Two Parameter Laplace Type Bimodal Distribution

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Abstract

This paper is on the two parameter Laplace type Bimodal distribution. After discussing distributional properties, order statistics were developed and discussed. Inferential aspects were discussed and estimates of the parameters were obtained through Method of Moments and Maximum Likelihood Estimation techniques. Minimum unbiased estimator of the location parameter and best linear unbiased estimator of the location and scale parameter were also obtained.

Key words: *Two parameter Laplace, Bimodal, Order statistics, Estimation of Parameters*

I. INTRODUCTION

The Laplace distribution has received considerable attention as an appropriate model in reliability theory and life testing models.The statistical data in the fields like agriculture, meteorology and population studiesare appearing as if it is generated from a Laplace distribution, but have the kurtosis lies between 3 and 6. In this paper we introduce a two parameter Laplace type bimodal distribution which suits the distributionarising out of the situations mentioned above. The various distributional properties and inferential aspects of this distribution are discussed. This distribution can be viewed as a reflected gamma type distribution and a hybridization of Laplace and reflected two parameter gamma distribution.

II. TWO PARAMETER LAPLACE TYPE BIMODAL DISTRIBUTION

A continuous random variable X is said to have a two parameter Laplace type bimodal distribution if its probability density function is of the form,

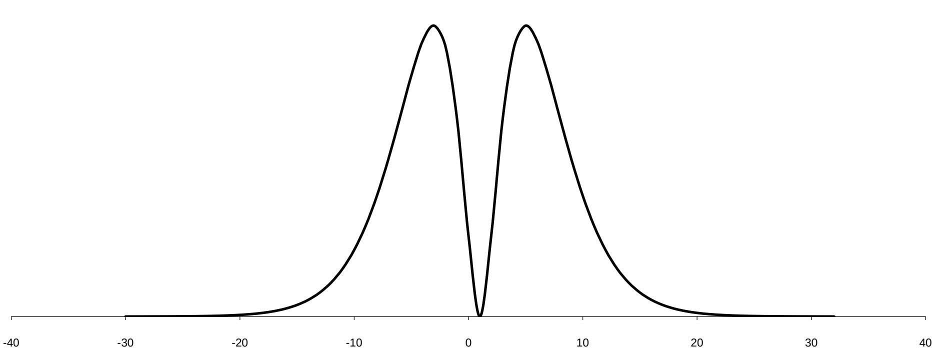
$$f(x,\mu,\beta) = \frac{1}{4\beta}\left(\frac{x-\mu}{\beta}\right)^2 e^{-\left|\frac{x-\mu}{\beta}\right|}, \quad -\infty < x < \infty, \quad \infty < \mu < \infty, \quad \beta > 0 \qquad \longrightarrow 1$$

Making the transformations, $y = \frac{x-\mu}{\beta}$in equation (1), one can get

$$f(y) = \frac{1}{4}y^2 e^{-|y|}, \quad -\infty < x < \infty \qquad \longrightarrow 2$$

which can be called as the standard Laplace type bimodal distribution.
The various shapes of the frequency curves are shown in figure 1.

Case 1:$\mu = 1, \beta = 2$



Case 2:$\mu = 1, \beta = 1$

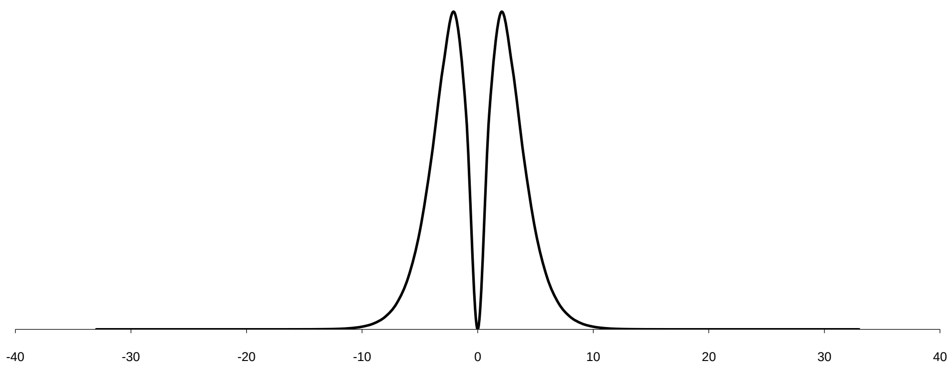


Fig 1
Various shapes of theLaplace type bimodal distribution

III. PROPERTIES OF DISTRIBUTION

The distribution function is

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{4\beta}\left(\frac{x-\mu}{\beta}\right)^2 e^{-\left|\frac{x-\mu}{\beta}\right|} dx = 1 - \left[\frac{1}{2} + \left(\frac{x-\mu}{\beta}\right) + \left(\frac{x-\mu}{\beta}\right)^2\right] e^{-\left(\frac{x-\mu}{\beta}\right)}, \quad for\ x \geq \mu$$

$$= \left[\frac{1}{2} + \left(\frac{x-\mu}{\beta}\right) + \left(\frac{x-\mu}{\beta}\right)^2\right] e^{-\left(\frac{x-\mu}{\beta}\right)}, \quad for\ x < \mu \rightarrow 3$$

