On Finite Double Integrals Involving a General Multivariable Polynomial and Generalized Multivariable Gimel-Function

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ABSTRACT
In this paper we evaluate three general finite double integrals involving the product of algebraic and exponential functions, a general multivariable polynomials and generalized multivariable polynomials. Some new and interesting special cases of our main integrals have been considered briefly.

Keywords : multivariable Gimel-function, hypergeometric function, finite double integrals, multivariable polynomials.

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1. Introduction and preliminaries.

We define a generalized transcendental function of several complex variables.

$$\gimel(z_1,\cdots,z_r)=\gimel^{m_2,n_2;m_3,n_3;\cdots;m_r,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|} \mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\end{array}\right.$$

$$[(\mathrm{a}_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[(\mathrm{b}_{2j};\beta^{(1)}_{2j},\beta^{(2)}_{2j};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta^{(1)}_{3j},\beta^{(2)}_{3j},\beta^{(3)}_{3j};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj};B_{rj})_{1,m_r}],$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{m_r+1,q_r}]:[(\mathrm{d}^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}]\\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{m_2}\Gamma^{B_{2j}}(b_{2j}-\sum_{k=1}^{2}\beta^{(k)}_{2j}s_k)\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha^{(k)}_{2ji_2}s_k)\prod_{j=m_2+1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{m_3}\Gamma^{B_{3j}}(b_{3j}-\sum_{k=1}^{3}\beta^{(k)}_{3j}s_k)\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha^{(k)}_{3j}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha^{(k)}_{3ji_3}s_k)\prod_{j=m_3+1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta^{(k)}_{3ji3}s_k)]}$$

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$$\frac{\prod_{j=1}^{m_r}\Gamma^{B_{rj}}(b_{rj}-\sum_{k=1}^{r}\beta_{rj}^{(k)}s_k)\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=m_r+1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \quad (1.2)$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \quad (1.3)$$

1) $[(c_j^{(1)};\gamma_j^{(1)})]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $m_2,n_2,\cdots,m_r,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2\leqslant q_{i_2},0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant m_r\leqslant q_{i_r},0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r)$;

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kj}^{(l)},B_{kj}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n_k);(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r)$.

$b_{kji_k}\in\mathbb{C};(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r)$.

$d^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$\gamma^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour L_k is in the $s_k(k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha^{(k)}_{2j} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha^{(k)}_{3j} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha^{(i)}_{rj}\right) (j = 1, \cdots, n_r)$, $\Gamma^{C^{(k)}_j}\left(1 - c^{(k)}_j + \gamma^{(k)}_j s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$to the right of the contour L_k and the poles of $\Gamma^{B_{2j}}\left(b_{2j} - \sum_{k=1}^{2} \beta^{(k)}_{2j} s_k\right) (j = 1, \cdots, m_2), \Gamma^{B_{3j}}\left(b_{3j} - \sum_{k=1}^{3} \beta^{(k)}_{3j} s_k\right) (j = 1, \cdots, m_3)$ $, \cdots, \Gamma^{B_{rj}}\left(b_{rj} - \sum_{i=1}^{r} \beta^{(i)}_{rj}\right) (j = 1, \cdots, m_r)$, $\Gamma^{D^{(k)}_j}\left(d^{(k)}_j - \delta^{(k)}_j s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A^{(k)}_i \pi$ where

$$A^{(k)}_i = \sum_{j=1}^{m^{(k)}} D^{(k)}_j \delta^{(k)}_j + \sum_{j=1}^{n^{(k)}} C^{(k)}_j \gamma^{(k)}_j - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q^{(k)}_i} D^{(k)}_{ji^{(k)}} \delta^{(k)}_{ji^{(k)}} + \sum_{j=n^{(k)}+1}^{p^{(k)}_i} C^{(k)}_{ji^{(k)}} \gamma^{(k)}_{ji^{(k)}} \right) +$$

