On the Diameter of Middle Graphs and Total Graphs

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Abstract

Let G = (V,E) be a simple connected graph. The diameter of a connected graph G is denoted by diam(G) and is the maximum eccentricity in a graph G. In this article, we establish the results for diameter of Middle graph [M(G)] of a graph G and the equality relation between the diameter of graph G, Middle graph [M(G)] and Total graph [T(G)] of a graph G.

Keywords - *Diameter, Middle graph, Total graph, Distance.*

I. INTRODUCTION

Let $G(V,E)$ be a simple connected graph with n-vertices and m-edges. For undefined terminologies we refer [7]. Let the vertex set of graph G be denoted as $V(G) = v_1,v_2,v_3,...,v_n$, $\forall n \in N$ and the edge set $E(G) = e_1,e_2,e_3,...,e_m$, or $e_k= v_iv_j$, $\forall i,j,k \in N, i\neq j$. The length of the shortest path joining the two vertices v_i and $v_j$in G is known as distance between the vertices v_i and $v_j$and is denoted as $d_G(v_i,v_j)$. The diameter of graph G is denoted by $diam(G)$ or $d(G)$ and is defined as the maximum eccentricity [$e(v_i)$] in a connected graph G, whereas the eccentricity is the maximum of $d_G(v_i,v_j)$ or also the diameter is defined as shortest longest path in graph G.The concept of distance provides the simplest and most natural metric in graph theory and is one of the popular areas of research in discrete mathematics. Details on distance in graph theory can be found in [1, 2, 3, 4, 5, 7, 8, 9, 10].

The Line graph of a graph G is denoted as $L(G)$ whose vertices are corresponds to edges of graph G, if two vertices of graph $L(G)$ are adjacent if and only if the corresponding edges of graph G are adjacent. One of the important variation of the line graph is the Middle graph and it is studied in [6]. The Middle graph [$M(G)$] of a graph G is the graph whose vertex set can be put in one-to-one correspondence with the set of vertices and edges of G in such a way that two vertices of $M(G)$ are adjacent if and only if they are adjacent edges of G and one is vertex and the other is on edge of G incident to it [6]. The vertices and edges of a graph G are called as its elements. Two elements of a graph are neighbors if they are either incident or adjacent. The Total graph [$T(G)$] has vertex set $V(G)\cup E(G)$ and two vertices of $T(G)$ are adjacent whenever they are neighbors in G [7].

Motivated by the work of H. S. Ramane, et al. in [11], we established the results for diameter of middle graph [$M(G)$] of graph G and also established the results for equality relation between the diameter of graph G, diameter of middle graph [$M(G)$] and diameter of total graph [$T(G)$] of graph G.

II. RESULTS

Theorem 2.1. *Let G be any connected graph, then*

$diam[M(G)] = diam[G] + 1.$

Proof.Consider a connected graph G. Let $v_1,v_2,...,v_n$, $\forall n \in N$ are the vertices and $e_1,e_2,...,e_m$, $\forall m \in N$ are the edges of graph G.

Suppose the maximum distance between any two vertices v_i and $v_j$is k and is the maximum eccentricity in graph G,

i.e., $diam[G] = k, \qquad \forall k = 1, 2, 3,... \qquad (1)$

Now the graph $M(G)$ contains the line graph [$L(G)$] as its subgraphs. Hence the distance between the vertices $e_1, e_2,...,e_m$, $\forall m \in N$ [which are edges in graph G] in graph $L(G)$ remains same in graph M(G).

Now consider the vertices v_i and $v_j(\forall i, j \in N, i\neq j)$ in graph G which having the maximum distance k and is maximum eccentricity in graph G [as our supposition]. But the same two vertices v_i and $v_j$of graph G have their maximum distance $(k + 1)$ in middle graph [$M(G)$]. Because any two vertices v_i and $v_j(\forall i, j \in N, i\neq j)$ of graph G are not adjacent in graph $M(G)$ [because by definition of graph $M(G)$], for this reason, if any two vertices v_i and $v_j$have their distance d in graph G then those two vertices have their distance $(d + 1)$ in graph $M(G)$.

