(V)$ is s-open.

Theorem 4.2Let $(G,*,\tau)$ be an extremally disconnected irresolute paratopological group, R its invariant subgroup. Then $(G/R,*,s\tau_Q)$ is an irresolute paratopological group.

Proof.First, we observe that $s\tau_Q$ is a topology on $s\tau_Q$. Let $(x * R), (y * R) \in G/R$ and let $W \subset G/R$ be a semi open neighbourhoodof $(x * R) * (y * R)$. By the definition of $s\tau_Q$ (induced by π), the set $\pi^{\leftarrow}(W) = W * R$ is a semi open neighbourhood of $x * y$in G, and since G is an irresolute paratopological group, there are semi open sets $U \in SO(G,x)$ and $V \in SO(G,y)$ such that $U * V \subset W * R$. By Theorem 30, the sets $\pi(U) = U * R$ and $\pi(V) = V * R$ are semi open in G/R, which contain $x * R$ and $y * R$, respectively, and satisfy $(U * R) * (V * R) = (U * V) * R \subset W * R$. This just means that $(G/R,*,s\tau_Q)$ is an irresolute paratopological group.

Theorem 4.3 If $(G,*,\tau)$ is an extremally disconnected irresolute paratopological group, R a subgroup of G, and $a \in G$, then the mapping λ_a is a semi homeomorphism and $\pi \circ l_a = \lambda_a \circ \pi$ holds.

Proof.Since G is a group, it is easy to see that λ_a is a (well defined) bijection on G/R. We prove that $\lambda_a \circ \pi = \pi \circ l_a$. Indeed, for each $x \in G$ we have, $(\pi \circ l_a)(x) = \pi(a * x) = (a * x) * R = a * (x * R) = \lambda_a(\pi(x)) = (\lambda_a \circ \pi)(x)$. This is required. It remains to prove that λ_a is irresolute and pre semi open.

This follows from the following facts. Let $x * R \in G/R$. For any semi open neighbourhood $U$of e_G, $\pi(x * U * R)$ is a semi open neighbourhood of $x * R$ in G/R.

Similarly, the set $\pi(a * x * U * R)$ is a semi open neighbourhood of $a * x * R$in G/R. Since

$$\lambda_a\big(\pi(x * U * R)\big) = \pi\big(\lambda_a(x * U * R)\big) = \pi(a * x * U * R)$$

It follows that λ_a is a semi-homeomorphism.

Theorem 4.4 Let $(G,*,\tau_G)$ and $(R,\cdot,\tau_R)$ be extremally disconnected irresolute paratopological groups and $f: G \to R$ be a semi-isomorphism. If G_0 is an invariant subgroup of $G$and $R_0 = f(G_0)$, then the semi quotient irresolute paratopological groups $(G/G_0, s\tau_Q)$ and$(R/R_0, s\tau_Q)$are semi-isomorphic.

Proof.Let $\pi: G \to G/G_0,\ x \mapsto x * G_0$, and $g: R \to R/R_0$, $f(x_0) \mapsto f(x_0) \cdot R_0$ $(x_0 \in G_0)$ be the canonical projections. Consider the mapping $\phi: G/G_0 \to R/R_0$ defined by

$\phi(x * G_0) = f(x) \cdot f(G_0), x \in G, y = f(x)$

Then for $x_1 * G_0, x_2 * G_0 \in G/G_0$ we have,

$$\phi(x_1 * G_0 * x_2 * G_0) = \phi(x_1 * x_2 * G_0)$$

$$= f(x_1 * x_2) \cdot f(G_0)$$

$$= y_1 \cdot y_2 \cdot R_0$$

$$= \phi(x_1 * G_0) \cdot \phi(x_2 * G_0)$$

i.e. ϕ is a homomorphism. Let us prove that ϕ is one-to-one. Let $x * G_0$ be an arbitrary element of G/G_0. Set $y = f(x)$. If $\phi(x * G_0) = R_0$, then $g(y) = R_0$, which implies $x \in G_0, y \in R_0$, and $ker\phi = G_0$. So, ϕ is one-to-one. Next, we have $\phi(x * G_0) = y \cdot R_0$, i.e. $\phi(\pi(x)) = g(y) = g(f(x))$. This implies $\phi \circ \pi = g \circ f$. Since f is a semi homeomorphism, and π, g are s-open, semi continuous homomorphisms, we conclude that ϕ is open and continuous. Hence ϕ is a semi homeomorphism and a semi-isomorphism.

Theorem 4.5Let $(G,*,\tau)$ be an extremally disconnected irresolute paratopological group, R an invariant subgroup of G, M an open subgroup of G, and $\pi: G \to G/R$ the canonical projection. Then the semi quotient group $M * R/R$ is semi-isomorphic to the subgroup $\pi(M)$of G/R.

