$\mathcal{I}_{wgp}$-NORMAL AND $\mathcal{I}_{wgp}$-REGULAR SPACES

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Abstract. $\mathcal{I}_{wgp}$-normal and $\mathcal{I}_{wgp}$-regular spaces are introduced and various characterizations and properties are given. Characterizations of normal, mildly normal, wgp-normal and regular spaces are also given.

1. Introduction and Preliminaries

Throughout this paper, by a space X, we always mean a topological space (X,τ) with no separation properties assumed. Let H be a subset of X. We denote the interior, the closure and the complement of a subset H by int(H), cl(H) and X\H or H^c, respectively.

An ideal $\mathcal{I}$ on a space X is a non-empty collection of subsets of X which satisfies (i)P$\in\mathcal{I}$ and Q$\subseteq$P$\Rightarrow$Q$\in\mathcal{I}$ and (ii) P$\in\mathcal{I}$ and Q$\in\mathcal{I}\Rightarrow$P$\cup$Q$\in\mathcal{I}$. Given a space X with an ideal $\mathcal{I}$ on X and if $\wp$(X) is the set of all subsets of X, a set operator $(.)^*$: $\wp(X)\rightarrow\wp(X)$, called a local function [7] of H with respect to τ and $\mathcal{I}$ is defined as follows: for H$\subseteq$X,

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$H^*(\mathcal{I},\tau)=\{x\in X|U\cap H\notin\mathcal{I}$ for every $U\in\tau(x)\}$ where $\tau(x)=\{U\in\tau|x\in U\}$. We will make use of the basic facts about the local functions [[6], Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(\mathcal{I},\tau)$, called the $\star$-topology, finer than τ is defined by $cl^*(H)=H\cup H^*(\mathcal{I},\tau)$ [20]. When there is no chance for confusion, we will simply write H^* for $H^*(\mathcal{I},\tau)$ and τ^* for $\tau^*(\mathcal{I},\tau)$. If $\mathcal{I}$ is an ideal on X, then $(X,\tau,\mathcal{I})$ is called an ideal space. $\mathcal{N}$ is the ideal of all nowhere dense subsets in (X,τ). A subset H of an ideal space $(X,\tau,\mathcal{I})$ is called $\star$-closed [6] (resp. $\star$-dense in itself [5]) if $H^*\subseteq H$ (resp. $H\subseteq H^*$). The complement of a $\star$-closed set is called $\star$-open. A subset H of an ideal space $(X,\tau,\mathcal{I})$ is called $\mathcal{I}_g$-closed [3] if $H^*\subseteq U$ whenever $H\subseteq U$ and U is open.

$int^*(H)$ will denote the interior of H in (X,τ^*). A subset A of a space (X,τ) is said to be regular open [19] if $A=int(cl(A))$ and A is said to be regular closed [19] if $A=cl(int(A))$. A subset H of a space X is called an α-open [13] (resp. preopen [10]) set if $H\subseteq int(cl(int(H)))$ (resp. $H\subseteq int(cl(H))$). The family of all α-open sets in (X,τ), denoted by τ^α, is a topology on X finer than τ. The complement of an α-open set is called α-closed. The interior of a subset H in (X,τ^α) is denoted by $int_\alpha(H)$. The closure of a subset H in (X,τ^α) is denoted by $cl_\alpha(H)$. A subset H of a space X is said to be g-closed [8] if $cl(H)\subseteq U$ whenever $H\subseteq U$ and U is open. H is said to be g-open if $X-H$ is g-closed.

An ideal $\mathcal{I}$ is said to be codense [4] or τ-boundary [12] if $\tau\cap\mathcal{I}=\{\emptyset\}$. $\mathcal{I}$ is said to be completely codense [4] if $PO(X)\cap\mathcal{I}=\{\emptyset\}$, where PO(X) is the family of all preopen sets in (X,τ). Every completely codense ideal is codense but not conversely [4]. The following Lemmas will be useful in the sequel.

Lemma 1.1. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If $\mathcal{I}$ is completely codense, then $\tau^*\subseteq\tau^\alpha$ [[16], Theorem 6].*

Let us say that $w\subseteq P$ is a weak structure (briefly WS) on X iff $\emptyset\in w$. Clearly each generalized topology and each minimal structure is a WS [2].

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Each member of w is said to be w-open and the complement of a w-open set is called w-closed.

Let w be a weak structure on X and H⊆X. We define (as in the general case) i_w(H) is the union of all w-open subsets contained in H and c_w(H) is the intersection of all w-closed sets containing H [2].

Let w be a WS on a space X and H⊆X. Then H∈$\pi(w)$ if H⊆$i_w(c_w$(H)) [2].

Let w be a WS on a space X. Then H⊆X is said to be wgp-closed [17] if cl(H)⊆U whenever H⊆U and U∈$\pi(w)$. A subset H of a space X is said to be rαg-closed [14] if cl_α(H)⊆U whenever H⊆U and U is regular open. H is said to be wgp-open (resp. rαg-open) if X−H is wgp-closed (resp. rαg-closed).