Mean of the distribution is

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \frac{1}{4\beta}\int_{-\infty}^{\infty} x\left(\frac{x-\mu}{\beta}\right)^2 e^{-\left|\frac{x-\mu}{\beta}\right|} dx$$

On simplification, one can have Mean= μ

The Median 'M 'of this distribution is such that

$$\frac{1}{4\beta}\int_{-\infty}^{M} x\left(\frac{x-\mu}{\beta}\right)^2 e^{-\left|\frac{x-\mu}{\beta}\right|} dx + \frac{1}{4\beta}\int_{M}^{\infty} x\left(\frac{x-\mu}{\beta}\right)^2 e^{-\left|\frac{x-\mu}{\beta}\right|} dx = 1$$

Solving this one can get Median = M= μ. Hence the distribution is symmetric.
Taking the derivative of (1) with respect to x and equating to zero and solving for x, one can get the mode and, in this case,, the modes as $\mu - 2\beta$and$\mu + 2\beta$. These two modes are separated by the distance of4β. That is the distribution is Bimodal.

The moment generating function of this distribution is

$$M_X(t) = E(e^{tX}) = \frac{1}{4\beta}\int_{-\infty}^{\infty} e^{tx}\left(\frac{x-\mu}{\beta}\right)^2 e^{-\left|\frac{x-\mu}{\beta}\right|} dx$$

On simplification, one can get

$$M_X(t) = \frac{e^{t\mu}}{2}\left[\frac{1}{(1+\beta t)^3} + \frac{1}{(1-\beta t)^3}\right] = \frac{e^{t\mu}}{2}\left[\frac{2(1+3\beta^2 t^2)}{(1-\beta^2 t^2)^3}\right] \quad -\!\!\rightarrow 4$$

The characteristic function of this distribution is

$$\emptyset_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{4\beta}\left(\frac{x-\mu}{\beta}\right)^2 e^{-\left|\frac{x-\mu}{\beta}\right|} dx$$

Using the transformation, $y = x - \mu$,one can get

$$\emptyset_X(t) = \frac{1}{4\beta} e^{it\mu}\int_{-\infty}^{\infty} e^{ity}\left(\frac{\mu}{\beta}\right)^2 e^{-\left|\frac{\mu}{\beta}\right|} dy = \frac{e^{it\mu}}{2}\left[\frac{2-6\beta^2 t^2}{(1+\beta^2 t^2)^3}\right] \quad -\!\!\rightarrow 5$$

To verify the additive property of the distribution,consider two independent Laplace type bimodal variables $X_1$and $X_2$with probability density function$f(x_1,\mu_1,\beta_1)$ and $f(x_2,\mu_2,\beta_2)$ respectively and their characteristic functions be$\emptyset_{X_1}(t)$and$\emptyset_2(t)$. For finding the distribution of X_1+X_2 , one can obtain the characteristic function ofX_1+X_2. Since X_1 and $X_2$are independent,

$$\emptyset_{X_1+X_2}(t)=\emptyset_{X_1}(t)\emptyset_{X_2}(t)$$

Since

$$\emptyset_{X_j}(t)=\frac{e^{it\mu_j}}{2}\left[\frac{1}{(1+i\beta_j t)^3}+\frac{1}{(1-i\beta_j t)^3}\right] \quad , j=1,2$$

then

$$\emptyset_{X_1+X_2}(t)=\frac{e^{it(\mu_1+\mu_2)}}{2}\left[\frac{1}{(1+i\beta_1 t)^3}+\frac{1}{(1-i\beta_1 t)^3}\right]+\left[\frac{1}{(1+i\beta_2 t)^3}+\frac{1}{(1-i\beta_2 t)^3}\right] \qquad --\rightarrow 6$$

Since (6) is not in the form of (5) one can say that the distribution of $X_1+X_2$is not a Laplace type bimodal distribution. Hence the additive property does not hold good for this distribution. The central moments of this distribution are

$$\mu_{2n}=\frac{1}{4\beta}\int_{-\infty}^{\infty}(x-\mu)^{2n}\left(\frac{x-\mu}{\beta}\right)^2 e^{-\left|\frac{x-\mu}{\beta}\right|}dx$$

Making the necessary transformation and by integration, one can get

$$\mu_{2n}=\frac{[(2n+2)!]\beta^{2n}}{2} \quad and\ \mu_{2n+1}=0 \qquad --\rightarrow 7$$

Further, the first four central moments of this distribution are

$\mu_1=0$, Variance= $\mu_2=12\beta^2$, $\mu_3=0, \mu_4=360\beta^4$

Since, the distribution is symmetric, its skewness is zero.

