Fuzzy Shortest Path with α- Cuts

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Abstract

In this paper, we present an algorithm for computing a shortest path in an acyclic network in which each edge is assigned to a non-trapezoidal fuzzy number. α-cuts are used to find the fuzzy path lengths. In a proposed algorithm, Euclidean distance is used to find the shortest path. Consequently, a shortest path is obtained from source node to destination node.

Key words - *Non-trapezoidal fuzzy number, α-cut, Euclidean distance, shortest path*

I. INTRODUCTON

The shortest path problem is one of the most fundamental and well-known combinatorial optimization problems that appear in many applications as a sub problem. The length of arcs in the network represents travelling time, cost, distance or other variables. In real life applications, these arc lengths could be uncertain and to determine the exact value of these arc lengths is very difficult or sometimes difficult for decision maker. In such a situation fuzzy shortest path problem seems to be more realistic, where the arc lengths are characterized by fuzzy numbers. Dubois and Prade [3] first introduce fuzzy shortest path problem. Okada and Soper [7] developed an algorithm based on multiple labeling approach which is useful to generate number of non-dominated paths. Applying fuzzy min concept they have introduced an order relation between fuzzy numbers. Applying extension principle Klein[5] has given an algorithm which results dominated path on a acyclic network.

The remainder of the paper is organized as follows. In section 2, basic concepts and definitions are given. It also explains ways of computing α-cuts for fuzzy numbers. In section 3, we present an algorithm for finding fuzzy shortest path in a acyclic network in which each edge is assigned to a non-trapezoidal fuzzy number. An illustration is given for the proposed algorithm in section 4.

II. CONCEPTS

A. *Fuzzy Set*

A fuzzy set A of a universal set X is defined by its membership function $\mu_A : X \to [0,1]$ which assigns a real number $\mu_A(x)$ in the interval [0,1] to each element $x \in X$, where the value of $\mu_A(x)$ at x shows the grade of membership of x in A.

B. *α-cut*

Given a fuzzy set A in X and any real number $\alpha \in [0,1]$, then the α-cut or α-level or cut worthy set of A, denoted by α_A is the crisp set

$$\alpha_A = \{x \in X / \mu_A(x) \geq \alpha\}$$

1) **Example:** Let A be a fuzzy set whose membership function is given as

$$\mu_A(x) = \begin{cases} \frac{x-a}{b-a}, & a \leq x \leq b \\ \frac{c-x}{c-b}, & b \leq x \leq c \end{cases}$$

To find the α–cut of A, we first set $\alpha \in [0,1]$ to both left and right reference function of A. That is,

$$\alpha = \frac{x-a}{b-a} \quad \text{and} \quad \alpha = \frac{c-x}{c-b}$$

Expressing x in terms of α we have $x = (b-a)\alpha + a$ and $x = c - (c-b)\alpha$ which gives the α-cuts of A as

$$\alpha_A = [(b-a)\alpha + a, \; c - (c-b)\alpha]$$

C. *Non-trapezoidal fuzzy number*

Let X be the universal set. Then non-trapezoidal fuzzy number $A(a^n, b^n, c^n, d^n)$ is defined by the membership function

$$\mu_A(x) = \begin{cases} \frac{\sqrt[n]{x}-a}{b-a}, & a^n \le x \le b^n \\ 1, & b^n \le x \le c^n \\ \frac{d-\sqrt[n]{x}}{d-c}, & c^n \le x \le d^n \end{cases}$$

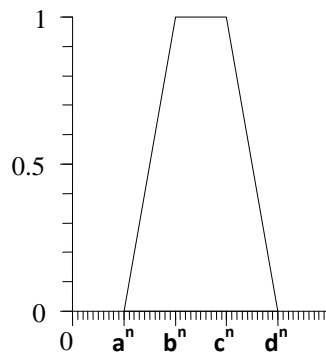The diagrammatic representation of non-trapezoidal fuzzy number is given by

Fig. 1

D. *α–cut for non-trapezoidal fuzzy number*

To find the α–cut of $A(a^n, b^n, c^n, d^n)$, we first set $\alpha \in [0,1]$ to both left and right reference function of *A*. That is, $\alpha = \frac{\sqrt[n]{x}-a}{b-a}$ and $\alpha = \frac{d-\sqrt[n]{x}}{d-c}$

Expressing x in terms of α, we have $x = \left((b-a)\alpha + a\right)^n$ and $x = (d - (d-c)\alpha)^n$ which gives the α-cuts of *A* as

$$\alpha_A = \left[\left((b-a)\alpha + a\right)^n, (d - (d-c)\alpha)^n\right]$$

E. *Addition of two fuzzy numbers*

Let $X = [a, b, c]$ and $Y = [p, q, r]$ be two fuzzy numbers whose membership functions are

$$\mu_X(x) = \begin{cases} \frac{x-a}{b-a}, & a \le x \le b \\ \frac{c-x}{c-b}, & b \le x \le c \end{cases}$$

$$\mu_Y(x) = \begin{cases} \frac{x-p}{q-p}, & p \le x \le q \\ \frac{r-x}{r-q}, & q \le x \le r \end{cases}$$