Let w be a WS on an ideal space (X,τ,$\mathcal{I}$). Then H⊆X is called $\mathcal{I}_{wgp}$-closed if H*⊆U whenever H⊆U and U∈$\pi(w)$ [17] . In [17], every ∗-closed and hence every closed set is $\mathcal{I}_{wgp}$-closed. A subset H of an ideal space (X,τ,$\mathcal{I}$) is said to be $\mathcal{I}_{wgp}$-open [17] if X−H is $\mathcal{I}_{wgp}$-closed. In this paper, we define $\mathcal{I}_{wgp}$-normal, $_{wgp}\mathcal{I}$-normal and $\mathcal{I}_{wgp}$-regular spaces using $\mathcal{I}_{wgp}$-open sets and give characterizations and properties of such spaces. Also, characterizations of normal, mildly normal, wgp-normal and regular spaces are given.

Remark 1.2. [1] *If w is a WS on X, then $i_w(\emptyset)=\emptyset$ and $c_w(X)=X$.*

Theorem 1.3. [2] *If w is a WS on X and A,B ∈ w then*

(1) $i_w(A)\subseteq A\subseteq c_w(A)$,
(2) $A\subseteq B\Rightarrow i_w(A)\subseteq i_w(B)$ *and* $c_w(A)\subseteq c_w(B)$,
(3) $i_w(i_w(A))=i_w(A)$ *and* $c_w(c_w(A))=c_w(A)$,
(4) $i_w(X-A)=X-c_w(A)$ *and* $c_w(X-A)=X-i_w(A)$.

Lemma 1.4. [1] *If w is a WS on X, then*

(1) $x\in i_w(A)$ *if and only if there is a w-open set* $G\subseteq A$ *such that* $x\in G$,
(2) $x\in c_w(A)$ *if and only if* $G\cap A\neq\emptyset$ *whenever* $x\in G\in w$,
(3) *If* $A\in w$, *then* $A=i_w(A)$ *and if A is w-closed then* $A=c_w(A)$.

Lemma 1.5 ([17], Theorem 2.47)**.** *Let $(X,\tau,\mathcal{I})$ be an ideal space where $\tau\subseteq w$ and $\mathcal{I}$ is completely codense. Then the following are equivalent.*

(1) *X is normal.*

(2) *For disjoint closed sets M and N, there exist disjoint $\mathcal{I}_{wgp}$-open sets P and Q such that $M\subseteq P$, $N\subseteq Q$.*

(3) *For a closed set M and an open set Q containing M, there exists an $\mathcal{I}_{wgp}$-open set P such that $M\subseteq P\subseteq cl^*(P)\subseteq Q$.*

Definition 1.6. [17] *A topological space (X,τ) is said to have the property C if $P^c\in\tau$ and $Q^c\in\pi(w)$ then $(P\cap Q)^c\in\pi(w)$.*

Lemma 1.7 ([17], Theorem 2.15)**.** *If $(X,\tau,\mathcal{I})$ is an ideal space with the property C and $H\subseteq X$, then the following are equivalent.*

(1) *H is $\mathcal{I}_{wgp}$-closed.*

(2) *$cl^*(H)\subseteq U$ whenever $H\subseteq U$ and $U\in\pi(w)$.*

Lemma 1.8 ([17], Theorem 2.42)**.** *Let $(X,\tau,\mathcal{I})$ be an ideal space with the property C and $H\subseteq X$. Then H is $\mathcal{I}_{wgp}$-open if and only if $F\subseteq int^*(H)$ whenever $F^c\in\pi(w)$ and $F\subseteq H$.*

Lemma 1.9 ([17], Theorem 2.46)**.** *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then every subset of X is $\mathcal{I}_{wgp}$-closed if and only if every subset of $\pi(w)$ is $*$-closed.*

Proposition 1.10. [17] *If $H\in\tau$ then $H\in\pi(w)$.*

Definition 1.11. [17] *Let w be a WS on a space X. A subset H of a space X is said to be $w\alpha gp$-closed if $cl_\alpha(H)\subseteq U$ whenever $H\subseteq U$ and $U\in\pi(w)$.*
The complement of an $w\alpha gp$-closed set is called $w\alpha gp$-open.

Lemma 1.12. *If $(X,\tau,\mathcal{I})$ is an ideal space and $H\subseteq X$. If $\mathcal{I}=\{\emptyset\}$, then H is $\mathcal{I}_{wgp}$-closed if and only if H is wgp-closed [[17], Corollary 2.28].*

2. $\mathcal{I}_{wgp}$-normal spaces

Let w be a WS on an ideal space (X,τ,$\mathcal{I}$). Then (X,τ,$\mathcal{I}$) is said to be an $\mathcal{I}_{wgp}$-normal space if for every pair of disjoint closed sets M and N, there exist disjoint $\mathcal{I}_{wgp}$-open sets P and Q such that M⊆P and N⊆Q. Since every open set is an $\mathcal{I}_{wgp}$-open set, every normal space is $\mathcal{I}_{wgp}$-normal. The following Example 2.1 shows that an $\mathcal{I}_{wgp}$-normal space is not necessarily a normal space. Theorem 2.2 below gives characterizations of $\mathcal{I}_{wgp}$-normal spaces. Theorem 2.3 below shows that the two concepts coincide for completely codense ideal spaces.