Since the distance k is maximum in graph G and that is maximum eccentricity also [as our supposition]. Then the distance $(k + 1)$ acts as maximum distance and maximum eccentricity in graph $M(G)$. Therefore the diameter of graph $M(G)$ is $(k + 1)$.

i.e. $diam[M(G)] = k + 1,\ \forall k = 1, 2, 3,...$(2)

from (1) and (2),

$diam[M(G)] = diam[G] + 1$

Theorem 2.2. *Let G be a graph,* $\forall G = C_n$ *with n-vertices* $\forall n \in N$*, then*

(i)$diam[T(C_n)] = diam(C_n), \forall n = 2n, n = 2, 3, 4,...$

(ii)$diam[T(C_n)] = diam[M(C_n)], \forall n = 2n + 1, n = 1, 2, 3,...$

Proof.Consider a graph G, $\forall G = C_n$ with n-vertices and to prove the above two conditions we consider the following two cases.

Case i. Let C_n be the cycle, $\forall n = 2n$, $n = 2,3,4,....$ Let $v_1, v_2,...,v_n$are the vertices of graph C_n. Then the distance from the vertex v_1 to vertex $v_{\lceil\frac{1+n}{2}\rceil}$ is $\frac{n}{2}$ and the distance $\frac{n}{2}$ is the maximum distance and also maximum eccentricity in graph C_n. Hence the diameter of graph C_n is $\frac{n}{2}$.

i.e., $diam[C_n] = \frac{n}{2}, \forall n = 2n$ and $n = 2, 3, 4,...$ (3)

Now from theorem 1, we know that the maximum eccentricity or diameter of middle graph of C_n i.e., $M(C_n)$ is $diam[C_n] + 1$.

i.e., $diam[M(C_n)] = \frac{n}{2}, \forall n = 2n$ and $n = 2, 3, 4,...$(4)

Now consider the total graph of graph C_n, the maximum distance and maximum eccentricityin total graph of graph C_n i.e., $T(C_n)$ is $\frac{n}{2}$. Because the graph C_n contains $v_1,v_2,...,v_n$are as vertices and $v_1v_2,v_2v_3,...,v_nv_1$ are as edges. Then these vertices and edges of graph C_n acts as vertices in graph $T(C_n)$.

i. The distance from vertex v_1 to vertex $v_{\lceil\frac{1+n}{2}\rceil}$ is $\frac{n}{2}$

ii. The distance from vertex v_1 to vertex $v_{\lceil\frac{1+n}{2}\rceil}v_{(\frac{n}{2})}$ [which is an edge in C_n] is $\frac{n}{2}$ in $T(C_n)$.

iii. The distance from vertex v_1 to vertex $v_{\lceil\frac{1+n}{2}\rceil}v_{\lceil\frac{4+n}{2}\rceil}$ [which is an edge in C_n] is also $\frac{n}{2}$ in $T(C_n)$.

Hence from above mentioned 3-conditions the distance $\frac{n}{2}$ is maximum and maximum eccentricity in graph $T[C_n]$. Therefore the diameter of graph $T(C_n)$ is $\frac{n}{2}$.

i.e., $diam[T(C_n)] = \frac{n}{2}$(5) from (3) and (5),

$diam[T(C_n)] = diam(C_n), \forall n = 2n$ *and* $n = 2, 3, 4,...$

Case ii. Consider a graph G, $\forall G = C_n$, $n = 2n + 1$, $n \in N$. Let $v_1,v_2,...,v_n$are the vertices of graph C_n. Then the distance between some vertices in graph C_n are as follows.

i. The distance from vertex v_1 to vertex $v_{(\frac{1+n}{2})}$ is $\lfloor\frac{n}{2}\rfloor$.

ii. The distance from vertex v_1 to vertex $v_{(\frac{3+n}{2})}$ is $\lfloor\frac{n}{2}\rfloor$.

From above two conditions, $\lfloor\frac{n}{2}\rfloor$ is acts as maximum distance and maxim um eccentricity in graph C_n. Hence the diameter of graph C_n is $\lfloor\frac{n}{2}\rfloor$.

i.e., $diam(C_n) = \lfloor\frac{n}{2}\rfloor$ (6)

Now from theorem 1, we know that the maximum eccentricity or diameter of middle graph of C_n i.e., $M(C_n)$ is $diam[C_n] + 1$.

i.e., $diam[M(C_n)] = \left\lfloor \frac{n}{2} \right\rfloor + 1$, $\forall n = 2n + 1$ and $n \in N$. (7)

The graph C_n contains $v_1, v_2, \ldots, v_n$ are as vertices and $v_1v_2, v_2v_3, \ldots, v_nv_1$ are as edges and the total graph of C_n i.e., $T(C_n)$ contains $v_1, v_2, \ldots, v_n, v_1v_2, v_2v_3, \ldots, v_nv_1$ are as vertices. Now the distance from vertex v_1 to vertex $v_{\left(\frac{1+n}{2}\right)}v_{\left(\frac{3+n}{2}\right)}$ [which is an edge in C_n] is $\left\lfloor \frac{n}{2} \right\rfloor + 1$ in $T(C_n)$. And this distance is maximum in $T(C_n)$.

