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# $\hat{g}$w-Closed Sets in Weak Structure Spaces

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Abstract. In this paper, we introduce the concepts of $\hat{g}w$-closed sets and $\hat{g}w$-open sets in weak structure spaces. Further, we study some of their properties.

## 1. Introduction

In 1970, Levine [8] introduced the notion of generalized closed (briefly, g-closed) sets in general topology. Császár [4] introduced a new notion of structures called weak structures. Al-Omari and Noiri [1] introduced generalized closed sets in weak structures. In this paper we introduce the notions of $\hat{g}w$-closed sets and $\hat{g}w$-open sets in weak structure spaces. The relation of the class of generalized closed sets with the class of $\hat{g}w$-closed sets are to be given. Also we study some of their properties.

## 2. Preliminaries

Throughout this paper, by a space X, we always mean a topological space (X, $\tau$) with no separation properties assumed. Let H be a subset of X. We denote the interior, the closure and the complement of a set H by int(H), cl(H) and X\H or $H^c$, respectively.

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**Definition 2.1.** [9] *Let X be a space. A subset H of a space X is said to be semi-open if* $H \subseteq cl(int(H))$.
*The family of all semi-open sets in X is denoted by SO(X).*
*The complement of a semi-open set is called semi-closed.*

**Definition 2.2.** [3] *The semi-closure of the subset H of a space X is the intersection of all semi-closed subsets of X containing H and it is denoted by scl(H).*

**Definition 2.3.** [2] *A subset H of a space X is called a semi-generalized closed set (briefly sg-closed) if* $scl(H) \subseteq U$ *whenever* $H \subseteq U$ *and U is semi-open in* $(X,\tau)$.

**Theorem 2.4.** [2] *Every semi-closed set is sg-closed but not conversely.*

**Definition 2.5.** [12] *Let X be a space and H a subset of X. A point* $x \in X$ *is called a* $\theta$*-cluster point of H if* $cl(V) \cap H \neq \emptyset$ *for every open set V containing x. The set of all* $\theta$*-cluster points of H is called the* $\theta$*-closure of H and is denoted by* $cl_\theta(H)$.
*A subset H of a space X is said to be* $\theta$*-closed if* $cl_\theta(H)=H$*. The complement of a* $\theta$*-closed set is called* $\theta$*-open. The collection of all* $\theta$*-open sets in X is denoted by* $\tau_\theta$.
$\tau_\theta$ *forms a topology on X.*

**Definition 2.6.** [8] *Let X be a space. A subset H of a space X is said to be generalized closed (briefly, g-closed) if* $cl(H) \subseteq U$ *whenever* $H \subseteq U$ *and U is open in* $(X, \tau)$.

The complement of a g-closed set is called g-open.

**Definition 2.7.** [7] *Let X be a space. A subset H of a space X is said to be* $\theta w$*-closed if* $cl_\theta(H) \subseteq U$ *whenever* $H \subseteq U$ *and* $U \in SO(X)$.

**Remark 2.8.** [9, 12] *For a subset of a space, we have the following implications.*

$$\theta\text{-open} \rightarrow \text{open} \rightarrow \text{semi-open}.$$

**Definition 2.9.** [13] *Let X be a space. A subset H of X is said to be* $\hat{g}$*-closed if* $cl(H) \subseteq U$ *whenever* $H \subseteq U$ *and* $U \in SO(X)$.
*The complement of a* $\hat{g}$*-closed set is called* $\hat{g}$*-open.*

**Theorem 2.10.** [8] *A subset H is g-open iff* $F \subseteq int(H)$*, whenever F is closed and* $F \subseteq H$.

**Definition 2.11.** [4, 10] *Let $X$ be a nonempty set and $w \subseteq P(X)$ where $P(X)$ is the power set of $X$. Then $w$ is called a weak structure (WS in short) on $X$ if $\emptyset \in w$.*

*A non-empty set $X$ with a weak structure $w$ is called a weak structure space (WSS in short) and is denoted by $(X, w)$. Each member of $w$ is said to be $w$-open and the complement of a $w$-open set is called $w$-closed.*