Example 2.1. *Let X={a, b, c}, τ={∅, {b}, {a, b}, {b, c}, X}, w={∅, {c}, X} and $\mathcal{I}$={∅, {b}}. Then $\pi(w)$={∅, {c}, {a, c}, {b, c}, X} and ⋆-closed sets=P(X), the power set of X. Here every subset of $\pi(w)$ is ⋆-closed and so, by Lemma 1.9, every subset of X is $\mathcal{I}_{wgp}$-closed and hence every subset of X is $\mathcal{I}_{wgp}$-open. This implies that (X,τ,$\mathcal{I}$) is $\mathcal{I}_{wgp}$-normal. Now {a} and {c} are disjoint closed subsets of X which are not separated by disjoint open sets and so (X,τ) is not normal.*

Theorem 2.2. *Let w be a WS on an ideal space (X,τ,$\mathcal{I}$) where τ⊆w. Then the following are equivalent.*

(1) *X is $\mathcal{I}_{wgp}$-normal.*
(2) *For every pair of disjoint closed sets M and N, there exist disjoint $\mathcal{I}_{wgp}$-open sets P and Q such that M⊆P and N⊆Q.*
(3) *For every closed set M and an open set Q containing M, there exists an $\mathcal{I}_{wgp}$-open set P such that M⊆P⊆cl*(P)⊆Q.*

Proof. (1)⇒(2). The proof follows from the definition of $\mathcal{I}_{wgp}$-normal spaces.

(2)⇒(3). Let M be a closed set and Q be an open set containing M. Since M and X−Q are disjoint closed sets, there exist disjoint $\mathcal{I}_{wgp}$-open sets P and R such that M⊆P and X−Q⊆R. Again, P∩R=∅ implies that P∩int*(R)=∅. Also P⊆X−R⇒cl*(P)⊆cl*(X−R)=X−int*(R). Since Q∈$\pi(w)$ and R is $\mathcal{I}_{wgp}$-open, X−Q ⊆R implies that X−Q⊆int*(R) and so X−int*(R)⊆Q. Thus, we have M⊆P⊆cl*(P) ⊆X−int*(R)⊆Q which proves (3).

(3)⇒(1). Let M and N be two disjoint closed subsets of X. By hypothesis, there exists an $\mathcal{I}_{wgp}$-open set P such that M⊆P⊆cl*(P)⊆X−N. If R=X−cl*(P), then P and R are the required disjoint $\mathcal{I}_{wgp}$-open sets containing M and N respectively. So, (X,τ,$\mathcal{I}$) is $\mathcal{I}_{wgp}$-normal.

Theorem 2.3. *Let (X,τ,$\mathcal{I}$) be an ideal space where $\tau\subseteq w$ and $\mathcal{I}$ is completely codense and w a WS on (X,τ,$\mathcal{I}$). If (X,τ,$\mathcal{I}$) is $\mathcal{I}_{wgp}$-normal, then it is a normal space.*

Proof. Suppose that $\mathcal{I}$ is completely codense. By Theorem 2.2, (X,τ,$\mathcal{I}$) is $\mathcal{I}_{wgp}$-normal if and only if for each pair of disjoint closed sets M and N, there exist disjoint $\mathcal{I}_{wgp}$-open sets P and Q such that M⊆P and N⊆Q if and only if X is normal, by Lemma 1.5.

Theorem 2.4. *Let w be a WS on an ideal space (X,τ,$\mathcal{I}$) where $\tau\subseteq w$ and X is $\mathcal{I}_{wgp}$-normal space. If G is closed and H is a wgp-closed set such that H∩G=∅, then there exist disjoint $\mathcal{I}_{wgp}$-open sets P and Q such that H⊆P and G⊆Q.*

Proof. Since H∩G=∅, H⊆X−G where X−G$\in\tau\subseteq\pi(w)$. Therefore, by hypothesis, cl(H)⊆ X−G. Since cl(H)∩G=∅ and X is $\mathcal{I}_{wgp}$-normal, there exist disjoint $\mathcal{I}_{wgp}$-open sets P and Q such that H⊆cl(H)⊆P and G⊆Q.

Proposition 2.5. *For a WS w on a space X, every closed subset is $w\alpha$gp-closed.*

Proof. Let H be a closed set such that H⊆U and U$\in\pi(w)$. Then cl(H)=H and cl_α(H)⊆cl(H)=H⊆U. Hence H is $w\alpha$gp-closed. □

Example 2.6. *Let $X=\{a, b, c\}$, $\tau=\{\emptyset, \{c\}, X\}$ and $w=\{\emptyset, \{a\}, \{a, b\}, X\}$. Then the $w\alpha$gp-closed sets are $\emptyset$, $\{a\}$, $\{b\}$, $\{a, b\}$, $\{b, c\}$, X and the closed sets are $\emptyset$, X, $\{a, b\}$. It is clear that $\{b\}$ is $w\alpha$gp-closed set but it is not closed.*

The following Corollaries 2.7 and 2.8 give properties of normal spaces. If $\mathcal{I}=\{\emptyset\}$ in Theorem 2.4, then we have the following Corollary 2.7, the proof of which follows from Theorem 2.3 and Lemma 1.12, since $\{\emptyset\}$ is a completel codense ideal. If $\mathcal{I}=\mathcal{N}$ in

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Theorem 2.4, then we have the Corollary 2.8 below, since $\tau^*(\mathcal{N})=\tau^\alpha$ and $\mathcal{I}_{wgp}$-open sets coincide with $w\alpha$gp-open sets.