Proof.It is clear that $M * R = \pi^{\leftarrow}(\pi(M))$. As π is s-open and semi continuous, the restriction f of π to $M * R$ is an s-open and semi continuous mapping of $M * R$ onto $\pi(M)$ by Lemma 2.17. Since M is a subgroup of G and π is a homomorphism it follows that $\pi(M)$ is a subgroup of $G/R, M * R$ is a subgroup of G, and $f: M * R \to \pi(M)$ is a homomorphism. We have $f^{\leftarrow}(f(e_G)) = \pi^{\leftarrow}(\pi(e_G)) = R$, i.e. $ker\, f = R$. It is easy now to conclude that $M * R/R$ and $\pi(M)$are semi-isomorphic.

Theorem 4.6Let$G$ is an extremely disconnected irresolute paratopological group and let R be a invariant subgroup of G.If R and $(G/R,*,s\tau_Q)$ are semi connected, then G is semi connected.

Proof.Assume that $G = U \cup V$, where U and V are disjoint novoid semi open sets. Since H is semi connected, each coset of H is either a subset of U or a subset of V by Lemma 14 and is semi connectd by Lemma 15. Thus the relation

$$G/H = \{xH : xH \subset U\} \cup \{xH: xH \subset V\}$$

$$= \{xH : x \in U\} \cup \{xH : x \in V\}$$

It expresses G/H as the union of disjoint nonvoid semi open sets . This contradicts the hypothesis that G/H is semi connected.

Theorem 4.7Let$G$ be an extremely disconnected irresolute paratopological group and let C be a semi component of identity in G. Then $(G/C,*,s\tau_Q)$ is totally semi disconnected semi T_2 irresolute paratopological group.

Proof.Let C be semi connected suset of G/C. We show that C is a singleton set, equivalently $f^{\leftarrow}(C)$ is a coset of Cin , where $f: G \to G/C$ is the canonical map. Since the semi component of G are cosets of Cin G. It is suffices to prove that $f^{\leftarrow}(C)$ is semi connected. Assume on the contrary that $f^{\leftarrow}(C)$ is semi disconnected and $f^{\leftarrow}(C) = A \cup B$ be a semi disconnection. Then both A and B are non empty semi open subsets of $f^{\leftarrow}(C)$and $A \cap B = \emptyset$. So, there exist semi open subsets U and V in G such that $A = U \cap f^{\leftarrow}(C)$, $B = V \cap f^{\leftarrow}(C)$.Obviously, we have $f(A) = f(U) \cap C$ and $f(B) = f(V) \cap C$.Since$f$ is s- open, so it is pre semi open. $f(A)$and $f(B)$ are both non empty semi open subsets of C such that $C = f(A) \cup f(B)$. Since $C$is semi connected, we must have $f(A \cap f(B) \neq \emptyset$. So, there exist points $a \in A$ and $b \in B$ such that $f(a) = f(b)$. Then $E = aC = bC \subset f^{\leftarrow}(C)$. Consequently $E = (aC \cap A) \cup (bC \cap B)$ is semi disconnectedness for E.This contradiction for E is obviously semi connected. Therefore $f^{\leftarrow}(C)$ is semi conneccted. This completes the theorem.

Theorem 4.8Let$G$ be an extremely disconnected irresolute paratopological group and R is a subgroup of G, Then $(G/R, s\tau_Q)$ is a discrete space if and only if R is semi open in G.

Proof.If R is a semi open in G, then aR is semi open in G for all $a \in G$, and so, $\varphi^{\leftarrow}(\{aR\}) = aR$ is semi open in G, for every point $aR \in G/R$. that is, every point of G/R is open set, and hence every subset of G/R is open. Conversely, if G/R is discrete, then the set $\{R\}$ is open in G/R and thus $\varphi^{-1}(\{R\}) = R$ is semi open in G.

Theorem 4.9Let G be an irresolute paratopological group and R be an invariant subgroup of G. For $a \in G$, let Ψ_a be the mapping defined by $\Psi_a(xR) = (ax)R$ for $xR \in G/R$.Then Ψ_a is a semi homeomorphism of G/R. Then $(G/R, s\tau_Q)$ is semi homogeneous space.

Proof. Since Ψ_a is a one to one mapping of G/R on itself and $(\Psi_a)^{-1} = \Psi_{a^{-1}}$and we need only to show that $\Psi_a$is a pre semi open mapping. Let $\{uR: u \in U\}$ be a semi open subset of G/R, where U is a semi open subset of G.