The kurtosis of this distribution is

$$\beta_2=\frac{\mu_4}{\mu_2^2}=\frac{360\beta^4}{12\beta^2}=2.5<3$$

Hence the distribution is platykurtic having the peak less than the Laplace distribution.

The recurrence relation between the central moments is obtained as

$$\mu_{2n}=\frac{[(2n+1)(2n+2)]}{2n(2n-1)}\beta^2\mu_{2n-2} \qquad --\rightarrow 8$$

The p^{th} order absolute moments of the distribution are

$$E[|X-\mu|^p]=\frac{1}{4\beta}\int_{-\infty}^{\infty}|x-\mu|^p\left(\frac{x-\mu}{\beta}\right)^2 e^{-\left|\frac{x-\mu}{\beta}\right|}dx$$

On simplification, one can have

$$E[|X-\mu|^p]=\left[\frac{(p+2)!}{2}\right]\beta^p \qquad --\rightarrow 9$$

The Mean deviation about meanμ of the distribution is$M\ D=3\beta$

The ratio of the mean deviation to standard deviation is$\frac{3\beta}{\sqrt{12}\beta}=0.866$

The distribution of the square of the variate $Y=X^2$ can be obtained as

$$f(y)=\frac{1}{4\beta\sqrt{y}}\left(\frac{y}{\beta}\right)^2 e^{\frac{-\sqrt{y}}{\beta}}, \qquad 0\le y<\infty$$

For obtaining the sampling distribution, assume$\mu=0$ without loss of generality. Let $x_1,x_2,\ldots.x_n$ bea sample drawn from the distribution

$$f(x) = \frac{1}{4\beta}\left(\frac{x}{\beta}\right)^2 e^{\frac{-|x|}{\beta}}, \quad -\infty < x < \infty \qquad \longrightarrow 10$$

The sample observations $x_1, x_2, \ldots . x_n$ are independently identically distributed as given in (10).
Since $x_1, x_2, \ldots . x_n$ are independently distributed, the characteristic function of $x_1, x_2, \ldots . x_n$ is given by

$$\emptyset(t) = \frac{1}{4\beta}\int_{-\infty}^{\infty} e^{it|x|}\left(\frac{x}{\beta}\right)^2 e^{\frac{-|x|}{\beta}}\, dx \qquad \longrightarrow 11$$

If $u = \sum_{i=0}^{n} x_i$ then the distribution function of, namely $p(u)$, is given by

$$p(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itu}\left\{\frac{1}{4\beta}\int_{-\infty}^{\infty} e^{it|x|}\left(\frac{x}{\beta}\right)^2 e^{\frac{-|x|}{\beta}}\, dx\right\}^n du \qquad \longrightarrow 12$$

On simplification

$$p(u) = \frac{1}{2\pi}\frac{1}{2^n\beta^n}\int_{-\infty}^{\infty} e^{itu}\frac{1}{(1-i\beta t)^{3n}}du \qquad \longrightarrow 13$$

The poles of the integrand are of the 3n[th] order and one of those is $(1-i\beta t)^{3n}$. From the well-known residue theorem of Cauchy Macro Bert (1933), one can have

$$p(u) = \frac{1}{2^n\beta^n}\frac{1}{i^{3n-1}}\frac{d^{3n-1}}{dt^{3n-1}}\left.\frac{\left[\frac{e^{-i\mu t}}{(1-i\beta t)^{3n}}\right]}{t}\right|_{t=\frac{1}{\beta i}} \qquad \longrightarrow 14$$

The distribution of arithmetic mean of the sample mean of the sample size n is

$$p(|\bar{x}|) = \frac{n}{3^n\beta^n}\sum_{j=1}^{n}\frac{(-1)^{n+2j-1}}{(n+2j-i)!}\frac{1}{i(n+2j-1)}\frac{d^{n+2j-1}}{dt^{n+2j-1}}\left.\frac{\left[\frac{e^{-itn|\bar{x}|}}{(1+i\beta t)^{n+2j}}\right]}{t}\right|_{t=\frac{1}{\beta i}} \qquad \longrightarrow 15$$

defined for all values of on the range $(-\infty < \bar{x} < \infty)$.