Corollary 2.7. *Let (X,τ) be a normal space and w a WS on X such that $\tau\subseteq w$. If G is a closed set and H is a wgp-closed set disjoint from G, then there exist disjoint wgp-open sets P and Q such that $H\subseteq P$ and $G\subseteq Q$.*

Corollary 2.8. *Let $(X,\tau,\mathcal{I})$ be a normal ideal space where $\mathcal{I}=\mathcal{N}$ and w a WS on $(X,\tau,\mathcal{I})$ such that $\tau\subseteq w$. If G is a closed set and H is a wgp-closed set disjoint from G, then there exist disjoint $w\alpha$gp-open sets P and Q such that $H\subseteq P$ and $G\subseteq Q$.*

Lemma 2.9. *If w is a WS on an ideal space $(X,\tau,\mathcal{I})$ and $H\subseteq X$, then the following hold. If $\mathcal{I}=\mathcal{N}$, then H is $\mathcal{I}_{wgp}$-closed if and only if H is $w\alpha$gp-closed.*

Proof. It follows from cl*(H)=cl_α(H) for any subset H of X. □

Theorem 2.10. *Let w be a WS on an ideal space $(X,\tau,\mathcal{I})$ where $\tau\subseteq w$ and X is $\mathcal{I}_{wgp}$-normal. Then the following hold.*

(1) *For every closed set M and every wgp-open set N containing M, there exists an $\mathcal{I}_{wgp}$-open set P such that M⊆int*(P)⊆P⊆N.*
(2) *For every wgp-closed set M and every open set N containing M, there exists an $\mathcal{I}_{wgp}$-closed set P such that M⊆P⊆cl*(P)⊆N.*

Proof. (1) Let M be a closed set and N be a *wgp*-open set containing M. Then M∩(X−N)=∅, where M is closed and X−N is *wgp*-closed. By Theorem 2.4, there exist disjoint $\mathcal{I}_{wgp}$-open sets P and Q such that M⊆P and X−N⊆Q. Since P∩Q=∅, we have P⊆X−Q. By Lemma 1.8, M⊆int*(P). Therefore, M⊆int*(P)⊆P⊆X−Q⊆N. This proves (1).

(2) Let M be a *wgp*-closed set and N be an open set containing M. Then X−N is a closed set contained in the *wgp*-open set X−M. By (1), there exists an $\mathcal{I}_{wgp}$-open set Q such that X−N⊆int*(Q)⊆Q⊆X−M. Therefore, M⊆X−Q⊆cl*(X−Q)⊆N. If P=X−Q, then M⊆P⊆cl*(P)⊆N and so P is the required $\mathcal{I}_{wgp}$-closed set.

The following Corollaries 2.11 and 2.12 give some properties of normal spaces. If $\mathcal{I}$=$\{\emptyset\}$ in Theorem 2.10, then we have the following Corollary 2.11. If $\mathcal{I}$=$\mathcal{N}$ in Theorem 2.10, then we have the Corollary 2.12 below.

Corollary 2.11. *Let (X,τ) be a normal space and w a WS on X such that $\tau \subseteq w$. Then the following hold.*

(1) *For every closed set M and every wgp-open set N containing M, there exists a wgp-open set P such that $M \subseteq int(P) \subseteq P \subseteq N$.*
(2) *For every wgp-closed set M and every open set N containing M, there exists a wgp-closed set P such that $M \subseteq P \subseteq cl(P) \subseteq N$.*

Corollary 2.12. *Let (X,τ) be a normal space and w a WS on X such that $\tau \subseteq w$. Then the following hold.*

(1) *For every closed set M and every wgp-open set N containing M, there exists an $w\alpha gp$-open set P such that $M \subseteq int_\alpha(P) \subseteq P \subseteq N$.*
(2) *For every wgp-closed set M and every open set N containing M, there exists an $w\alpha gp$-closed set P such that $M \subseteq P \subseteq cl_\alpha(P) \subseteq N$.*

Let w be a WS on an ideal space (X,τ,$\mathcal{I}$). Then (X,τ,$\mathcal{I}$) is said to be ${}_{wgp}\mathcal{I}$-normal if for each pair of disjoint $\mathcal{I}_{wgp}$-closed sets M and N, there exist disjoint open sets P and Q in X such that M⊆P and N⊆Q. Since every closed set is $\mathcal{I}_{wgp}$-closed, every ${}_{wgp}\mathcal{I}$-normal space is normal. But a normal space need not be ${}_{wgp}\mathcal{I}$-normal as the following Example 2.13 shows. Theorems 2.14 and 2.17 below give characterizations of ${}_{wgp}\mathcal{I}$-normal spaces.

Example 2.13. *Let $X=\{a, b, c\}$, $\tau=\{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $w=\{\emptyset, \{c\}, \{a, c\}, X\}$ and $\mathcal{I}=\{\emptyset, \{a\}\}$. Every subset of $\pi(w)$ is $*$-closed and so every subset of X is $\mathcal{I}_{wgp}$-closed. Now $A=\{a, b\}$ and $B=\{c\}$ are disjoint $\mathcal{I}_{wgp}$-closed sets, but they are not separated by disjoint open sets. So $(X,\tau,\mathcal{I})$ is not ${}_{wgp}\mathcal{I}$-normal. But $(X,\tau,\mathcal{I})$ is normal.*

Theorem 2.14. *Let w be a WS on an ideal space (X,τ,$\mathcal{I}$). Then the following are equivalent.*

(1) *X is ${}_{wgp}\mathcal{I}$-normal.*

(2) *For every $\mathcal{I}_{wgp}$-closed set M and every $\mathcal{I}_{wgp}$-open set N containing M, there exists an open set P of X such that M⊆P⊆cl(P)⊆N.*

Proof. (1)⇒(2). Let M be an $\mathcal{I}_{wgp}$-closed set and N be an $\mathcal{I}_{wgp}$-open set containing M. Since M and X−N are disjoint $\mathcal{I}_{wgp}$-closed sets, there exist disjoint open sets P and Q such that M⊆P and X−N⊆Q. Now P∩Q=∅ implies that cl(P)⊆X−Q. Therefore, M⊆P⊆cl(P)⊆X−Q⊆N. This proves (2).