**Definition 2.12.** [10] *Let $(X, w)$ be a WSS. Let $H \subseteq X$. Then the interior of $H$ (briefly $i_w(H)$ ) is the union of all $w$-open sets contained in $H$ and the closure of $A$ (briefly $c_w(H)$) is the intersection of all $w$-closed sets containing $H$.*

**Remark 2.13.** [1] *If $w$ is a WS on $X$, then $i_w(\emptyset)=\emptyset$ and $c_w(X)=X$.*

**Theorem 2.14.** [4] *If $w$ is a WS on $X$ and $A,B \in w$ then*

(1) $i_w(A) \subseteq A \subseteq c_w(A)$,

(2) $A \subseteq B \Rightarrow i_w(A) \subseteq i_w(B)$ *and* $c_w(A) \subseteq c_w(B)$,

(3) $i_w(i_w(A))=i_w(A)$ *and* $c_w(c_w(A))=c_w(A)$,

(4) $i_w(X-A)=X-c_w(A)$ *and* $c_w(X-A)=X-i_w(A)$.

**Lemma 2.15.** [1] *If $w$ is a WS on $X$, then*

(1) $x \in i_w(A)$ *if and only if there is a $w$-open set $G \subseteq A$ such that* $x \in G$,

(2) $x \in c_w(A)$ *if and only if* $G \cap A \neq \emptyset$ *whenever* $x \in G \in w$,

(3) *If $A \in w$, then $A=i_w(A)$ and if $A$ is $w$-closed then $A=c_w(A)$.*

**Definition 2.16.** [1] *Let $w$ be a WS on a space $X$. Then $H \subseteq X$ is called a generalized $w$-closed set (g$w$-closed in short) if $c_w(H) \subseteq U$ whenever $H \subseteq U \in \tau$.*

The complement of a g$w$-closed set is called g$w$-open.

**Lemma 2.17.** [1] *For a WS $w$ on a space $X$, every $w$-closed set is a g$w$-closed set but not conversely.*

**Definition 2.18.** [1] *A space $X$ is called a $w$-$T_{\frac{1}{2}}$-space if for every g$w$-closed set $H$ of $X$, $c_w(H)=H$.*

**Definition 2.19.** [1] *Let $X$ be a space and $w$ be a WS on $X$. Then $(X, \tau)$ is said to be $w$-regular if for each closed set $F$ of $X$ and each $x \notin F$, there exist disjoint $w$-open sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$.*

**Definition 2.20.** [1] *Let $X$ be a space and $w$ be a WS on $X$. Then $(X, \tau)$ is said to be $w$-normal if for any two disjoint closed sets $A$ and $B$ there exist two disjoint $w$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.*

## 3. Properties of $\hat{g}w$-closed sets

In this section we introduce $\hat{g}w$-closed sets and study some of its properties.

**Definition 3.1.** *Let $w$ be a WS on a space $X$. Then $H \subseteq X$ is called a $\hat{g}w$-closed set if $c_w(H) \subseteq U$ whenever $H \subseteq U \in SO(X)$.*

*The complement of a $\hat{g}w$-closed set is called $\hat{g}w$-open.*

**Remark 3.2.**
(1) *Let $w$ be a WS on a space $X$. Then $c_w(H) \subseteq cl(H)$ for any set $H \subseteq X$.*
(2) *Let $w$ be a WS on a space $X$. Then every $\hat{g}w$-closed set reduces to $\hat{g}$-closed (resp. sg-closed, $\theta w$-closed) if one takes $w$ to be $\tau$ (resp. $SO(X)$, $\tau_\theta$).*
(3) *For a WS $w$ on a space $X$, every $w$-closed set is $\hat{g}w$-closed. In fact, if $H$ is a $w$-closed set with $H \subseteq U \in SO(X)$ then $H = c_w(H) \subseteq U$, so that $H$ is $\hat{g}w$-closed. That the converse is not true is shown by the following Example 3.3.*
(4) *For a WS $w$ on a space $X$, every $\hat{g}w$-closed set is $gw$-closed set. In fact, if $H$ is a $\hat{g}w$-closed set with $H \subseteq U \in \tau \subseteq SO(X)$ then $c_w(H) \subseteq U$, so that $H$ is $gw$-closed set. That the converse is not true is shown by the following Example 3.4.*