Then $\alpha_X = [(b-a)\alpha + a, c - (c-b)\alpha]$ and $\alpha_Y = [(q-p)\alpha + p, \ r - (r-q)\alpha]$ are the α-cuts of fuzzy numbers X and Y respectively. To calculate addition of fuzzy numbers X and Y using interval arithmetic

$$\alpha_X + \alpha_Y = [(b-a)\alpha + a, c - (c-b)\alpha] + [(q-p)\alpha + p, r - (r-q)\alpha]$$
$$= [a + p + (b - a + q - p)\alpha, c + r - (c - b + r - q)\alpha] \quad \ldots\ldots\ldots\ldots\ldots$$
[1]

F. *Addition of two non-trapezoidal fuzzy numbers*

Let $X = [a^n, b^n, c^n, d^n]$ and $Y = [p^n, q^n, r^n, s^n]$ be two non-trapezoidal fuzzy numbers whose membership functions are

$$\mu_X(x) = \begin{cases} \frac{\sqrt[n]{x}-a}{b-a}, & a^n \le x \le b^n \\ 1, & b^n \le x \le c^n \\ \frac{d-\sqrt[n]{x}}{d-c}, & c^n \le x \le d^n \end{cases}$$

$$\mu_Y(x) = \begin{cases} \frac{\sqrt[n]{x}-p}{q-p}, & p^n \leq x \leq q^n \\ 1, & q^n \leq x \leq r^n \\ \frac{s-\sqrt[n]{x}}{s-r}, & r^n \leq x \leq s^n \end{cases}$$

Then $\alpha_X = \left[((b-a)\alpha + a)^n, (d-(d-c)\alpha)^n\right]$ and $\alpha_Y = \left[((q-p)\alpha + p)^n, (s-(s-r)\alpha)^n\right]$ are the α-cuts of non-trapezoidal fuzzy numbers X and Y respectively. To calculate addition of non-trapezoidal fuzzy numbers *X* and *Y* using interval arithmetic

$$\alpha_X + \alpha_Y = \left[((b-a)\alpha + a)^n, (d-(d-c)\alpha)^n\right] + \left[((q-p)\alpha + p)^n, (s-(s-r)\alpha)^n\right]$$
$$= \left[[((b-a)\alpha + a)^n + ((q-p)\alpha + p)^n], [(d-(d-c)\alpha)^n + (s-(s-r)\alpha)^n]\right] \dots\dots\dots\dots [2]$$

G. Euclidean Distance

Let $A = [x_1, y_1]$ and $B = [x_2, y_2]$ be two intervals. Then the Euclidean distance $D(A, B)$ is defined as

$$D(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

H. Minimum value of α-cuts

Let $\alpha_A = [a, b]$ and $\alpha_B = [p, q]$ be two α-cuts. The minimum value of α_A and α_B is given by

$$MV = \min[\alpha_A, \alpha_B]$$
$$= [\min(a, p), \min(b, q)]$$

I. Network Terminology

Consider a directed network $G(V, E)$, consisting of a finite set of nodes $V = \{1, \dots, n\}$ and a set of m directed edges $E \subseteq V \times V$. Each edge is denoted by an ordered pair (i, j), where $i, j \in V$. Each edge is assigned to a non-trapezoidal fuzzy number. We calculate the α-cuts for each and every edge in the network by the formula $\alpha_X = \left[((b-a)\alpha + a)^n, (d-(d-c)\alpha)^n\right]$ where $X(a^n, b^n, c^n, d^n)$ is a non-trapezoidal fuzzy number and $\alpha \in [0,1]$.

III.   ALGORITHM

Step: 1 Finding possible paths

   (i)   Find all the possible paths P_i from source node to destination node in the given acyclic network.
   (ii)   Assign the number of possible paths in the given acyclic network to N.

Step: 2 Computation of length of paths

   (i)   Set α value between 0 and 1.
   (ii)   Find α-cuts for every edge.
   (iii) Find the lengths L_i of all possible paths by adding the α-cuts of the corresponding edges.

Step: 3 Comparison of paths

   (i)   Let $L_{min} = L_1$
   (ii)   For i = 2 to N
           $MV = \min[L_{min}, L_i]$
           $D_1 = D(MV, L_{min})$
           $D_2 = D(MV, L_i)$
           If $D_1 < D_2$ then $L_{min} = L_{min}$
           Otherwise $L_{min} = L_i$

Step:4 Shortest Path

       Shortest path is the corresponding path of L_{min}

IV. NUMERICAL EXAMPLE
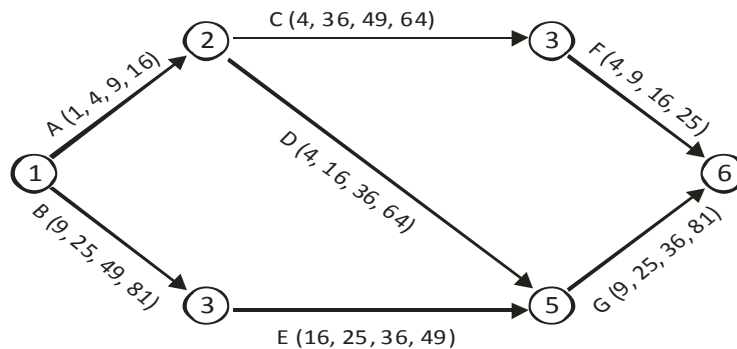
The following fuzzy network explains the proposed algorithm.