(2)⇒(1). Suppose M and N are disjoint $\mathcal{I}_{wgp}$-closed sets, then the $\mathcal{I}_{wgp}$-closed set M is contained in the $\mathcal{I}_{wgp}$-open set X−N. By hypothesis, there exists an open set P of X such that M⊆P⊆cl(P)⊆X−N. If Q=X−cl(P), then P and Q are disjoint open sets containing M and N respectively. Therefore, (X,τ,$\mathcal{I}$) is ${}_{wgp}\mathcal{I}$-normal.

Definition 2.15. *A space X is said to be wgp-normal if for every disjoint wgp-closed sets A and B, there exist disjoint open sets U and V such that A⊆U and B⊆V.*

If $\mathcal{I}$={∅}, then ${}_{wgp}\mathcal{I}$-normal spaces coincide with *wgp*-normal spaces and so if we take $\mathcal{I}$={∅}, in Theorem 2.14, then we have the following characterization for *wgp*-normal spaces.

Corollary 2.16. *Let w be a WS on a space X. Then the following are equivalent.*

(1) *X is wgp-normal.*

(2) *For every wgp-closed set M and every wgp-open set N containing M, there exists an open set P of X such that M⊆P⊆cl(P)⊆N.*

Theorem 2.17. *Let w be a WS on an ideal space (X,τ,$\mathcal{I}$). Then the following are equivalent.*

(1) *X is ${}_{wgp}\mathcal{I}$-normal.*

(2) *For each pair of disjoint $\mathcal{I}_{wgp}$-closed subsets M and N of X, there exists an open set P of X containing M such that cl(P)∩N=∅.*

(3) *For each pair of disjoint $\mathcal{I}_{wgp}$-closed subsets M and N of X, there exist an open set P containing M and an open set Q containing N such that cl(P)∩cl(Q)=∅.*

Proof. (1)⇒(2). Suppose that M and N are disjoint $\mathcal{I}_{wgp}$-closed subsets of X. Then the $\mathcal{I}_{wgp}$-closed set M is contained in the $\mathcal{I}_{wgp}$-open set X−N. By Theorem 2.14, there exists an open set P of X such that M⊆P⊆cl(P)⊆X−N. Therefore, P is the required open set containing M such that cl(P)∩N=∅.

(2)⇒(3). Let M and N be two disjoint $\mathcal{I}_{wgp}$-closed subsets of X. By hypothesis, there exists an open set P of X containing M such that cl(P)∩N=∅. Also, cl(P) and N are disjoint $\mathcal{I}_{wgp}$-closed sets of X. By hypothesis, there exists an open set Q of X containing N such that cl(P)∩cl(Q)=∅.

(3)⇒(1). The proof is clear.

If $\mathcal{I}$={∅}, in Theorem 2.17, then we have the following characterizations for *wgp*-normal spaces.

Corollary 2.18. *Let (X,τ) be a space and w a WS on X. Then the following are equivalent.*

(1) *X is wgp-normal.*
(2) *For each pair of disjoint wgp-closed subsets M and N of X, there exists an open set P of X containing M such that cl(P)∩N=∅.*
(3) *For each pair of disjoint wgp-closed subsets M and N of X, there exist an open set P containing M and an open set Q containing N such that cl(P)∩cl(Q)=∅.*

Theorem 2.19. *Let w be a WS on a $_{wgp}\mathcal{I}$-normal ideal space (X,τ,$\mathcal{I}$) with the property C. If M and N are disjoint $\mathcal{I}_{wgp}$-closed subsets of X, then there exist disjoint open sets P and Q such that cl*(M)⊆P and cl*(N)⊆Q.*

Proof. Suppose that M and N are disjoint $\mathcal{I}_{wgp}$-closed sets. By Theorem 2.17(3), there exist an open set P containing M and an open set Q containing N such that cl(P)∩cl(Q)=∅. Since M is $\mathcal{I}_{wgp}$-closed, M⊆P implies that cl*(M)⊆P. Similarly cl*(N)⊆Q.

If $\mathcal{I}$={∅}, in Theorem 2.19, then we have the following property of disjoint *wgp*-closed sets in *wgp*-normal spaces.

Corollary 2.20. *Let w be a WS on a wgp-normal space X with the property C. If M and N are disjoint wgp-closed subsets of X, then there exist disjoint open sets P and N such that cl(M)⊆P and cl(N)⊆Q.*

Theorem 2.21. *Let w be a WS on an ideal $_{wgp}\mathcal{I}$-normal space (X,τ,$\mathcal{I}$) with the property C. If M is an $\mathcal{I}_{wgp}$-closed set and N is an $\mathcal{I}_{wgp}$-open set containing M, then there exists an open set P such that M⊆cl*(M)⊆P⊆int*(N)⊆N.*

Proof. Suppose M is an $\mathcal{I}_{wgp}$-closed set and N is an $\mathcal{I}_{wgp}$-open set containing M. Since M and X−N are disjoint $\mathcal{I}_{wgp}$-closed sets, by Theorem 2.19, there exist disjoint open sets P and Q such that cl*(M)⊆P and cl*(X−N)⊆Q. Now, X−int*(N)=cl*(X−N)⊆Q implies that X−Q⊆int*(N). Again, P∩Q=∅ implies P⊆X−Q and so M⊆cl*(M)⊆P ⊆X−Q⊆int*(N)⊆N.