**Example 3.3.** *Let $X=\{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. If $w = \{\phi, \{b\}, \{a,b\}\}$, then $w$ is a WS on $X$. It is easy to check that the subset $\{b, c\}$ is $\hat{g}w$-closed but not $w$-closed.*

**Example 3.4.** *Let $X=\{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. If $w = \{\phi, \{a\}\}$, then $w$ is a WS on $X$. It is easy to check that the subset $\{a, c\}$ is $gw$-closed but not $\hat{g}w$-closed.*

**Remark 3.5.** *The Union (resp. the Intersection) of two $\hat{g}w$-closed sets is not in general $\hat{g}w$-closed.*

**Example 3.6.** *Let X={a, b, c} and $\tau$ = {$\phi$, {b}, {b, c}, X}. If w = {$\phi$, {a, b}, {a, c}}, then w is a WS on X. It is easy to check that M ={b} and N ={c} are $\hat{g}w$-closed sets and M∪N={b, c} is not a $\hat{g}w$-closed set in X.*

**Example 3.7.** *Let X={a, b, c} and $\tau$ = {$\phi$, {b}, {b, c}, X}. If w = {$\phi$, {a}, {a, c}}, then w is a WS on X. It is easy to check that S={b} and T={c} are $\hat{g}w$-closed sets and S∩T=∅ is not a $\hat{g}w$-closed set in X.*

**Theorem 3.8.** *Let w be a WS on a space X. If H is $\hat{g}w$-closed, then $c_w$(H)−H does not contain any non-empty semi-closed set.*

*Proof.* Let F be a semi-closed subset of X such that F⊆$c_w$(H)−H, where H is $\hat{g}w$-closed. Since X−F is semi-open, H⊆X−F and H is $\hat{g}w$-closed, $c_w$(H)⊆X−F and thus F⊆X−$c_w$(H). Thus F⊆(X−$c_w$(H))∩$c_w$(H)=∅ and hence F=∅.

If $c_w$(H)−H does not contain any non-empty semi-closed subset of X, then H need not be $\hat{g}w$-closed in general.

**Example 3.9.** *In Example 3.6, let H={a}. Then $c_w$(H)−H=X−{a}={b, c} does not contain any non-empty semi-closed set, but H is not a $\hat{g}w$-closed set in X.*

**Corollary 3.10.** *Let w be a WS on a space X and H⊆X be a $\hat{g}w$-closed set. Then $c_w$(H)=H if and only if $c_w$(H)−H is semi-closed.*

*Proof.* Let H be a $\hat{g}w$-closed set. If $c_w$(H)=H, then $c_w$(H)−H=∅, and $c_w$(H)−H is a semi-closed set.
Conversely, let $c_w$(H)−H be a semi-closed set, where H is $\hat{g}w$-closed. Then by Theorem 3.8, $c_w$(H)−H does not contain any non-empty semi-closed set. Since $c_w$(H)−H is a semi-closed subset of itself, $c_w$(H)−H=∅ and hence $c_w$(H)=H.

**Theorem 3.11.** *A subset H of a space X with a WS w on it is $\hat{g}w$-closed if and only if scl({x})∩H $\neq$ ∅ for every x∈$c_w$(H).*

*Proof.* Let H be a $\hat{g}w$-closed set in X and suppose if possible that there exists $x \in c_w(H)$ such that $scl(\{x\}) \cap H = \emptyset$. Therefore, $H \subseteq X - scl(\{x\})$, and so $c_w(H) \subseteq X - scl(\{x\})$. Hence $x \notin c_w(H)$, which is a contradiction.

Conversely, suppose that the condition of the theorem holds and let U be any semi-open set containing H. Let $x \in c_w(H)$. Then by hypothesis $scl(\{x\}) \cap H \neq \emptyset$, so there exists $z \in scl(\{x\}) \cap H$ and so $z \in H \subseteq U$ and U is semi-open. Thus $\{x\} \cap U \neq \emptyset$. Hence $x \in U$, which implies that $c_w(H) \subseteq U$. This shows that H is $\hat{g}w$-closed.