IV. ORDER STATISTICS OF TWO PARAMETER LAPLACE TYPE BIMODAL DISTRIBUTION

The simple explicit form of the distribution function as given in (3) leads to derive the order statistics connected with this two parameter Laplace type bimodal distribution. For simplicity, let us assume $\mu = 0, \beta = 1$ then the probability density function (1) reduces to

$$f(x) = \frac{1}{4}x^2 e^{-|x|}, \quad -\infty < x < \infty \qquad \longrightarrow 16$$

Let $X_{1:n} \le X_{2:n} \le \cdots X_{n:n}$ denote the order statistics obtained from a random sample of size 'n' from the standardized Laplace distribution having the probability density function of the form given in (1).
The probability density function of the m[th] order statistics is given by [David (1991)].
$f_{m:n}(x) = D_{m:n}[f(x)]^{m-1}[1-F(x)]^{n-m}f(x), \quad -\infty < x < \infty$
where

$$D_{m:n} = \frac{n!}{(m-1)!\,(n-m)!} \qquad \longrightarrow 17$$

Substituting $F(x)$ values in the above equation (17), one can get

$$f_{m:n}(x) = D_{m:n}\frac{1}{4}x^2 e^{-|x|}\sum_{q=0}^{n-m}\binom{n-m}{q}(-1)^q\left[1-\sum_{k=0}^{l}\binom{l}{k}(2k)!\sum_{i=0}^{2k}\frac{x^i}{i!}e^{-|x|}\right]^{m+q-1} \quad for\ x \ge 0$$

$$f_{m:n}(x) = D_{m:n}\frac{1}{4}x^2 e^{-|x|}\sum_{q=0}^{n-m}\binom{n-m}{q}(-1)^q\left[\frac{\sum_{k=0}^{l}\binom{l}{k}(2k)!\sum_{i=0}^{2k}\frac{x^i}{i!}e^{-|x|}}{4}\right]^{m+q-1} \quad for\ x < 0$$

$$\rightarrow (18)$$

where, $D_{m:n}$ is as given in (17).
The a [th] moment of $X_{m:n}$ is given by

$\alpha_{m:n}(x) =$

$$\int_0^{\infty}(-x)^a\, D_{m:n} x^2 e^{-x} \sum_{q=0}^{n-m}\binom{n-m}{q}(-1)^q\left[\frac{\sum_{k=0}^{l}\binom{l}{k}(2k)!\sum_{i=0}^{2k}\frac{x^i}{i!}e^{-|x|}}{4}\right]^{m+q-1} dx +$$

$$\int_0^{a}(x)^a\, D_{m:n} x^2 e^{-x} \sum_{q=0}^{n-m}\binom{n-m}{q}(-1)^q\left[1-\frac{\sum_{k=0}^{l}\binom{l}{k}(2k)!\sum_{i=0}^{2k}\frac{x^i}{i!}e^{-|x|}}{4}\right]^{m+q-1} dx \quad \rightarrow 19$$

Define $D_{m:n} = \frac{D_{m:n}}{4}$ and after simplifying $\alpha_{m:n}(a)$

$$= (-1)^a D_{m:n}\sum_{k=0}^{l}\sum_{q=0}^{n-m}\sum_{j=0}^{m+q-1}\sum_{l=0}^{j}\sum_{p=0}^{l}\binom{l}{k}\binom{n-m}{q}\binom{m+q-1}{j}\binom{j}{l}\binom{l}{p}2^{j-1+l-p}(-1)^q\frac{(a+2k+l+p)!}{(m+q)^{a+2k+l+p+1}}$$

$$+D_{m:n}\sum_{k=0}^{l}\sum_{q=0}^{n-m}\sum_{j=0}^{m+q-1}\sum_{l=0}^{j}\sum_{p=0}^{l}\binom{l}{k}\binom{n-m}{q}\binom{m+q-1}{j}\binom{j}{l}\binom{l}{p}2^{jl-p}\frac{(a+2k+i+p)!}{(1+l)^{a+2k+i+p+l}} \rightarrow 20$$

Probability distribution of first ordered statistics is

$$f_{1:n}(x) = \frac{n}{4}x^2e^{-x}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^q\left[\frac{1-\sum_{k=0}^{l}\binom{l}{k}(2k)!\sum_{i=0}^{2k}\frac{x^i}{i!}e^{-x}}{4}\right]^q \quad for\ x \geq 0$$

$$f_{1:n}(x) = \frac{n}{4}x^2e^{-|x|}\sum_{q=0}^{n-1}\binom{n-1}{q}(-1)^q\left[\frac{\sum_{k=0}^{l}\binom{l}{k}(2k)!\sum_{i=0}^{2k}\frac{x^i}{i!}e^{-|x|}}{4}\right]^q \quad for\ x < 0 \ \longrightarrow 21$$

The expected value of the first order statistics is

$$E[X_{(1)}] = n\left[-\sum_{k=0}^{l}\sum_{q=0}^{n-m}\sum_{j=0}^{m+q-1}\sum_{l=0}^{j}\sum_{p=0}^{l}\binom{l}{k}\binom{n-1}{q}\binom{q}{j}\binom{j}{l}\binom{l}{p}2^{l-p}\frac{(a+2k+1+p)!}{(1+q)^{a+2k+i+p+l+1}}\right]$$

$$+\sum_{k=0}^{l}\sum_{q=0}^{n-1}\sum_{l=0}^{m+q-1}\sum_{i=0}^{l}\sum_{p=0}^{i}\binom{l}{k}\binom{n-1}{q}\binom{q}{l}\binom{l}{i}\binom{i}{p}2^{l-p}\frac{(a+2k+i+p)!}{(1+l)^{a+2k+i+p+l}} \longrightarrow 22$$

The nth order statistics of the distribution is same as the first order statistic with negative variate because the distribution is symmetric.