If $\mathcal{I}$={∅}, in Theorem 2.21, then we have the following Corollary 2.22.

Corollary 2.22. *Let w be a WS on a wgp-normal space X with the property C. If M is a wgp-closed set and N is a wgp-open set containing M, then there exists an open set P such that M⊆cl(M)⊆P⊆int(N)⊆N.*

The following Theorem 2.23 gives a characterization of normal spaces in terms of *wgp*-open sets which follows from Lemma 1.5 if $\mathcal{I}$={∅}.

Theorem 2.23. *Let (X,τ) be a space and w a WS on X such that τ⊆w. Then the following are equivalent.*

(1) *X is normal.*

(2) *For any disjoint closed sets M and N, there exist disjoint wgp-open sets P and Q such that M⊆P and N⊆Q.*

(3) *For any closed set M and an open set Q containing M, there exists a wgp-open set P such that M⊆P⊆cl(P)⊆Q.*

The rest of the section is devoted to the study of mildly normal spaces in terms of $\mathcal{I}_{wgp}$-open sets, $\mathcal{I}_g$-open sets and $\mathcal{I}_{rg}$-open sets. A space (X,τ) is said to be a mildly normal space [18] if disjoint regular closed sets are separated by disjoint open sets. A subset H of a space (X,τ) is said to be αg-closed [9] if cl_α(H)⊆U whenever H⊆U and U is open. A subset H of a space (X,τ) is said to rg-closed [15] if cl(H)⊆U whenever H⊆U and U is regular open in X. The complements of the above closed sets are called their respective open sets.

A subset H of an ideal space (X,τ,$\mathcal{I}$) is said to be a regular generalized closed set with respect to an ideal $\mathcal{I}$ ($\mathcal{I}_{rg}$-closed) [11] if H*⊆U whenever H⊆U and U is regular open. H is called $\mathcal{I}_g$-open (resp. $\mathcal{I}_{rg}$-open) if X−H is $\mathcal{I}_g$-closed (resp. $\mathcal{I}_{rg}$-closed). Clearly, every $\mathcal{I}_{wgp}$-closed set is $\mathcal{I}_g$-closed if $\tau \subseteq w$ and every $\mathcal{I}_g$-closed set is $\mathcal{I}_{rg}$-closed but the separate converses are not true. Theorem 2.25 below gives characterizations of mildly normal spaces. Corollary 2.26 below gives characterizations of mildly normal spaces in terms of $w\alpha$gp-open, αg-open and rαg-open sets. Corollary 2.26 below gives characterizations of mildly normal spaces in terms of wgp-open, g-open and rg-open sets. The following Lemma 2.24 is essential to prove Theorem 2.25.

Lemma 2.24. [11] *Let (X,τ,$\mathcal{I}$) be an ideal space. A subset H⊆X is $\mathcal{I}_{rg}$-open if and only if F⊆int*(H) whenever F is regular closed and F⊆H.*

Theorem 2.25. *Let w be a WS on an ideal space (X,τ,$\mathcal{I}$) where $\tau \subseteq w$ and $\mathcal{I}$ is completely codense. Then the following are equivalent.*

(1) *X is mildly normal.*

(2) *For disjoint regular closed sets M and N, there exist disjoint $\mathcal{I}_{wgp}$-open sets P and Q such that M⊆P and N⊆Q.*

(3) *For disjoint regular closed sets M and N, there exist disjoint $\mathcal{I}_g$-open sets P and Q such that M⊆P and N⊆Q.*

(4) *For disjoint regular closed sets M and N, there exist disjoint $\mathcal{I}_{rg}$-open sets P and Q such that M⊆P and N⊆Q.*

(5) *For a regular closed set M and a regular open set N containing M, there exists an $\mathcal{I}_{rg}$-open set P of X such that M⊆P⊆cl*(P)⊆N.*

(6) *For a regular closed set M and a regular open set N containing M, there exists an ∗-open set S of X such that M⊆S⊆cl*(S)⊆N.*

(7) *For disjoint regular closed sets M and N, there exist disjoint ∗-open sets P and Q such that M⊆P and N⊆Q.*

Proof. (1)⇒(2). Suppose that M and N are disjoint regular closed sets. Since X is mildly normal, there exist disjoint open sets P and Q such that M⊆P and N⊆Q. But every open set is an $\mathcal{I}_{wgp}$-open set. This proves (2).

(2)⇒(3). The proof follows from the fact that every $\mathcal{I}_{wgp}$-open set is an $\mathcal{I}_g$-open set.

(3)⇒(4). The proof follows from the fact that every $\mathcal{I}_g$-open set is an $\mathcal{I}_{rg}$-open set.

