On the Cubic Diophantine Equation with Five Unknowns $x^3 + y^3 + (x+y)(x-y)^2 = 32(z+w)p^2$

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Abstract

The cubic Diophantine equation with five unknowns represented by $x^3 + y^3 + (x+y)(x-y)^2 = 32(z+w)p^2$ *is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.*

Keywords : *Cubic Equation, Integral Solutions, Special Polygonal Numbers, Pyramidal Numbers*

I. INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, cubic equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity (Dickson,1952; Mordell,1969; Carmichael, 1959).For illustration, one may refer (Gopalan and premalatha,2009;Gopalan and Pandichelvi,2010;Gopalan and Sivagami,2010;Gopalan and premalatha, 2010; Gopalan et al., 2012)for homogeneous and non-homogeneous cubic equations with three, four and five unknowns. This paper concerns with the problem of determining non-trivial integral of the non-homogeneous cubic equation with five unknowns given by $x^3 + y^3 + (x+y)(x-y)^2 = 32(z+w)p^2$. A few relations between the solutions and the special numbers are presented.

Notations Used

- $t_{m,n}$ –Polygonal number of rank n with size m.
- gn_a – Gnomonic number of rank n.
- J_n - Jacobsthal number of rank n.
- J_n - Jacobsthal-Lucas number of rank n.
- Pr_n - Pronic number of rank n.
- SO_n - Stella Octangular number of rank n.
- PP_n - Pentagonal Pyramidal number of rank n.

II. METHOD OF ANALYSIS

The cubic Diophantine equation with five unknowns to be solved for its non-zero distinct integral solutions is given by

$$x^3 + y^3 + (x+y)(x-y)^2 = 32(z+w)p^2 \tag{1}$$

Introducing the linear transformations

$$x = u+v, y = u-v, z = u+R, w = u-R \tag{2}$$

In (1), leads to

$$u^2 + 7v^2 = 32p^2 \tag{3}$$

We present below different methods of solving (3) and thus obtain different patterns of integral solutions to (1).

Pattern-I

Assume $p = p(r,s) = r^2 + 7s^2$ (4)

Where r and s are non zero distinct integers.

Write 32 as $32 = (5+i\sqrt{7})(5-i\sqrt{7})$ (5)

Substituting (4) and (5) in (3) and employing factorization, define

$(u+i\sqrt{7}v)=(3+i\sqrt{7})(r+i\sqrt{7}s)^2$

Equating the real and imaginary parts, we have

$u=u(r,s)=5r^2-35s^2-4rs$

$v=v(r,s)=r^2-7s^2+10rs$

Hence in view of (2) and (4), the non-zero distinct integral solutions of (1) are

$x=x(r,s)=6r^2-42s^2-4rs$

$y=y(r,s)=4r^2-28s^2-24rs$

$z=z(r,s,R)=5r^2-35s^2-14rs+R$

$w=w(r,s,R)=5r^2-35s^2-14rs-R$

$p=p(r,s)=r^2+7s^2$

Properties

1. $4x(r,s)-6y(r,s)=128rs$
   - $(i)4x(r,2r^2-1)-6y(r,2r^2-1)=128SO_r$
   - $(ii)4x(2s^2+1,s)-6y(2s^2+1,s)=384OH_s$
   - $(iii)4x(2r+1,1)-6y(2r+1,1)=128gn_r$
2. $x(1,r)-z(1,r,-5)+t_{16,r}-OH_2\equiv 0(\bmod 3)$
3. $y(2s^2-1,s)-w(2s^2-1,s,-8)-14\Pr_s-CP_s^6+10SO_s\equiv 0(\bmod 14$
4. $x(r,2r^2-1)+6p(r,2r^2-1)-t_{26,r}+4SO_r\equiv 0(\bmod 11)$
5. $5x(r,r+1)-6z(r,r+1,5)-64\Pr_r+30=0$
6. $5y(r,r^2)-4w(r,r^2,R)-64CP_r^6-4R=0$

Pattern-II

Instead of (5),write 32 as $32=(-5+i\sqrt{7})(-5-i\sqrt{7})$

Following the procedure similar to pattern-I,the corresponding non-zero distinct integer solutions of (1)

$x=x(r,s)=-4r^2+28s^2-24rs$

$y=y(r,s)=-6r^2+42s^2-4rs$

$z=z(r,s,R)=-5r^2+35s^2-14rs+R$

$w=w(r,s,R)=-5r^2+35s^2-14rs-R$

$p=p(r,s)=r^2+7s^2$

Properties

1. $y(2r-1,1)+6p(2r-1,1)+4gn_r-J_7-j_5=10$
2. $5x(r,r^2)-4z(r,r^2,R)-CP_{4r}^6+4R=0$
3. $y(1,s)-42\Pr_s\equiv -6(\bmod 46)$
4. $6x(s^2,s)-4y(s^2,s)-128CP_s^6=0$
5. $w(r,2r^2-1,R)-5p(r,2r^2-1)+14SO_r+10\Pr_r=-R(\bmod 10)$

Pattern-III

Rewrite (3) as $u^2=32p^2-7v^2$ (6)

