On $Z_2 \times Z_2$ - Cordial Graphs

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Abstract

For any abelian group A, a graph G is said to be A - cordial if there is a labeling f of $V(G)$ with elements of A so that for all $a,b \in A$, the edge ab is labeled with $f(a) + f(b)$ then the number of vertices labeled with a and the vertices labeled with b differ by at most 1and the number of edges labeled with a and edges labeled with b differ by at most 1. In this paper we determine some classes of $Z_2 \times Z_2$ cordial graphs, a necessary condition for the sum of two $Z_2 \times Z_2$ - cordial graphs to be $Z_2 \times Z_2$ - cordial and we prove that every graph is an induced sub graph of $Z_2 \times Z_2$ - cordial graph.

Key words - *A- Cordial labeling, Abelian group.*

I. INTRODUCTION

Mark Hovey [1] has introduced A – cordial labeling as a generalization of harmonious and cordial labeling. It is well known that $Z_2 = \{0,1\}$ is the residue classes modulo 2, where 0 denote the set of all even integers and 1 denote the set of all odd integers and consequently $Z_2 \times Z_2 = \{(0,0),(1,0),(0,1),(1,1)\}$ is an abelian group. Without loss of generality, we may assume that $e = (0,0)$, $a = (1,0)$, $b = (0,1)$, $c = (1,1)$. Then $a + b = c$, $a + c = b$ and $b + c = a$. In this paper we determine some classes of $Z_2 \times Z_2$ cordial graphs and a necessary condition for sum of two $Z_2 \times Z_2$ - cordial graphs to be $Z_2 \times Z_2$ - cordial. Also we prove that every graph is an induced subgraph of $Z_2 \times Z_2$ - cordial graph.

Definition 1. For any abelian group A, a graph $G = (V, E)$ is said to be A - cordial , if there is a labeling f of V with elements of A so that for all $a, b \in A$, the edge ab is labeled with $f(a) + f(b)$ then $v_f(a)$ and $v_f(b)$ differ by at most 1 and $e_f(a)$ and $e_f(b)$ differ by at most 1, where $v_f(a)$ and $e_f(b)$ are respectively the number of vertices labeled with a and the number of edges labeled with b.

II. MAIN RESULTS

Theorem 2: *The cycle C_n is $Z_2 \times Z_2$ - cordial for all n except $n = 4,5$ and $n \equiv 2 (\text{mod } 4)$.*

Proof: *Let $V(C_n) = \{v_i : 1 \le i \le n\}$ be the set of all vertices of C_n, and $E(C_n) = \{e_i = v_i v_{i+1} : 1 \le i \le n-1\}$ be the set of all edges of C_n. Define $f : V(C_n) \to Z_2 \times Z_2$ as follows:* Case(i) If $n \equiv 0,1,3,7 (\text{mod } 8)$: Label the vertices of C_n as $f(v_i) = e$, if $i = 4k, 4k+1$ where k is odd, $f(v_i) = a$, if $i = 1, 8k, 8k+1$ where k is an integer, $f(v_i) = b$, if $i \equiv 2 (\text{mod } 4)$ and $f(v_i) = c$, if $i \equiv 3 (\text{mod } 4)$.

Case(ii) If $n \equiv 4$ *or* $5 (\text{mod } 8)$ where $n \ne 4,5$: Label the vertices of C_n as $f(v_i) = e$ if $i = 4,5,8,8k,8k+1$ where $k > 1$ is an integer, $f(v_i) = a$, if $i = 1,7,4k,4k+1$ where $k > 1$ is odd, $f(v_i) = b$ if $i = 2,9,2k$ where $k > 3$ is odd, $f(v_i) = c$, if $i = 3,6,4k+3$ where $k > 1$ is an integer. Table 1 shows the values of $v_f(x)$ for all $x \in Z_2 \times Z_2$ and Table 2 shows the values of $e_f(x)$ for all $x \in Z_2 \times Z_2$. Also $|v_f(i) - v_f(j)| \le 1$ and $|e_f(i) - e_f(j)| \le 1$ for all $i, j \in Z_2 \times Z_2$. Hence C_n is $Z_2 \times Z_2$ - cordial for all n except $n = 4,5$ and $n \equiv 2 (\text{mod } 4)$.