**Theorem 3.12.** *Let $w$ be a WS on a space X and $H \subseteq G \subseteq c_w(H)$, where H is $\hat{g}w$-closed. Then G is $\hat{g}w$-closed.*

*Proof.* Let $G \subseteq U \in SO(X)$. Since H is $\hat{g}w$-closed and $H \subseteq U$, $c_w(H) \subseteq U$. Now, $G \subseteq c_w(H)$, $c_w(G) \subseteq c_w(H)$ and hence $c_w(G) \subseteq U$.

**Theorem 3.13.** *Let X be a space and $w$ be a WS on X. Then H is $\hat{g}w$-open if and only if $F \subseteq i_w(H)$ whenever $F \subseteq H$ and F is semi-closed.*

*Proof.* Let H be a $\hat{g}w$-open set and $F \subseteq H$, where F is semi-closed. Then $X - H$ is $\hat{g}w$-closed set contained in a semi-open set $X - F$. Hence $c_w(X-H) \subseteq X - F$, that is $X - i_w(H) \subseteq X - F$. So $F \subseteq i_w(H)$.

Conversely, suppose that $F \subseteq i_w(H)$ for any semi-closed set F whenever $F \subseteq H$. Let $X - H \subseteq U$, where $U \in SO(X)$. Then $X - U \subseteq H$ and $X - U$ is semi-closed. By assumption, $X - U \subseteq i_w(H)$ and hence $c_w(X-H) = X - i_w(H) \subseteq U$. Therefore $X - H$ is $\hat{g}w$-closed and hence H is $\hat{g}w$-open.

**Theorem 3.14.** *Let $w$ be a WS on a space X. If H is semi-open and $\hat{g}w$-closed subset of X, then $c_w(H) = H$.*

*Proof.* Obvious.

**Theorem 3.15.** *Let $w$ be a WS on a space X. Then the following are equivalent:*

(1) *For every semi-open set U of X, $c_w(U) \subseteq U$.*

(2) *Every subset of X is $\hat{g}w$-closed.*

*Proof.* (1) ⇒ (2). Let H be any subset of X and H⊆U∈SO(X). Then by (1) $c_w$(U)⊆U and hence $c_w$(H)⊆$c_w$(U)⊆U. Thus H is $\hat{g}w$-closed.
(2) ⇒ (1). Let U∈SO(X). Then by (2), U is $\hat{g}w$-closed and hence $c_w$(U)⊆U.

**Theorem 3.16.** *Let w be a WS on a space X. If a subset H of X is $\hat{g}w$-open, then U=X whenever U is semi-open and $i_w$(H)∪(X−H)⊆U.*

*Proof.* Let U∈SO(X) and $i_w$(H)∪(X−H)⊆U for a $\hat{g}w$-open set H. Then X−U⊆(X−$i_w$(H))∩H. That is X−U⊆ $c_w$(X−H)−(X−H). Since X−H is $\hat{g}w$-closed, by Theorem 3.8, X−U=∅ and hence X=U.

**Theorem 3.17.** *Let w be a WS on a space X. If a subset H of X is $\hat{g}w$-open and $i_w$(H)⊆G⊆H, then G is $\hat{g}w$-open.*

*Proof.* We have X−H⊆X−G⊆X−$i_w$(H)=$c_w$(X−H). Since X−H is $\hat{g}w$-closed, it follows from Theorem 3.12 that X−G is $\hat{g}w$-closed and hence G is $\hat{g}w$-open.

Let us introduce $\hat{g}w$-$T_{\frac{1}{2}}$-space.

**Definition 3.18.** *A space X is called a $\hat{g}w$-$T_{\frac{1}{2}}$-space if for every $\hat{g}w$-closed set H of X, $c_w$(H)=H.*

**Example 3.19.** *Let X={a, b, c}, $\tau$ = {$\phi$, {a}, {a, b}, X} and w = {$\phi$, {a}}. Then $\hat{g}w$-closed sets are {b, c}, X. Therefore (X, $\tau$) is a $\hat{g}w$-$T_{\frac{1}{2}}$-space.*

**Example 3.20.** *Let X={a, b, c}, $\tau$ = {$\phi$, {a}, {a, b}, {a, c}, X} and w = {$\phi$, {a, b}, {a, c}}. Then $\hat{g}w$-closed sets are {b}, {c}, {b, c}, $\phi$, X. Therefore (X, $\tau$) is not a $\hat{g}w$-$T_{\frac{1}{2}}$-space.*

**Proposition 3.21.** *Every $\hat{g}w$-$T_{\frac{1}{2}}$-space is w-$T_{\frac{1}{2}}$-space.*

*Proof.* It follows from Remark 3.2(3).