Introducing the linear transformations

$$p=X+7T,\ v=X+32T \qquad (7)$$

In (6), it leads to

$$u^2 = 25X^2 - 5600T^2 \quad (8)$$

Replacing u by 5U,we get

$$X^2 = 224T^2 + U^2 \quad (9)$$

Which is satisfied by

$$T = 2rs$$

$$U = 224r^2 - s^2$$

$$X = 224r^2 + s^2$$

In view of (7) and (8), we have

$$u = 1120r^2 - 5s^2$$

$$v = 224r^2 + s^2 + 64rs$$

$$p = 224r^2 + s^2 + 14rs$$

Substituting the value of u, v and p in (2), the corresponding non-zero integer solutions are given by

$$x = 1344r^2 - 4s^2 + 64rs$$

$$y = 896r^2 - 6s^2 - 64rs$$

$$z = 1120r^2 - 5s^2 + R$$

$$w = 1120r^2 - 5s^2 - R$$

$$p = 224r^2 + s^2 + 14rs$$

Properties

1) $x(r,1) + y(r,1) - 2240\,\Pr_r \equiv -10 \pmod{2240}$
2) $z(2s^2-1, s, R) - 5p(2s^2-1, s) + 70SO_s + 10\,\Pr_s - R \equiv 0 \pmod{10}$
3) $y(2r-1,1) - 4p(2r-1,1) + 120GnO_r = -10$
4) $x(s+1, s) - 6(s+1, s) + 10PP_s + 20\,\Pr_s \equiv 0 \pmod{10}$
5) $y(r,1) - 896\,\Pr_r \equiv -6 \pmod{960}$

Note 1

The linear transformation (7) can also be taken as

$$p = X - 7T \ , \ v = X - 32T$$

By following the procedure as in the above pattern, we get the non-zero distinct integer solutions are given by

$$x = x(r,s) = 1344r^2 - 4s^2 - 64rs$$

$$y = y(r,s) = 896r^2 - 6s^2 + 64rs$$

$$z = z(r,s) = 1120r^2 - 5s^2 + R$$

$$w = w(r,s) = 1120r^2 - 5s^2 - R$$

$$p = p(r,s) = 224r^2 + s^2 - 14rs$$

Pattern –IV

Equation (9) can be written in the form

$$X^2 - U^2 = 224T^2$$

Define $(X+U)(X-U) = 224T^2$ (10)

Now consider
$$X + U = T^2$$
$$X - U = 224$$

Solving the above two equations, we obtain

$$X = \frac{T^2 + 224}{2}$$

$$U = \frac{T^2 - 224}{2}$$

Since our interest is on finding integer solutions, it is noted that the values of X and U are integers when T is even.

In other words, choosing T=2k and proceeding as in pattern-III the corresponding non-zero integer solutions are

$$x = x(k) = 12k^2 + 64k - 448$$

$$y = y(k) = 8k^2 - 64k - 672$$

$$z = z(k) = 10k^2 - 560 + R$$

$$w = w(k) = 10k^2 - 560 - R$$

$$p = p(k) = 2k^2 + 14k + 112$$

Properties

1) $x(k) + y(k) - 20\,\mathrm{Pr}_k \equiv -1120 \pmod{20}$

2) $y(k) + p(k) - 10PP_k \equiv -560 \pmod{60}$

3) *Each of the following expressions represents the nasty number*

 a) $x(1) + y(1) + p(1) + 996$

 b) $x(1) + 3p(1)$

 c) $x(1) - y(1) - 380$

4) *Each of the following expression represents a perfect square*

 a) $4y(1) - 8x(1)$

 b) $-[z(k,R) + w(k,R)] + 20t_{4,k} - 96$

Note 2

The system (10) can also be written as

$$X + U = 2T^2$$

$$X - U = 112$$

Following the procedure similar to pattern-IV, the corresponding integral solutions are obtained to be

$$x = x(T) = 6T^2 + 32T - 224$$

$$y = y(T) = 4T^2 - 32T - 336$$

$$z = z(T) = 5T^2 - 280 + R$$

$$w = w(T) = 5T^2 - 280 - R$$

$$p = p(T) = T^2 + 7T - 224$$

Note 3

Rewrite (10),

$$X + U = 2T^2$$

$$X - U = 112$$

By repeating process as in pattern-IV, the non-zero distinct integer solutions are found to be

$$x = x(T) = 12T^2 + 32T - 112$$
$$y = y(T) = 8T^2 - 32T - 168$$
$$z = z(T) = 10T^2 - 140 + R$$
$$w = w(T) = 10T^2 - 140 - R$$
$$p = p(T) = 2T^2 + 7T + 28$$

Note 4

Rewrite (10),

$$X + U = 8T^2$$
$$X - U = 28$$

By repeating process as in pattern-IV, the non-zero distinct integer solutions are found to be

$$x = x(T) = 24T^2 + 32T - 56$$
$$y = y(T) = 16T^2 - 32T - 84$$
$$z = z(T) = 20T^2 - 70 + R$$
$$w = w(T) = 20T^2 - 70 - R$$
$$p = p(T) = 4T^2 + 7T + 14$$

III. CONCLUSION

To conclude, one may search for other patterns of integral solutions of (1).

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