$n=8r+i$	$v_f(e)$	$v_f(a)$	$v_f(b)$	$v_f(c)$
$i=0$	$2r$	$2r$	$2r$	$2r$
$i=1$	$2r$	$2r+1$	$2r$	$2r$
$i=3$	$2r$	$2r+1$	$2r+1$	$2r+1$
$i=4$	$2r+1$	$2r+1$	$2r+1$	$2r+1$
$i=5$	$2r+1$	$2r+2$	$2r+1$	$2r+1$
$i=7$	$2r+2$	$2r+1$	$2r+2$	$2r+2$

Table 1. Labelings of the vertices of C_n

$n=8r+i$	$e_f(e)$	$e_f(a)$	$e_f(b)$	$e_f(c)$
$i=0$	$2r$	$2r$	$2r$	$2r$
$i=1$	2r+1	$2r$	$2r$	$2r$
$i=3$	$2r$	$2r+1$	$2r+1$	$2r+1$
$i=4$	$2r+1$	$2r+1$	$2r+1$	$2r+1$
$i=5$	$2r+2$	$2r+1$	$2r+1$	$2r+1$
$i=7$	$2r+1$	$2r+2$	$2r+2$	$2r+2$

Table 2. Labelings of the edges of C_n

Theorem3. The complete bipartite graph $K_{m,n}$ where $m \le n$ is $Z_2 \times Z_2$- cordial for all m and n except m & $n \equiv 2(\text{mod } 4)$.

Proof: Let $V_1 = \{u_i : 1 \le i \le m\}$ and $V_2 = \{v_j : 1 \le j \le n\}$ be the set vertices of $K_{m,n}$ and $E(K_{m,n}) = \{e_i = u_i v_j : 1 \le i \le m, 1 \le j \le n\}$ be the set of all edges of $K_{m,n}$. Define $f : V(K_{m,n}) \to Z_2 \times Z_2$ as follows: $f(u_i) = e$ if $i \equiv 1(\text{mod } 4)$, $f(u_i) = a$ if $i \equiv 2(\text{mod } 4)$, $f(u_i) = b$ if $i \equiv 3(\text{mod } 4)$**,** $f(u_i) = c$ if $i \equiv 0(\text{mod } 4)$.

Case(i) If $m \equiv 0,1,2(\text{mod } 4)$ $\left(Note\ that\ if\ m \equiv 2(\text{mod } 4), choose\ n\ such\ that\ n \not\equiv 2(mod 4)\right)$, define $f(v_i) = e$ if
$i \equiv 0(\text{mod } 4)$, $f(v_j) = a$ if $i \equiv 3(\text{mod } 4)$, $f(v_i) = b$ if $i \equiv 1(\text{mod } 4)$, $f(v_j) = c$ if $i \equiv 2(\text{mod } 4)$.

If $m = 4r$, Table 3 and 4 shows the values of $v_f(x)$ and $e_f(x)$ for all $x \in Z_2 \times Z_2$.

$n=4k+i$	$v_f(e)$	$v_f(a)$	$v_f(b)$	$v_f(c)$
$i=0$	$r+k+1$	$r+k$	$r+k$	$r+k$
$i=1$	$r+k$	$r+k$	$r+k+1$	$r+k$
$i=2$	$r+k$	$r+k$	$r+k+1$	$r+k+1$
$i=3$	$r+k$	$r+k+1$	$r+k+1$	$r+k+1$

Table 3. Labelings of the vertices of $K_{m,n}$

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$n=4k+i$	$e_f(e)$	$e_f(a)$	$e_f(b)$	$e_f(c)$
$i=0$	nr	nr	nr	nr
$i=1$	nr	nr	nr	nr
$i=2$	nr	nr	nr	nr
$i=3$	nr	nr	nr	nr

Table 4. Labelings of the edges of $K_{m,n}$

If $m = 4r + 1$ **Table 5 and 6 shows the values of** $v_f(x)$ **and** $e_f(x)$ **for all** $x \in Z_2 \times Z_2$.