**Theorem 3.22.** *Let w be a WS on a space X. Then the implication (1) ⇒ (2) holds. If $i_w$({x})∈w for every x∈X, then the following statements are equivalent:*

(1) *X is a $\hat{g}w$-$T_{\frac{1}{2}}$-space.*

(2) *Every singleton is either semi-closed or* $\{x\}=i_w(\{x\})$.

*Proof.* (1) ⇒ (2). Suppose {x} is not semi-closed subset for some x∈X. Then X−{x} is not semi-open and hence X is the only semi-open set containing X−{x}. Therefore X−{x} is $\hat{g}w$-closed. Since X is a $\hat{g}w$-$T_{\frac{1}{2}}$-space, $c_w(X-\{x\})=X-i_w(\{x\})=X-\{x\}$ and thus $\{x\}=i_w(\{x\})$.
(2) ⇒ (1). Let H be a $\hat{g}w$-closed subset of X and $x\in c_w(H)$. We show that x∈H. If {x} is semi-closed and x∉H, then $x\in(c_w(H)-H)$. Then {x}⊆X−H and hence H⊆X−{x}. Since H is a $\hat{g}w$-closed set and X−{x} is a semi-open subset of X, $c_w(H)\subseteq X-\{x\}$ and hence $\{x\}\subseteq X-c_w(H)$. Therefore, $\{x\}\in c_w(H)\cap(X-c_w(H))=\emptyset$. This is a contradiction. Therefore, x∈H. If $\{x\}=i_w(\{x\})$, since $x\in c_w(H)$, then for every $w$-open set U containing x, we have U∩H≠∅. But $\{x\}=i_w(\{x\})$ is $w$-open and {x}∩H≠∅. Hence x∈H. Therefore, in both cases we have x∈H. Therefore, $c_w(H)=H$ and hence X is a $\hat{g}w$-$T_{\frac{1}{2}}$-space.

## 4. $\hat{g}w$-regular spaces and $\hat{g}w$-normal spaces

**Definition 4.1.** *Let X be a space and w be a WS on X. Then (X, τ) is said to be $\hat{g}w$-regular if for each semi-closed set F of X and each x∉F, there exist disjoint w-open sets U and V such that x∈U and F⊆V.*

**Example 4.2.** *Let* $X=\{a, b, c\}$, $\tau=\{\phi, \{a\}, \{b, c\}, X\}$ *and* $w=\{\phi, \{a\}, \{b, c\}, \{b\}, X\}$. *Then semi-closed sets are* $\{a\}, \{b, c\}, \phi, X$. *Therefore (X, τ) is a $\hat{g}w$-regular.*

**Theorem 4.3.** *Let w be a WS on a space X. Consider the following statements:*

(1) *X is $\hat{g}w$-regular.*

(2) *For each x∈X and U∈SO(X) with x∈U; there exist V∈w such that* $x\in V\subseteq c_w(V)\subseteq U$.

*Then the implication (1) ⇒ (2) holds. If* $i_w(H)\in w$ *for every w-closed set H of a space X, then the following statements are equivalent.*

*Proof.* (1) ⇒ (2). Let x∉(X−U), where U∈SO(X). Then by (1) there exist disjoint G, V∈$w$ such that X−U⊆G and x∈V. Thus V⊆X−G and hence $x\in V\subseteq c_w(V)\subseteq c_w(X-G)=X-G\subseteq U$.

(2) $\Rightarrow$ (1). Let F be a semi-closed set and x$\notin$F. Then x$\in$X$-$F$\in$SO(X) and hence there exist V$\in w$ such that x$\in$V$\subseteq$$c_w$(V)$\subseteq$X$-$F. Therefore, F$\subseteq$X$-$$c_w$(V)=$i_w$(X$-$V)$\in w$.

The implication (2) $\Rightarrow$ (1) in the above theorem need not be true in general.