(4)⇒(5). Suppose M is a regular closed and N is a regular open set containing M. Then M and X−N are disjoint regular closed sets. By hypothesis, there exist disjoint $\mathcal{I}_{rg}$-open sets P and Q such that M⊆P and X−N⊆Q. Since X−N is regular closed and Q is $\mathcal{I}_{rg}$-open, by Lemma 2.24, X−N⊆int*(Q) and so X−int*(Q)⊆N. Again, P∩Q=∅ implies that P∩int*(Q)=∅ and so cl*(P)⊆X−int*(Q)⊆N. Hence P is the required $\mathcal{I}_{rg}$-open set such that M⊆P⊆cl*(P)⊆N.

(5)⇒(6). Let M be a regular closed set and N be a regular open set containing M. Then there exists an $\mathcal{I}_{rg}$-open set P of X such that M⊆P⊆cl*(P)⊆N. By Lemma 2.24, M⊆int*(P). If S=int*(P), then S is an ∗-open set and M⊆S⊆cl*(S)⊆cl*(P)⊆N. Therefore, M⊆S⊆cl*(S)⊆N.

(6)⇒(7). Let M and N be disjoint regular closed subsets of X. Then X−N is a regular open set containing M. By hypothesis, there exists an ∗-open set P of X such that M⊆P⊆cl*(P)⊆X−N. If Q=X−cl*(P), then P and Q are disjoint ∗-open sets of X such that M⊆P and N⊆Q.

(7)⇒(1). Let M and N be disjoint regular closed sets of X. Then there exist disjoint ∗-open sets P and Q such that M⊆P and N⊆Q. Since $\mathcal{I}$ is completely co-dense, by Lemma 1.1, $\tau^*\subseteq\tau^\alpha$ and so P, Q$\in\tau^\alpha$. Hence M⊆P⊆int(cl(int(P)))=S and N⊆Q⊆int(cl(int(Q)))=T. S and T are the required disjoint open sets containing M and N respectively. This proves (1).

If $\mathcal{I}$=$\mathcal{N}$, in the above Theorem 2.25, then $\mathcal{I}_{rg}$-closed sets coincide with rαg-closed sets and so we have the following Corollary 2.26.

Corollary 2.26. *Let w be a WS on a space X where $\tau\subseteq w$. Then the following are equivalent.*

1. *X is mildly normal.*
2. *For disjoint regular closed sets M and N, there exist disjoint wαgp-open sets P and Q such that M⊆P and N⊆Q.*
3. *For disjoint regular closed sets M and N, there exist disjoint αg-open sets P and Q such that M⊆P and N⊆Q.*
4. *For disjoint regular closed sets M and N, there exist disjoint rαg-open sets P and Q such that M⊆P and N⊆Q.*
5. *For a regular closed set M and a regular open set N containing M, there exists an rαg-open set P of X such that M⊆P⊆cl_α(P)⊆N.*
6. *For a regular closed set M and a regular open set N containing M, there exists an α-open set S of X such that M⊆S⊆cl_α(S)⊆N.*
7. *For disjoint regular closed sets M and N, there exist disjoint α-open sets P and Q such that M⊆P and N⊆Q.*

If $\mathcal{I}$={∅} in the above Theorem 2.25, we get the following Corollary 2.27

Corollary 2.27. *Let w be a WS on a space X where $\tau\subseteq w$. Then the following are equivalent.*

(1) *X is mildly normal.*

(2) *For disjoint regular closed sets M and N, there exist disjoint wgp-open sets P and Q such that M⊆P and N⊆Q.*

(3) *For disjoint regular closed sets M and N, there exist disjoint g-open sets P and Q such that M⊆P and N⊆Q.*

(4) *For disjoint regular closed sets M and N, there exist disjoint rg-open sets P and Q such that M⊆P and N⊆Q.*

(5) *For a regular closed set M and a regular open set N containing M, there exists an rg-open set P of X such that M⊆P⊆cl(P)⊆N.*

(6) *For a regular closed set M and a regular open set N containing M, there exists an open set S of X such that M⊆S⊆cl(S)⊆N.*

(7) *For disjoint regular closed sets M and N, there exist disjoint open sets P and Q such that M⊆P and N⊆Q.*

3. $\mathcal{I}_{wgp}$-regular spaces

Let w be a WS on (X,τ,$\mathcal{I}$). Then (X,τ,$\mathcal{I}$) is said to be an $\mathcal{I}_{wgp}$-regular space if for each pair consisting of a point x and a closed set N not containing x, there exist disjoint $\mathcal{I}_{wgp}$-open sets P and Q such that x∈P and N⊆Q. Every regular space is $\mathcal{I}_{wgp}$-regular, since every open set is $\mathcal{I}_{wgp}$-open. The following Example 3.1 shows that an $\mathcal{I}_{wgp}$-regular space need not be regular. Theorem 3.2 gives a characterization of $\mathcal{I}_{wgp}$-regular spaces.

Example 3.1. *Consider the Example 2.1. Since every subset of $\pi(w)$ is $*$-closed, every subset of X is $\mathcal{I}_{wgp}$-closed and so every subset of X is $\mathcal{I}_{wgp}$-open. This implies that (X,τ,$\mathcal{I}$) is $\mathcal{I}_{wgp}$-regular. Now, $\{c\}$ is a closed set not containing a∈X, $\{c\}$ and a are not separated by disjoint open sets. So (X,τ,$\mathcal{I}$) is not regular.*

Theorem 3.2. *Let w be a WS on an ideal space (X,τ,$\mathcal{I}$) with the property C. Then the following are equivalent.*

(1) *X is $\mathcal{I}_{wgp}$-regular.*

(2) *For every closed set N not containing x∈X, there exist disjoint $\mathcal{I}_{wgp}$-open sets P and Q such that x∈P and N⊆Q.*

(3) *For every open set Q containing x∈X, there exists an $\mathcal{I}_{wgp}$-open set P of X such that x∈P⊆cl*(P)⊆Q.*

Proof. (1) and (2) are equivalent by the definition.