$n=4k+i$	$v_f(e)$	$v_f(a)$	$v_f(b)$	$v_f(c)$
$i=0$	$r+k+1$	$r+k$	$r+k$	$r+k$
$i=1$	$r+k+1$	$r+k$	$r+k+1$	$r+k$

$i=2$	$r+k+1$	$r+k$	$r+k+1$	$r+k+1$
$i=3$	$r+k+1$	$r+k+1$	$r+k+1$	$r+k+1$

Table5. Labelings of the vertices of $K_{m,n}$

$n=4k+i$	$e_f(e)$	$e_f(a)$	$e_f(b)$	$e_f(c)$
$i=0$	mk	mk	mk	mk
$i=1$	$mk+r$	$mk+r$	$mk+r+1$	$mk+r$
$i=2$	$mk+2r$	$mk+2r$	$mk+2r+1$	$mk+2r+1$
$i=3$	$mk+3r$	$mk+3r+1$	$mk+3r+1$	$mk+3r+1$

Table 6. Labelings of the edges of $K_{m,n}$

If $m=4r+2$ Table 7 and 8 shows the values of $v_f(x)$ and $e_f(x)$ for all $x \in Z_2 \times Z_2$.

$n=4k+i$	$v_f(e)$	$v_f(a)$	$v_f(b)$	$v_f(c)$
$i=0$	$r+k+1$	$r+k+1$	$r+k$	$r+k$
$i=1$	$r+k+1$	$r+k+1$	$r+k+1$	$r+k$
$i=3$	$r+k$	$r+k+1$	$r+k$	$r+k+1$

Table7. Labelings of the vertices of $K_{m,n}$

$n=4k+i$	$e_f(e)$	$e_f(a)$	$e_f(b)$	$e_f(c)$
$i=0$	mk	mk	mk	mk
$i=1$	$mk+r$	$mk+r$	$mk+r+1$	$mk+r+1$
$i=3$	$mk+3r+1$	$mk+3r+1$	$mk+3r+2$	$mk+3r+2$

Table8. Labelings of the edges of $K_{m,n}$

If $m=4r+3$ Table 9 and 10 shows the values of $v_f(x)$ and $e_f(x)$ for all $x \in Z_2 \times Z_2$.

$n=4k+i$	$v_f(e)$	$v_f(a)$	$v_f(b)$	$v_f(c)$
$i=0$	$r+k+1$	$r+k+1$	$r+k+1$	$r+k$
$i=1$	$r+k+1$	$r+k+1$	$r+k+1$	$r+k+1$
$i=2$	$r+k+2$	$r+k+1$	$r+k+1$	$r+k+1$
$i=3$	$r+k+2$	$r+k+2$	$r+k+1$	$r+k+1$

Table 9. Labelings of the vertices of $K_{m,n}$

$n=4k+i$	$e_f(e)$	$e_f(a)$	$e_f(b)$	$e_f(c)$
$i=0$	mk	mk	mk	mk
$i=1$	$mk+r$	$mk+r+1$	$mk+r+1$	$mk+r+1$
$i=2$	$mk+2r+1$	$mk+2r+2$	$mk+2r+2$	$mk+2r+1$
$i=3$	$mk+3r+2$	$mk+3r+3$	$mk+3r+2$	$mk+3r+2$

Table 10. Labelings of the edges of $K_{m,n}$

Case(ii) If $m \equiv 3(\text{mod } 4)$, define $f(v_i)=e$ if $i \equiv 2(\text{mod } 4)$, $f(v_i)=a$ if $i \equiv 3(\text{mod } 4)$, $f(v_i)=b$ if $i \equiv 0(\text{mod } 4)$,
$f(v_i)=c$ if $i \equiv 1(\text{mod } 4)$.

If $m=4r+3$ Table 11 and 12 shows the values of $v_f(x)$ and $e_f(x)$ for all $x \in Z_2 \times Z_2$

$n=4k+i$	$v_f(e)$	$v_f(a)$	$v_f(b)$	$v_f(c)$
$i=0$	$r+k+1$	$r+k+1$	$r+k+1$	$r+k$
$i=1$	$r+k+1$	$r+k+1$	$r+k+1$	$r+k+1$