$$\sum_{j=1}^{n_2} A_{2j} \alpha^{(k)}_{2j} + \sum_{j=1}^{m_2} B_{2j} \beta^{(k)}_{2j} - \tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha^{(k)}_{2ji_2} + \sum_{j=m_2+1}^{q_{i_2}} B_{2ji_2} \beta^{(k)}_{2ji_2} \right) + \cdots +$$

$$\sum_{j=1}^{n_r} A_{rj} \alpha^{(k)}_{rj} + \sum_{j=1}^{m_r} B_{rj} \beta^{(k)}_{rj} - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha^{(k)}_{rji_r} + \sum_{j=m_r+1}^{q_{i_r}} B_{rji_r} \beta^{(k)}_{rji_r} \right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \to 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \to \infty$ where $i = 1, \cdots, r$:

$\alpha_i = \min_{\substack{1 \leqslant k \leqslant m_i \\ 1 \leqslant j \leqslant m^{(i)}}} Re \left(\sum_{h=2}^{r} \sum_{h'=1}^{h} B_{hj} \frac{b_{hj}}{\beta^{h'}_{hj}} + D^{(i)}_k \frac{d^{(i)}_k}{\delta^{(i)}_k} \right)$ and $\beta_i = \max_{\substack{1 \leqslant k \leqslant n_i \\ 1 \leqslant j \leqslant n^{(i)}}} Re \left(\sum_{h=2}^{r} \sum_{h'=1}^{h} A_{hj} \frac{a_{hj} - 1}{\alpha^{h'}_{hj}} + C^{(i)}_k \frac{c^{(i)}_k - 1}{\gamma^{(i)}_k} \right)$

Remark 1.
If $m_2 = n_2 = \cdots = m_{r-1} = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the generalized multivariable Gimel-function reduces in the generalized multivariable Aleph- function (extension of multivariable Aleph-function defined by Ayant [1]).

Remark 2.
If $m_2 = n_2 = \cdots = m_r = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \, = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [7]).

Remark 3.
If$A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [6]).

Remark 4.

If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and Panda [11,12].
In your investigation, we shall use the following notations.

$$\mathbb{A} = [(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A = [(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B} = [(b_{2j};\beta_{2j}^{(1)},\beta_{2j}^{(2)};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta_{3j}^{(1)},\beta_{3j}^{(2)},\beta_{3j}^{(3)};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{(r-1)j};\beta_{(r-1)j}^{(1)},\cdots,\beta_{(r-1)j}^{(r-1)};B_{(r-1)j})_{1,m_{r-1}}],$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{m_{r-1}+1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B} = [(\mathrm{b}_{rj};\beta_{rj}^{(1)},\cdots,\beta_{rj}^{(r)};B_{rj})_{1,m_r}],[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{m_r+1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B} = [(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U = m_2,n_2;m_3,n_3;\cdots;m_{r-1},n_{r-1};V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

The general class of polynomials $S_n^m(x)$ introduced by Srivastava [9] has been further generalized by Srivastava and Garg [10] to a multivariable polynomial in the following manner [10] :

$$S_N^{M_1,\cdots,M_s}(x_1,\cdots,x_s) = \sum_{K_1,\cdots,K_s=0}^{M_1K_1+\cdots+M_sK_s\leqslant n}(-N)_{M_1K_1+\cdots+M_sK_s}A(N,K_1,\cdots,K_s)\frac{x_1^{K_1}\cdots x_s^{K_s}}{K_1!\cdots K_s!} \tag{1.13}$$

where $M_1,\cdots,M_s$ are arbitrary positive integers and coefficients $A(N;K_1,\cdots,K_s)(N,K_i\geqslant 0,i=1,\cdots,s)$

are arbitrary constants, real or complex.

We shall note

$$A = (-N)_{M_1K_1+\cdots+M_sK_s}A(N,K_1,\cdots,K_s) \tag{1.14}$$

2. Required results.

We shall require the following integral ([4], p. 450), ([3], p. 10), ([5], p. 71) and ([8], p. 254) for the evaluation of our main integrals :

Lemma 1.

$$\int_0^{\frac{\pi}{2}} e^{\omega(a+b)\theta}(\sin\theta)^{a-1}(\cos\theta)^{b-1}\mathrm{d}\theta = \frac{e^{\frac{\pi}{2}\omega a}\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{2.1}$$

provided $Re(a), Re(b) > 0$.