(2)⇒(3). Let Q be an open subset such that x∈Q. Then X−Q is a closed set not containing x. Therefore, there exist disjoint $\mathcal{I}_{wgp}$-open sets P and W such that x∈P and X−Q⊆W. Now, X−Q⊆W implies that X−Q⊆int*(W) and so X−int*(W)⊆Q. Again, P∩W=∅ implies that P∩int*(W)=∅ and so cl*(P)⊆X−int*(W). Therefore, x∈P⊆cl*(P)⊆Q. This proves (3).

(3)⇒(1). Let N be a closed set not containing x. By hypothesis, there exists an $\mathcal{I}_{wgp}$-open set P such that x∈P⊆cl*(P)⊆X−N. If W=X−cl*(P), then P and W are disjoint $\mathcal{I}_{wgp}$-open sets such that x∈P and N⊆W. This proves (1).

Theorem 3.3. *Let w be a WS on an ideal space (X,τ,$\mathcal{I}$) with the property C and τ⊆w. If (X,τ,$\mathcal{I}$) is an $\mathcal{I}_{wgp}$-regular, T_1-space where $\mathcal{I}$ is completely codense, then X is regular.*

Proof. Let N be a closed set not containing x∈X. By Theorem 3.2, there exists an $\mathcal{I}_{wgp}$-open set P of X such that x∈P⊆cl*(P)⊆X−N. Since X is a T_1-space, {x} is closed and $\{x\}^c \in \pi(w)$ and so {x}⊆int*(P), by Lemma 1.8. Since $\mathcal{I}$ is completely codense, $\tau^* \subseteq \tau^\alpha$ and so int*(P) and X−cl*(P) are α-open sets. Now, x∈int*(P) ⊆int(cl(int(int*(P))))=G and N⊆X−cl*(P)⊆int(cl(int(X−cl*(P))))=H. Then G and H are disjoint open sets containing x and N respectively. Therefore, X is regular.

If $\mathcal{I}=\mathcal{N}$ in Theorem 3.2, then we have the following Corollary 3.4 which gives characterizations of regular spaces, the proof of which follows from Theorem 3.3.

Corollary 3.4. *If w is a WS on a T_1-space X with the property C and τ⊆w, then the following are equivalent.*

(1) *X is regular.*

(2) *For every closed set N not containing $x \in X$, there exist disjoint $w\alpha gp$-open sets P and Q such that $x \in P$ and $N \subseteq Q$.*

(3) *For every open set Q containing $x \in X$, there exists an $w\alpha gp$-open set P of X such that $x \in P \subseteq cl_{\alpha}(P) \subseteq Q$.*

If $\mathcal{I}=\{\emptyset\}$ in Theorem 3.2, then we have the following Corollary 3.5 which gives characterizations of regular spaces.

Corollary 3.5. *If w is a WS on a T_1-space X with the property C and $\tau \subseteq w$, then the following are equivalent.*

(1) *X is regular.*

(2) *For every closed set N not containing $x \in X$, there exist disjoint wgp-open sets P and Q such that $x \in P$ and $N \subseteq Q$.*

(3) *For every open set Q containing $x \in X$, there exists a wgp-open set P of X such that $x \in P \subseteq cl(P) \subseteq Q$.*

Theorem 3.6. *Let w be a WS on an ideal space $(X,\tau,\mathcal{I})$. If every subset of $\pi(w)$ is $*$-closed, then $(X,\tau,\mathcal{I})$ is $\mathcal{I}_{wgp}$-regular.*

Proof. Suppose every subset of $\pi(w)$ is $*$-closed. Then by Lemma 1.9, every subset of X is $\mathcal{I}_{wgp}$-closed and hence every subset of X is $\mathcal{I}_{wgp}$-open. If N is a closed set not containing x, then {x} and N are the required disjoint $\mathcal{I}_{wgp}$-open sets containing x and N respectively. Therefore, (X,τ,$\mathcal{I}$) is $\mathcal{I}_{wgp}$-regular.

The following Example 3.7 shows that the reverse direction of the above Theorem 3.6 is not true.

Example 3.7. *Consider the real line $\mathcal{R}$ with the usual topology τ_u. Let $\mathcal{I}=\{\emptyset\}$ and $\tau_u \subseteq w$. Then $\mathcal{R}$ is regular and hence $\mathcal{I}_{wgp}$-regular. Let H=(0,1) be an open set and hence $H \in \pi(w)$. Then $cl^*(H)=cl(H)=[0,1] \neq H$ and consequently H is not $\star$-closed. Thus the subset H of $\pi(w)$ is not $\star$-closed.*

4. conclusion

We introduced and investigated $\mathcal{I}_{wgp}$-normal and $\mathcal{I}_{wgp}$-regular spaces. Some examples are given to illustrate the results. Also we discussed wgp-normal and $wgp\mathcal{I}$-normal. We arrive at some characterizations of $\mathcal{I}_{wgp}$-regular spaces.

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