Thus the maximum distance and maximum eccentricity of graph $T(C_n)$ is $\left\lfloor \frac{n}{2} \right\rfloor + 1$.

i.e., $diam[T(C_n)] = \left\lfloor \frac{n}{2} \right\rfloor + 1$.(8)

from (6) and (8),

$$diam[T(C_n)] = diam[M(C_n)], \quad \forall n = 2n + 1 \text{ and } n \in N$$

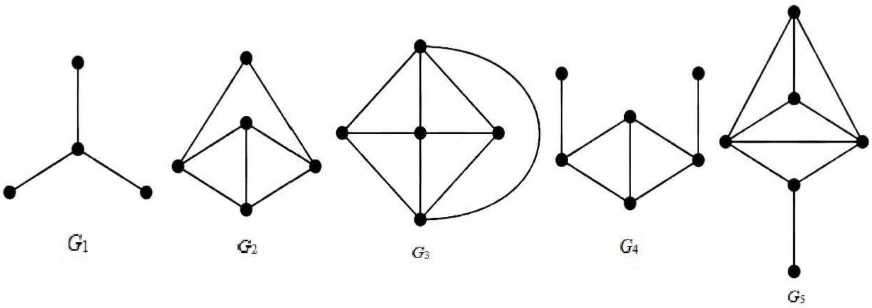


Figure 1. The graphs mentioned in Theorem 2.3.

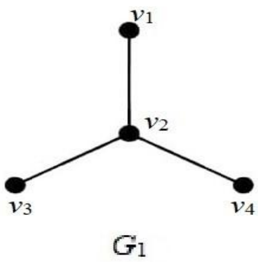
Theorem 2.3. Let G be a connected graph with n-vertices. If G is other than the cycle and G does not contain the graphs depicted in figure 1 as an induced subgraph of G, then

$$diam[M(G)] = diam[T(G)]$$

Proof. Consider a connected graph G except the cycle C_n. Let

$$diam[M(G)] = diam[T(G)].$$

Suppose the graph G_1 is an induced subgraph of graph G.



Then the distance from vertex v_1 to v_3 and distance from vertex v_1 to v_4 is 2 and this distance is maximum in graph G_1.

Therefore the maximum distance and maximum eccentricity, i.e., diameter of graph G_1 is 2.

i.e., $diam[G_1] = 2$ (9)

Now from theorem 1, we know that the maximum eccentricity i.e., diameter of middle graph of G_1 i.e., $M(G_1)$ is $diam[G_1] + 1$.

i.e., $diam[M(G_1)] = 3$(10)

Since the graph G_1 contains v_1,v_2,v_3,v_4 as the vertices and $v_1v_2,v_2v_3,v_3v_4,v_4v_1$ as the edges. The total graph of graph G_1 i.e., $T[G_1]$ contains $v_1,v_2,v_3,v_4,v_1v_2,v_2v_3,v_3v_4,v_4v_1$ as the vertices. The distance between the vertices in graph $T[G_1]$ is as follows:

i. The distance from vertex v_1 to v_3 is 2.
ii. The distance from vertex v_1 to v_4 is 2.
iii. The distance from vertex v_1 to v_2v_3 [which is an edge in G_1] is 2.
iv. The distance from vertex v_1 to v_2v_4 [which is an edge in G_1] is 2.

Now from above 4-conditions the distance 2 acts as the maximum distance and maximum eccentricity, i.e., diameter of graph $T(G_1)$. Therefore the diameter of graph $T(G_1)$ is 2.

$$diam[T(G_1)] = 2 \tag{11}$$

From (9), (10) and (11),

$$diam[G_1] = diam[T(G_1)]$$

and

$$diam[M(G_1)] \neq diam[T(G_1)]$$

a contradiction.

Therefore G_1 is not an induced subgraph of graph G. Similarly we can show that remaining G_2, G_3, G_4 and G_5 graphs are also not an induced subgraphs of G.

Hence the proof.

III. CONCLUSIONS

In this article, we have discussed about the diameter of Middle graph and Total graph of graph G, also we establish the results for relation between diameter of Middle graph $[M(G)]$ and Total graph $[T(G)]$ of a graph G. This study can be extended to various graph operations.

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