$i=2$	$r+k+2$	$r+k+1$	$r+k+1$	$r+k+1$
$i=3$	$r+k+2$	$r+k+2$	$r+k+1$	$r+k+1$

Table 11. Labelings of the vertices of $K_{m,n}$

$n=4k+i$	$e_f(e)$	$e_f(a)$	$e_f(b)$	$e_f(c)$
$i=0$	mk	mk	mk	mk
$i=1$	$mk+r$	$mk+r+1$	$mk+r+1$	$mk+r+1$
$i=2$	$mk+2r+1$	$mk+2r+2$	$mk+2r+2$	$mk+2r+1$
$i=3$	$mk+3r+2$	$mk+3r+3$	$mk+3r+2$	$mk+3r+2$

Table 12. Labelings of the edges of $K_{m,n}$

In all cases, $|v_f(i)-v_f(j)|\le 1$ and $|e_f(i)-e_f(j)|\le 1$ for all $i, j\in Z_2\times Z_2$. Hence the complete bipartite graph $K_{m,n}$ where $m\le n$ is $Z_2\times Z_2$- cordial for all m and n except $m\,\&\,n\equiv 2(\mathrm{mod}\,4)$.

Remark: $Z_2\times Z_2$- cordiality of $K_{m,n}$ IS found in [2] . However, the proof is different from ours.

Theorem 4. Let G_i are (p_i,q_i) $Z_2\times Z_2$- cordial labeled graph under f_i for $i=1,2$ respectively. Then G_1+G_2 is $Z_2\times Z_2$- cordial if (i) either p_1 or $p_2\equiv 0(\mathrm{mod}\,4)$ and (ii) either q_1 or $q_2\equiv 0(\mathrm{mod}\,4)$.

Proof: Case (i) Let $p_1\equiv 0(\mathrm{mod}\,4)$ and $q_1\equiv 0(\mathrm{mod}\,4)$. Then $v_{f_1}(i)=\frac{p_1}{4}$, $e_{f_1}(i)=\frac{q_1}{4}$ $\forall i=e,a,b,c$. Also, $|v_{f_2}(i)-v_{f_2}(j)|\le 1$ and $|e_{f_2}(i)-e_{f_2}(j)|\le 1$ $\forall i,j=e,a,b,c$. Let $v_{f_2}(e)=m_1$, $v_{f_2}(a)=m_2$, $v_{f_2}(b)=m_3$, $v_{f_2}(c)=m_4$, $e_{f_2}(e)=n_1$, $e_{f_2}(a)=n_2$, $e_{f_2}(b)=n_3$, and $v_{f_2}(c)=m_4$. Define $f:V(G_1+G_2)\to Z_2\times Z_2$ such that $f|_{V(G_i)}=f_i$ for $i=1,2$. Then clearly $|v_f(i)-v_f(j)|\le 1$ $\forall i,j=e,a,b,c$ and $e_f(e)=\frac{q_1}{4}+n_1+\frac{p_1p_2}{4}$, $e_f(a)=\frac{q_1}{4}+n_2+\frac{p_1p_2}{4}$, $e_f(b)=\frac{q_1}{4}+n_3+\frac{p_1p_2}{4}$, $e_f(c)=\frac{q_1}{4}+n_4+\frac{p_1p_2}{4}$ .Therefore, $|e_f(i)-e_f(j)|=|e_{f_2}(i)-e_{f_2}(j)|\le 1$ $\forall i,j=e,a,b,c$.