**Example 4.4.** *Let $X=\{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$ with a WS $w = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}\}$. Since for each $H\in w$, $c_w(H)=H$, then for each $U\in SO(X)$ with $x\in U$ there exist $V\in w$ such that $x\in V\subseteq c_w(V)\subseteq U$. But if $F=\{c, d\}$, $a\notin F$, it is clear that X is not $\hat{g}w$-regular.*

**Theorem 4.5.** *Let X be a space and w be a WS on X, and consider the following statements:*

(1) *X is $\hat{g}w$-regular.*

(2) *For each semi-closed set F and $x\notin F$, there exist $U\in w$ and a $\hat{g}w$-open set V such that $x\in U$, $F\subseteq V$ and $U\cap V=\emptyset$.*

(3) *For each $H\subseteq X$ and each semi-closed set F with $H\cap F=\emptyset$, there exist $U\in w$ and a $\hat{g}w$-open set V such that $H\cap U\neq\emptyset$, $F\subseteq V$ and $U\cap V=\emptyset$.*

*Then the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) hold. If $i_w(H)\in w$ for every $\hat{g}w$-open set H of X, then the statements are equivalent.*

*Proof.* (1) $\Rightarrow$ (2). Obvious.
(2) $\Rightarrow$ (3). Let H$\subseteq$X and F be a semi-closed set with H$\cap$F=$\emptyset$. Then for a$\in$H, a$\notin$ F, and hence by (2), there exist U$\in w$ and a $\hat{g}w$-open set V such that a$\in$U, F$\subseteq$V and U$\cap$V=$\emptyset$. Hence H$\cap$U$\neq\emptyset$, F$\subseteq$V and U$\cap$V=$\emptyset$.
(3) $\Rightarrow$ (1). Let x$\notin$F, where F is semi-closed in X. Since F$\cap$\{x\}=$\emptyset$, by (3) there exist U$\in w$ and a $\hat{g}w$-open set W such that x$\in$U, F$\subseteq$W and U$\cap$W=$\emptyset$. Then by Theorem 3.13 we have F$\subseteq$$i_w$(W)=V$\in w$ and hence U$\cap$V=$\emptyset$.

**Definition 4.6.** *Let X be a space and w be a WS on X. Then $(X, \tau)$ is said to be $\hat{g}w$-normal if for any disjoint semi-closed sets A and B there exist two disjoint w-open sets U and V such that $A\subseteq U$ and $B\subseteq V$.*

**Example 4.7.** *Let $X=\{a, b, c\}$, $\tau=\{\phi, \{b\}, \{b, c\}, X\}$ and $w=\{\phi, \{a, b\}, \{c\}, X\}$. Then semi-closed sets are $\{a\}$, $\{c\}$, $\{a, c\}$, $\phi$, X. Therefore $(X, \tau)$ is a $\hat{g}w$-normal.*

**Theorem 4.8.** *Let X be a space and $w$ be a WS on X, and consider the following statements:*

(1) *X is $\hat{g}w$-normal.*

(2) *For any pair of disjoint semi-closed sets M and N of X, there exist disjoint $\hat{g}w$-open sets P and Q of X such that M⊆P and N⊆Q.*

(3) *For each semi-closed set M and each semi-open set N containing M, there exist a $\hat{g}w$-open set P such that M⊆P⊆$c_w$(P)⊆N.*

*Then the implications (1) ⇒ (2) ⇒ (3) hold. If $i_w$(M)∈$w$ and $c_w$(M) is $w$-closed for every $\hat{g}w$-open set M of X, then the statements are equivalent.*

*Proof.* (1) ⇒ (2). Let M and N be a pair of disjoint semi-closed sets of X. Then by (1) there exist disjoint $w$-open sets P and Q of X such that M⊆P and N⊆Q. Then (2) follows from Remark 3.2(3).

(2) ⇒ (3). Let M be a semi-closed set and N be a semi-open set containing M. Then M and X−N are two disjoint semi-closed sets. Hence by (2) there exist disjoint $\hat{g}w$-open sets P and Q of X such that M⊆P and X−N⊆Q. Since Q is $\hat{g}w$-open and X−N is a semi-closed set with X−N⊆Q, by Theorem 3.13, X−N⊆$i_w$(Q). Hence $c_w$(X−Q)=X−$i_w$(Q)⊆N. Thus M⊆P⊆$c_w$(P)⊆ $c_w$(X−Q)⊆N.