A. Distribution of the Median

To obtain the distribution of the median, substitute $m = \frac{n+1}{2}$ if n is odd in (17).

Then one can get

$$f_{\frac{n+1}{2}:n}(x) = D_{\frac{n+1}{2}:n}\frac{1}{4}x^2e^{-|x|}\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{1-\sum_{k=0}^{l}\binom{l}{k}(2k)!\sum_{i=0}^{2k}\frac{x^i}{i!}e^{-|x|}}{4}\right]^{\frac{n+2q-1}{2}}, \quad for\ x \geq 0$$

$$= D_{\frac{n+1}{2}:n}\frac{1}{4}x^2e^{-|x|}\sum_{q=0}^{\frac{n-1}{2}}\binom{\frac{n-1}{2}}{q}(-1)^q\left[\frac{\sum_{k=0}^{l}\binom{l}{k}(2k)!\sum_{i=0}^{2k}\frac{x^i}{i!}e^{-|x|}}{4}\right]^{\frac{n+2q-1}{2}} \quad for\ x < 0 \longrightarrow 23$$

Where, $D_{\frac{n+1}{2}:n} = \frac{n!}{\left[\left(\frac{n-1}{2}\right)!\right]^2}$

When n is even, the distribution of the median is

$$f_{\frac{2n+1}{4}:n}(x) = D_{\frac{2n+1}{4}:n}\frac{1}{4}x^2e^{-|x|}\sum_{q=0}^{\frac{2n-1}{4}}\binom{\frac{2n-1}{4}}{q}(-1)^q\left[\frac{1-\sum_{k=0}^{l}\binom{l}{k}(2k)!\sum_{i=0}^{2k}\frac{x^i}{i!}e^{-|x|}}{4}\right]^{\frac{2n+2q-1}{4}}$$

for $x \geq 0$

$$= D_{\frac{2n+1}{4}:n}\frac{1}{4}x^2e^{-|x|}\sum_{q=0}^{\frac{2n-1}{4}}\binom{\frac{2n-1}{4}}{q}(-1)^q\left[\frac{\sum_{k=0}^{l}\binom{l}{k}(2k)!\sum_{i=0}^{2k}\frac{x^i}{i!}e^{-|x|}}{4}\right]^{\frac{2n+4q-3}{3}}$$

for $x < 0$ → 24

Where $D_{\frac{2n+1}{4}:n} = \frac{n!}{\left[\left(\frac{2n-3}{4}\right)!\right]^2}$

B. Joint Moments of the Order Statistics

The joint probability density function of the m^{th} and s^{th} order statistics $X_{m:n}$ and $X_{s:n}$, $m < s$ is given by [David (1981)]

$f_{m,s:n}(x) = D_{m,s:n}[F(x)]^{m-1}[F(y) - F(x)]^{s-m-1}[1 - F(y)]^{n-s}f(x)f(y)$

Where $D_{m,s:n} = \frac{n!}{(m-1)!(s-m-1)!(n-s)!}$

And $F(x)$ is the cumulative density function of the laplace type distribution as given in equation (3)

Writing $(x) = \frac{2+2x+x^2}{4}$, it is possible to express the joint density function of $X_{m:n}$ and $X_{s:n}$ as

$$f_{m,s:n}(x) = D_{m,s:n}[U(x)]^{m-1}[U(y) - U(x)]^{s-m-1}[1 - U(y)]^{n-s}f(x)f(y) \rightarrow 25$$

where $D_{m,s:n} = \frac{n!}{(m-1)!(s-m-1)!(n-s)!}$ $\quad for \quad [(x,y): -\infty < x < y < \infty]$

This region can be split into three mutually exclusive regions,

$R_1: [(x,y): -\infty < x < y < 0]$

$R_2: [(x,y): 0 < x < y < \infty]$

$R_3: [(x,y): -\infty < x < \infty, \qquad -\infty < y < \infty]$

With this splitting of the region of the product moments can be obtained as

$$E[X_{m:n}, X_{s:n}] = D_{m,s:n}\left\{\iint_{R_1} xy[U(-x)]^{m-1}[U(-y) - U(-x)]^{s-m-1}[1 - U(-y)]^{n-s}f(-x)f(-y)\,dxdy\right\}$$

$$+D_{m,s:n}\left\{\iint_{R_2} xy[1 - U(x)]^{m-1}[U(x) - U(y)]^{s-m-1}[1 - U(y)]^{m-s}f(x)f(y)\,dxdy\right\}$$

$$-D_{m,s:n}\left\{\iint_{R_3} (-xy)[U(x)]^{m-1}[1 - U(-x) - U(-y)]^{s-m-1}[U(y)]^{n-s}f(x)f(y)\,dxdy\right\} \rightarrow 26$$