Then

$$\Psi_a\ (\{uR: u \in U\}) = \{auR: u \in U\} = \{vR: v \in aU\}$$

is semi open, since aU is semi open in G.

Theorem 4.10If H is s-compact then the π is s-closed.If the space (G, τ) is semi-Hausdorff then the space $(G/H, s\tau_Q)$ is Hausdorff. If the space (G, τ) is s-regular then the space $(G/H, s\tau_Q)$ is regular.

Proof. Let F be a semi-closed subset of an extremely disconnected irresolute paratopological group G. Let $\tilde{x} \in G/H \setminus \pi(F)$. Consider an arbitrary point $x \in \pi^{\leftarrow}(\tilde{x})$.Then $xH \cap F = \varphi$. By Theorem 3.7, there exists an semi- open neighbourhood$U$ of the identity such that $UxH \cap F = \varphi$. Then $\tilde{x} \in \pi(Ux)$and $\pi(Ux) \cap \pi(F) = \varphi$, thus the map π is s-closed.

Let G be semi-Hausdorff and $x_1, x_2 \in G/H$.Consider arbitrary points $x_i \in \pi^{\leftarrow}(\tilde{x_i})$.Then $x_1H \cap x_2H = \emptyset$. By Theorem 3.6 there exists a semi-open neighbourhood$U$ of the identity such that $Ux_1H \cap Ux_2H = \emptyset$. Then $\tilde{x_i} \in \pi(Ux_i)$ and $\pi(Ux_1) \cap \pi(Ux_2) = \emptyset$.Then the space $(G/H, s\tau_Q)$ is Hausdorff.

Let G be s-regular, $\tilde{F}$ be a semi-closed subset of $G/H$and $\tilde{x} \in G/H \setminus \tilde{F}$. Consider an arbitrary point $x \in \pi^{\leftarrow}(\tilde{x})$. Then $x \notin \pi^{\leftarrow}(\tilde{F})$.Theorem 3.7 and s-regularity of G imply that there exists a semi-open neighbourhood U of identity such that $sCl(U) \cap \pi^{\leftarrow}(\tilde{F}) = \emptyset$. Then $\tilde{x} \in \pi(Ux)\ and\ Cl(\pi(Ux)) \cap \tilde{F} = \emptyset$, thus the space $(G/H, s\tau_Q)$ is regular.

Corollary 4.11Let H be a s-compact subgroup of an extremely disconnected irresolute paratopological group G, F be a semi-closed subset of G.Then FH is a semi-closed subset of G.

Proof.Let $\pi: G \to G/H$ be the standard projection. Then $FH = \pi^{\leftarrow}(\pi(F))$ is semi-closed subset of G.

Theorem 4.12If R is a s-compact subgroup of an irresolute paratopological group $(G,*,\tau)$, then for every semi closed set $F \subset G$, the set $\pi(G \setminus F)$ belongs to $s\tau_Q$. If $s\tau_Q$ is a topology, then π is an irresolute perfect mapping.

Proof.Let $F \subset G$ be semi closed. By Corollary 4.11, the set $\pi^{\leftarrow}(\pi(F)) = F * R \subset G$ is semi closed. By definition of $s\tau_Q$, $G/R \setminus F * R \in s\tau_Q$. Let now $s\tau_Q$ be a topology on G/R. Take any semi closed subset F of G. The set $F * R$ is semi closed in $G$and $F * R = \pi^{\leftarrow}(\pi(F))$. This implies, π(F) is closed in the semi quotient space G/R. Thus π is s-closed mapping. On the other hand, if $z * R \in G/R$ and $\pi(x) = z * R$ for some $x \in G$, then $\pi^{\leftarrow}(z * R) = \pi^{\leftarrow}(\pi(x)) = x * R$, and this set is s-compact in G. Therefore, π is irresolute perfect.

Corollary 4.13Let$(G,*,\tau)$ be an extremally disconnected irresolute paratopological group and R its s-compact subgroup. If the semi quotient space $(G/R, s\tau_Q)$ is compact, then G is s-compact.

Proof.By Theorem 4.12, the projection $\pi: G \to G/R$ is irresolute perfect. Then by Theorem 3.5, we obtain that $\pi^{\leftarrow}(\pi(G)) = G * R = G$ is s-compact.

V. CONCLUSION

In this paper, we have discussed semi-quotients of paratopologizedgroups. The results of the study will provide a deeper understanding as well as extension knowledge for the concept of semi-quotient mappings, which are stronger than semi-continuous mappings.

VI. ACKNOWLEDGEMENT

Author is thankful to the referee for his careful reading and suggestions for the improvement of this paper.

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