Fig. 2 Example Network

Step:1 Finding possible paths

(i) Find all the possible paths P_i from the source node to the destination node in the given acyclic network.

There are three possible paths

$P_1 : A \rightarrow C \rightarrow F$
$P_2 : B \rightarrow E \rightarrow G$
$P_3 : A \rightarrow D \rightarrow G$

(ii) Assign the number of possible paths in the given acyclic network to N.

Here the numbers of possible paths are 3. $\therefore$ N = 3

Step:2 Computation of length of paths

(i) Set α value between 0 and 1.

Let $\alpha = 0.73$

(ii) Find α-cuts for every edge

Now, $A = (1, 4, 9, 16)$
ie., $A = (1^2,\ 2^2,\ 3^2,\ 4^2)$

$\alpha_A = \left[\left((b-a)\alpha + a \right)^n, (d - (d-c)\alpha)^n \right]$

$\therefore \ \alpha_A = \left[\left((2-1)\alpha + 1 \right)^2, (4 - (4-3)\alpha)^2 \right]$

$\alpha_A = \left[(\alpha+1)^2, (4-\alpha)^2 \right] = [2.99, 10.69]$

Similarly,

$\alpha_B = \left[(2\alpha+3)^2, (9-2\alpha)^2 \right] = [19.89, 56.85]$
$\alpha_C = \left[(4\alpha+2)^2, (8-\alpha)^2 \right] = [24.21, 52.85]$
$\alpha_D = \left[(2\alpha+2)^2, (8-2\alpha)^2 \right] = [11.97, 42.77]$
$\alpha_E = \left[(\alpha+4)^2, (7-\alpha)^2 \right] = [22.37, 39.31]$
$\alpha_F = \left[(\alpha+2)^2, (5-\alpha)^2 \right] = [7.45, 18.23]$
$\alpha_G = \left[(2\alpha+3)^2, (9-3\alpha)^2 \right] = [19.89, 46.38]$

(iii) Find the lengths L_i of all possible paths by adding the α-cuts of the corresponding edges

$L_1 = [34.65, 81.77]$
$L_2 = [62.15, 142.54]$
$L_3 = [34.85, 99.84]$

Step:3 Comparison of paths

(i) Let $L_{min} = L_1$
$L_{min} = [34.65, 81.77]$

(ii) For $i = 2$ to N
$MV = \min[L_{min}, L_i]$

When $i = 2$,

$\text{MV} = \min[\text{L}_{min}, \text{L}_2]$

$\text{MV} = \min\big[[34.65, 81.77], [62.15, 142.54]\big]$

$\quad = [34.65, 81.77]$

$\text{D}_1 = \text{D(MV,L}_2)$

$\quad = \text{D}\big[[34.65, 81.77], [34.65, 81.77]\big]$

$\quad = \sqrt{(34.65 - 34.65)^2 + (81.77 - 81.77)^2}$

$\text{D}_1 = 0$

$\text{D}_2 = \text{D(MV, L}_2)$

$\quad = \text{D}\big[[34.65, 81.77], [62.15, 142.54\,]\big]$

$\quad = \sqrt{(34.65 - 62.15)^2 + (81.77 - 142.54)^2}$

$\text{D}_2 = 66.70$

If $\text{D}_1 < \text{D}_2$ then $\text{L}_{min} = \text{L}_{min}$. Otherwise $\text{L}_{min} = \text{L}_i$

Here $\text{D}_1 < \text{D}_2$.
$\quad \therefore \text{L}_{min} = \text{L}_{min}$
$\qquad\quad = [34.65, 81.77]$
When $i = 3$,
$\text{MV} = \min[\text{L}_{min}, \text{L}_3] = [34.65, 81.77]$

$\text{D}_1 = 0$
$\text{D}_2 = 18.07$
$\text{D}_1 < \text{D}_2$. Then $\text{L}_{min} = [34.65, 81.77] = \text{L}_1$

Step:4 Shortest Path

Shortest path is the corresponding path of L_{min}

$\therefore \text{P}_1$ is the shortest path. ie., $A \rightarrow C \rightarrow F$ is the shortest   path from source node to destination node.

V.   CONCLUSION

In this paper, we presented an algorithm for computing a shortest path in an acyclic network using the α-cuts in which each edge is assigned to a non-trapezoidal fuzzy number. Consequently, the shortest path is obtained from source node to destination node. But this algorithm doesn't work for a cyclic network.

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