Case(ii). Let $p_1\equiv 0(\mathrm{mod}\,4)$ and $q_2\equiv 0(\mathrm{mod}\,4)$. Then $v_{f_1}(i)=\frac{p_1}{4}$, $\forall i=e,a,b,c$ and $|e_{f_1}(i)-e_{f_1}(j)|\le 1$, $\forall i,j=e,a,b,c$. Let $e_{f_1}(e)=n_1$, $e_{f_1}(a)=n_2$, $e_{f_1}(b)=n_3$, $e_{f_1}(c)=n_4$ and $|v_{f_2}(i)-v_{f_2}(j)|\le 1$, $\forall i,j=e,a,b,c$ and $e_{f_2}(i)=\frac{q_2}{4}$ $\forall i=e,a,b,c$. Let $v_{f_2}(e)=m_1$, $v_{f_2}(a)=m_2$, $v_{f_2}(b)=m_3$ and $v_{f_2}(c)=m_4$. Define $f:V(G_1+G_2)\to Z_2\times Z_2$ such that $f|_{V(G_i)}=f_i$ for $i=1,2$ .Then $v_f(e)=\frac{p_1}{4}+m_1$, $v_f(a)=\frac{p_1}{4}+m_2$, $v_f(b)=\frac{p_1}{4}+m_3$ and $v_f(c)=\frac{p_1}{4}+m_4$ .Therefore, $|v_f(i)-v_f(j)|=|v_{f_2}(i)-v_{f_2}(j)|\le 1$ $\forall i,j=e,a,b,c$. Now, $e_f(e)=n_1+\frac{q_1}{4}+\frac{p_1}{4}(m_1+m_2+m_3+m_4)=n_1+\frac{q_1}{4}+\frac{p_1p_2}{4}$. Similarly $e_f(a)=n_2+\frac{q_1}{4}+\frac{p_1p_2}{4}$, $e_f(b)=n_3+\frac{q_1}{4}+\frac{p_1p_2}{4}$ and $e_f(c)=n_4+\frac{q_1}{4}+\frac{p_1p_2}{4}$ .Therefore, $|e_f(i)-e_f(j)|=|e_{f_1}(i)-e_{f_1}(j)|\le 1$ $\forall i,j=e,a,b,c$. In a similar manner we can consider the other two cases. Hence G_1+G_2 is $Z_2\times Z_2$- cordial under f .

Theorem5. Every connected graph is an induced sub graph of a $Z_2\times Z_2$- cordial graph.

Proof: Let G be a connected (p,q) graph with vertices $v_1,v_2,...,v_p$. If G itself is $Z_2\times Z_2$- cordial, there is nothing to prove. Otherwise, define $f:V(G)\to Z_2\times Z_2$ such that $|v_f(i)-v_f(j)|\le 1$ and $|e_f(i)-e_f(j)|\le \alpha$ $\forall i,j=e,a,b,c$ where α is possibly a small positive integer. Let $e_f(e)=m_1$, $e_f(a)=m_2$, $e_f(b)=m_3$,

$e_f(c) = m_4$. Let $m = \min\{m_1, m_2, m_3, m_4\}$ and $M = \max\{m_1, m_2, m_3, m_4\}$. Since G is not $Z_2 \times Z_2$ cordial, we have $M - m \geq 2$ and $p > 3$. Let u be a vertex in G whose label is x, and such that $v_f(x) = n_1$ is the minimum. Add a new vertex v and label it with x, then $v_f(x) = n_1 + 1$. Let $M - m = 2$. If $M - m_1 > 1$, join v with a vertex in G whose label is x, then $e_f(e) = m_1 + 1$. If $M - m_2 > 1$, join v with a vertex in G whose label is $x + a$, then $e_f(a) = m_2 + 1$. Similarly if $M - m_3 > 1$, join v with a vertex in G whose label is $x + b$, then $e_f(b) = m_3 + 1$ and if if $M - m_4 > 1$, join v with a vertex in G whose label is $x + c$, then $e_f(c) = m_4 + 1$. If $M - m > 2$, we repeat the above process with the new graph H_1. Since this process reduce the difference $M - m$, after a finite number of steps, we get a $Z_2 \times Z_2$ - cordial graph which contains G as an induced sub graph.

Notation : A (p, q) graph has p vertices and q edges. For basic concepts in graph theory, we follow [3] and for graph labeling follow [4].

CONCLUSION

The cycle C_n is $Z_2 \times Z_2$ - cordial for all n except $n = 4,5$ and $n \equiv 2 (\mathrm{mod}\ 4)$ and the complete bipartite graph $K_{m,n}$ where $m \leq n$ is $Z_2 \times Z_2$ - cordial for all m and n except m & $n \equiv 2 (\mathrm{mod}\ 4)$. Also if G_i are (p_i, q_i) $Z_2 \times Z_2$ - cordial labeled graph under f_i for $i = 1,2$ respectively then $G_1 + G_2$ is $Z_2 \times Z_2$ - cordial if (i) either p_1 or $p_2 \equiv 0 (\mathrm{mod}\ 4)$ and (ii) either q_1 or $q_2 \equiv 0 (\mathrm{mod}\ 4)$. Every connected graph is an induced sub graph of a $Z_2 \times Z_2$ - cordial graph.

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