(3) ⇒ (1). Let M and N be two disjoint semi-closed subsets of X. Then M is a semi-closed set and X−N is a semi-open set containing M. Thus by (3) there exists a $\hat{g}w$-open set P such that M⊆P⊆$c_w$(P) ⊆X−N. Thus by Theorem 3.13, M⊆$i_w$(P) and N⊆X−$c_w$(P), where $i_w$(P) and X−$c_w$(P)=$i_w$(X−P) are disjoint sets. Since P is $\hat{g}w$-open, $i_w$(P)∈$w$ and $i_w$(X−P)∈$w$. Hence X is $\hat{g}w$-normal.

The implication (3) ⇒ (1) in the above theorem need not be true in general.

**Example 4.9.** *Let X=$\{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, c\}, \{a, b\}, X\}$ . Let $w$ be a WS on a space X and $w = \{\phi, \{a, c\}, \{a, b\}\}$. Then it can be easily checked that (X, $\tau$) is not $\hat{g}w$-normal but for each semi-closed set M and each semi-open set N containing M, there exists a $\hat{g}w$-open set P such that M⊆P⊆$c_w$(P)⊆N.*

**Theorem 4.10.** *Let X be a space and $w$ be a WS on X and consider the following statements:*

(1) *For each g-closed set M and each open set N containing M, there exists a $w$-open set P such that cl(M)⊆P⊆$c_w$(P)⊆N.*

(2) *For each closed set M and each g-open set N containing M, there exists a $w$-open set P such that M⊆P⊆$c_w$(P)⊆int(N).*

(3) *For each g-closed set M and each open set N containing M, there exists a $\hat{g}w$-open set P such that cl(M)⊆P⊆$c_w$(P)⊆N.*

(4) *For each closed set M and each open set N containing M, there exists a $\hat{g}w$-open set P such that M⊆P⊆$c_w$(P)⊆N.*

(5) *For each closed set M and each g-open set N containing M, there exists a $\hat{g}w$-open set P such that M⊆P⊆$c_w$(P)⊆int(N).*

*Then the implications (1) ⇒ (2)⇒ (3)⇒ (4)⇒ (5) hold. If $i_w$(M)∈$w$ for every $\hat{g}w$-open set M of X, then the statements are equivalent:*

*Proof.* (1) ⇒ (2). Let M be a closed set and N be a g-open set containing M. Then by Theorem 2.10 M⊆int(N). Since M is g-closed and int(N) is open, by (1) there exists a $w$-open set such that P⊆$c_w$(P)⊆int(N).

(2) ⇒ (3) and (3) ⇒ (4) are obvious.

(4) ⇒ (5). Let M be a closed set and N be a g-open set containing M. Since N is g-open and M is closed, by Theorem 2.10 M⊆int(N). Thus by (4), there exists a $\hat{g}w$-open set P such that M⊆P⊆$c_w$(P)⊆int(N).

(5) ⇒ (1). Let M be a g-closed subset of X and N be an open set containing M. Then cl(M)⊆N, where N is g-open. Thus by (5), there exists a $\hat{g}w$-open set G such that cl(M)⊆G⊆$c_w$(G)⊆int(N)⊆ N. Since G is $\hat{g}w$-open and cl(M)⊆G, by Theorem 3.13, cl(M)⊆$i_w$(G). Put P=$i_w$(G). Then P∈$w$ and cl(M)⊆P⊆$c_w$(P)=$c_w$($i_w$(G))⊆$c_w$(G)⊆N.

## 5. Conclusion

A new class of generalized closed sets called $\hat{g}w$-closed sets in weak structure is defined and studied. Properties of $\hat{g}w$-closed sets are given. Also the new notion of $\hat{g}w$-open sets (or the complement of $\hat{g}w$-closed sets) is introduced and investigated. Properties of $\hat{g}w$-$T_{\frac{1}{2}}$-space, $\hat{g}w$-normal and $\hat{g}w$-regular are defined and studied. Properties and Characterizations of $\hat{g}w$-normal and $\hat{g}w$-regular are given.

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