Expanding the binomial expression in the integral, one can have

$$E[X_{m:n}, X_{s:n}] = D_{m,s:n}\left[\sum_{i=0}^{s-m-1}\sum_{j=0}^{n-s}\binom{s-m-1}{i}\binom{n-s}{j}(-1)^{s-m-1+i+j}\Psi(s-2-i, i+j)\right]$$

$$+D_{m,s:n}\left[\sum_{i=0}^{m-1}\sum_{j=0}^{s-m-1}\binom{s-m-1}{j}\binom{m-1}{j}(-1)^{i+j}\Psi(s-m-1+i+j, n-s+j)\right]$$

$$-D_{m,s:n}\left[\sum_{i=0}^{s-m-1}\sum_{j=0}^{n-s}\binom{s-m-1}{j}\binom{s-m-1-i}{i}(-1)^{i+j}\int_0^x x[U(x)]^{m+i-1}f(x)dx\int_0^\infty y[U(y)]^{n-s-j}f(y)dy\right]$$

where $D_{m,s:n}$ is as given in (25)

and $\Psi(a,b) = \int_0^\infty \int_0^y xy[U(x)]^a\,[U(y)]^b f(x)f(y)dxdy$ $\longrightarrow 27$

$$\Psi(a,b) = \int_0^\infty \int_0^y xy\left[\frac{\sum_{k=0}^{l}\binom{l}{k}(2k)!\sum_{i=0}^{2k}\frac{x^i}{i!}e^{-|x|}}{4}\right]^a \left[\sum_{k=0}^{l}\binom{l}{k}(2k)!\sum_{i=0}^{2k}\frac{y^i}{i!}e^{-|x|}\right]^b \left[\frac{2+x^2}{4}\right]e^{-x}\left[\frac{2+y^2}{4}\right]e^{-y}dxdy$$

$$= \frac{1}{4^{a+b+2}}\sum_{k=0}^{l}\binom{l}{k}\sum_{j=0}^{a}\binom{a}{r}\sum_{j=0}^{r}\binom{r}{j}\sum_{i=0}^{j}\binom{j}{i}2^{r-i}\left[\frac{\Gamma(2k+j+i+2)}{(r+1)(2k+j+i+3)} - \sum_{j_1}^{2k+j+i+2}\frac{(r+1)^{j_1}}{j!}\frac{\Gamma(2k+j+j_1+i+2}{[2(r+1)]^{2k+j+j_1+i+3}}\right] \longrightarrow 28$$

Substituting the value of $\Psi(a,b)$of (28) in (20), one can get the moments of the order statistics.

V. INFERENTIAL ASPECTS OF THE TWO PARAMETER LAPLACE TYPE BIMODAL DISTRIBUTION

In the earlier section the two parameter Laplace type bimodal distribution was studied its distributional properties were presented. Another aspect of any distributional study is to look into the inferential aspects of the distribution, in particular the estimation of the parameters involved in the distribution under study. In this section various methods of estimation using the method of moments, maximum likelihood method estimation and best linear unbiased estimation in estimating the parameters of the two parameter Laplace type bimodal distribution were discussed. The asymptotic behaviors of these estimators are also studied.

A. Method of moments

Let us consider a sample of size 'n' drawn from a population having the probability density function of the form given by

$$f(x,\mu,\beta) = \frac{1}{4\beta}\left(\frac{x-\mu}{\beta}\right)^2 e^{-\left|\frac{x-\mu}{\beta}\right|}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \quad \beta > 0$$

This distribution is having two parameters μ and β. Hence, we consider the first two moments of the sample and the population, which leads to the following equations.

$$\mu = \bar{x},\ \ 12\,\beta^2 = s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

These equations give us the moment estimators as

$\hat{\mu} = \bar{x}\ and\ \hat{\beta}^2 = \dfrac{s^2}{12} = 0.0833s^2$ $\longrightarrow 29$

The variance of $\hat{\mu}$ is

$Var(\bar{x}) = Var\left(\dfrac{1}{n}\sum_{i=1}^{n}x_i\right) = \dfrac{1}{n^2}\sum_{i=1}^{n}Var(x_i) = \dfrac{1}{n^2}(12\beta^2)$ $\longrightarrow 30$

and $E(\bar{x}) = E\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) = \frac{1}{n}\sum_{i=1}^{n}E(x_i) = \mu$

Therefore, the sample mean is an unbiased estimator ofμ.
For obtaining the unbiased estimator ofβ^2, consider

$$\hat{\beta}^{*2} = \frac{n}{n-1}(0.0833)s^2$$

$E(\hat{\beta}^{*2}) = \beta^2$
Therefore, $\hat{\beta}^{*2}$is an unbiased estimator of β^2.
The variance of $\hat{\beta}^{*2}$can be obtained as

$Var(\hat{\beta}^{*2}) = Var\left[\dfrac{n}{n-1}(0.0833)s^2\right] = \dfrac{n^2}{(n-1)^2}(0.0833)^2Var(s^2)$ $\longrightarrow 32$

Where $Var(s^2) = \frac{\mu_4-\mu_2^2}{n} - \frac{2(\mu_4-\mu_2^2)}{n^2} + \frac{(\mu_4-3\mu_2^2)}{n^3}$
and $\mu_i$is the i^{th} central moment (Cramer,1946)

Therefore,

$$Var(s^2) = \frac{(360-144)\,\beta^4}{n} - \frac{2(360-288)\,\beta^4}{n^2} + \frac{(360-432)\,\beta^4}{n^3} = \left[\frac{216}{n} - \frac{144}{n^2} - \frac{72}{n^3}\right]\beta^4 \longrightarrow 33$$

Equations (32) and (33) gives

$$Var(\hat{\beta}^{*2}) = \frac{n^2}{(n-1)^2}\left[\frac{216}{n} - \frac{144}{n^2} - \frac{72}{n^3}\right](0.0833)^2\,\beta^4 \qquad \longrightarrow 34$$

From the equations (32) and (34) it can be observed that as $n \to \infty$ the variances tend to zero. Hencethe estimators$\bar{x}$ and$\hat{\beta}^{*2}$are consistent. Since the sample observations are independently distributed, these estimators are strongly consistent. Since the covariance of $\bar{x}$ and$\hat{\beta}^{*2}$ is zero,these estimators $\bar{x}$ and$\hat{\beta}^{*2}$ are uncorrelated. This result can also be visualized from the fact that the sample is drawn from a symmetric distribution.

B. Maximum Likelihood Method of Estimation

Let $x_1, x_2, \ldots . x_n$ be a sample of size n drawn from a population having the probability density function of the form given in equation (1). Then the likelihood function of the sample is

$$L = \frac{1}{(4\beta)^n}\pi_{i=1}^{n}\left[\left(\frac{x_i-\mu}{\beta}\right)^2\right]e^{-\sum_{i=1}^{n}\left|\frac{x_i-\mu}{\beta}\right|} \qquad \longrightarrow 35$$

Taking logarithms on both sides of (35) gives

$$logL = -nlog(4\beta) + \sum_{i=1}^{n} log\left(\frac{x_i-\mu}{\beta}\right)^2 - \sum_{i=1}^{n}\left|\frac{x_i-\mu}{\beta}\right| \qquad \longrightarrow 36$$

For obtaining the maximum likelihood estimators of the parameters one has tomaximize L or $\log L$ with respect to the parameters μ andβ. Sinceequation (32) is not differentiable with respect to$\mu$the likelihood estimator ofμcan be obtained as follows.
Since the density function is symmetric about $x = \mu$i.e$f(x+\mu) = f(x-\mu)$it follows that$E(x) = \mu$. Hence$\frac{1}{n}\sum_{i=1}^{n}|x_i-\mu|$ is the mean deviationof the random sample from its mean. This shall be minimum if μ is the median of the sample. Since

$$Max(logL) = Max\left\{\sum_{i=1}^{n} log\left(\frac{x_i-\mu}{\beta}\right)^2 - \sum_{i=1}^{n}\left|\frac{x_i-\mu}{\beta}\right|\right\} = Min\sum_{i=1}^{n}|x_i-\mu|$$

it can be shown that $\hat{\mu}$ = Median of the sample if n is odd.
If n is even, then $X_{\frac{n}{2}-1} \le \hat{\mu} \le X_{\frac{n}{2}+1}$.where X_j is thej^{th} order statistic.
After obtaining the maximum likelihood estimation ofμ as$\hat{\mu}$ and substituting$\hat{\mu}$ in equation (32) the maximum likelihood estimation of β^2 can be obtained by differentiating log L with respect to β^2 and equating it to zero.After simplification one can obtain

$$\widehat{\beta^2} = \frac{-n+\sum_{i=1}^{n}|x_i-\hat{\mu}|}{2n}.$$

Also, since $E\left[\frac{-\partial^2 logL}{\partial(\beta^2)^2}\right]^{-1} = E\left[\frac{n}{2\beta^4} + \frac{1}{2}\frac{\sum_{i=1}^{n}|x_i-\mu|}{\beta^4}\right]^{-1} = \frac{2\beta^4}{n}$
The approximate expression to the asymptotic variance of $\widehat{\beta^2}$can be obtained as

$$Asyvar(\widehat{\beta^2}) = E\left[\frac{-\partial^2 logL}{\partial(\beta^2)^2}\right]^{-1} = \frac{2\beta^4}{n}$$

C. Minimum variance unbiased estimators

In this method the minimum variance unbiased estimators of the location parameter β^2 can be obtained. The general method of estimatingμ is to consider the linearcombination $l_1x_1 + l_2x_2 + \cdots + l_nx_n$of the random

sample and to determine the linear coefficients $l_1, l_2, \dots l_n$ so that the estimator is unbiased and minimum variance in this class of estimators. [C.R.Rao (1974)]. Here note that the random sample here means a set of i. i. d random variables.
The condition of unbiasedness leads to $l_1 + l_2 + \cdots + l_n = l$ and the condition of minimum variance leads to $l_1^2 + l_2^2 + \cdots + l_n^2$ being minimum.

Writing, $\hat{l} = \frac{l_1+l_2+\cdots+l_n}{n}$ one can have $\sum_{i=1}^{n}\left(l_i - \bar{l}\right)^2 \geq 0$
and this implies, $\sum_{i=1}^{n} {l_i}^2 \geq n\bar{l}^2 = \frac{1}{n}$, since $\sum_{i=1}^{n} l_i = 1$
Equality occurs only when all the l_i 's are equal.ie. $l_i = \frac{1}{n}$
Thus, the greatest lower bound of $\sum_{i=1}^{n} {l_i}^2$ for various l is $\frac{1}{n}$ and is attained.

Therefore, $\hat{\mu} = \bar{x}$, is the minimum variance unbiased estimator. The same holds for any value of μ. Hence it is also uniformly minimum variance unbiased estimator.
The minimum variance in this case is $\frac{var\,(\bar{x})}{n} = \frac{12\beta^2}{n}$

For obtaining the minimum variance unbiased estimator of β^2, consider the usual estimator $\widehat{\beta^{*2}}$ which is unbiased. It can also be observed that it is the minimum variance unbiased estimator of β^2

D. Best Linear Unbiased Estimators

Following the heuristic argument of Govindarajalu(1966) the best linear unbiased estimators of the location and scale parameters involved in the two parameter Laplace type bimodal distribution having the probability density function of the form given in (1) are obtained as follows.
Then the best linear unbiased estimators of μ and β are given by Lloyd (1952) as

$$\mu^* = \frac{l'\Omega X}{l'\Omega l} \; and \; \beta^* = \frac{\alpha'\Omega X}{\alpha'\Omega\alpha}$$

where $l' = (1,1,\dots,1)$, $X' = [X_{1:n},\ X_{2:n}, \dots X_{n:n}]$ $and\ \alpha' = [\alpha_{1:n},\ \alpha_{2:n}, \dots \alpha_{n:n}]$
and $\alpha_{1:n}$ is the first moment of the i^{th} order statistics and Ω is the variance matrix of vector X

The variance of these estimators are

$$var(\mu^*) = \frac{12\beta^2}{l'\Omega l}, var(\beta^*) = \frac{12\hat{\beta}^2}{\alpha'\Omega\alpha}$$

Where Ω is the standard deviation of the populationand $cov(\mu^*, \beta^*) = 0$. With the moments of the order statistics given in section (2.4) one can compute μ^* and β^* and their variances respectively.

VI. CONCLUSIONS

This paper is on the distributional properties of the two parameter Laplace type bimodal distribution. The order statistic was also studies. In addition, inferential aspects like estimation of the parameters by method of moments, Maximum likelihood method of estimation were explained. Minimum variance unbiased estimators and Best linear unbiased estimators for the parameters of the distribution were presented.

REFERENCES

[1] Cramer, H. (1946): Mathematical Methods of Statistics, Asia Publishing House.
[2] David. H.A. (1981): Order Statistics, John Wiley & Sons, Ins. New York
[3] Eisenberger, I. (1964): 'Genesis of Bimodal distributions' techno metrics, p.357-363., vol.6.
[4] Feller, W. (1966). An Introduction to Probability Theory and Its Applications, Vol.II, New York, John Wiley & sons, NC.
[5] Govindarajulu, Z. (1966): Best linear estimates under symmetric censoring of the parameters of a double exponential population,
[6] Johnson, N.L. and Kotz, S. (1970): Continuous Univariate Distributions, john Wiley Sons, New York.
[7] Kameda, T. (1928): On the reduction of frequency curves, SckandinaviskAktuarietidskrift, 11.
[8] Laplace, P.S. (1774): Memoire sur la pobabilite des causes par les evenemens, Memoiries de mathematique et de Physique, 6.
[9] Lloyd, E.H. (1952): Least square estimation of location and scale parameters using order statistics, Biometrica, 39
[10] Rao, C.R. (1965): Linear Statistical Inference and its applications, Wiley Eastern