Lemma 2.

$$\int_0^{\frac{\pi}{2}} (1+a\sin^2\theta)^{-\alpha-\beta}(\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}\mathrm{d}\theta = \frac{(1+a)^{-\alpha}\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{2.2}$$

provided $Re(\alpha), Re(\beta) > 0, a > -1$.

Lemma 3.

$$\int_0^{\frac{\pi}{2}} e^{\omega(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}{}_2F_1[a,b;\beta;e^{\omega\theta}\cos\theta]\mathrm{d}\theta = \frac{e^{\frac{pi}{2}\omega\alpha}\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta-a-b)}{\Gamma(\alpha+\beta-a)\Gamma(\alpha+\beta-b)} \tag{2.3}$$

provided $Re(\alpha), Re(\beta), Re(\alpha+\beta-a-b) > 0$.

Lemma 4.

$$\int_0^1 x^{\lambda-1}(1-x)^{a-2\lambda}(1+\mu x)^{\lambda-a-1}{}_2F_1\left[a,b;1+a-b;\frac{(1+\mu)x}{1+\mu x}\right]\mathrm{d}x =$$

$$\frac{2^{a-2\lambda}(1+\mu)^{-\lambda}\Gamma(\lambda)\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)\Gamma\left(\frac{1+a}{2}-\lambda\right)}{\sqrt{\pi}\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)\Gamma\left(1+\frac{a}{2}+b-\lambda\right)\Gamma\left(1+\frac{a}{2}-b-\lambda\right)} \tag{2.4}$$

provided $\mu > -1, Re(\lambda), Re(1+a-2), Re(1-2b) > 0$.

3. Main integrals.

We shall evaluate the following general and new integrals :

Theorem 1.

$$\int_0^1\int_0^1 x^{c-1}y^{\rho-1}(1-x)^{a-2c}(1+\mu x)^{c-a-1}(1-y^2)^{\sigma-1}\left[\sqrt{1-y^2}+\omega y\right]^{\rho+2\sigma}{}_2F_1\left[a,b;1+a-b;\frac{(1+\mu)x}{1+\mu x}\right]$$

$$S_N^{M_1,\cdots,M_s}\left[t_1y^{u_1}(1-y^2)^{v_1}\left[\sqrt{1-y^2}+\omega y\right]^{u_1+2v_1},\cdots,t_sy^{u_s}(1-y^2)^{v_s}\left[\sqrt{(1-y^2}+\omega y\right]^{u_s+2v_s}\right]$$

$$\gimel\left(z_1y^{\eta_1}(1-y^2)^{\zeta_1}\left[\sqrt{1-y^2}+\omega y\right]^{\eta_1+2\zeta_1}\left\{\frac{x(1+\mu x}{(1-x)^2}\right\}^{\theta_1},\cdots,z_ry^{\eta_r}(1-x^2)^{\zeta_r}\left[\sqrt{1-y^2}+\omega y\right]^{\eta_r+2\zeta_r}\left\{\frac{x(1+\mu x}{(1-x)^2}\right\}^{\theta_r}\right)$$

$$\mathrm{d}x\mathrm{d}y = \frac{2^{a-2c}(1+\mu)^{-c}\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)e^{\rho\omega\frac{\pi}{2}}}{\sqrt{\pi}\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)}\sum_{K_1,\cdots,K_s=0}^{M_1K_1+\cdots+M_sK_s\leqslant n} A$$

$$\prod_{j=1}^{s}\left[\frac{t_j^{K_j}e^{\frac{\pi}{2}\omega u_i K_i}}{K_i!}\right]\gimel^{U;m_r+2,n_r+3:V}_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1\frac{e^{\eta_1\frac{\pi}{2}\omega}}{(4(1+\mu))^{\theta_1}} & \mathbb{A};(1-c;\theta_1,\cdots,\theta_r;1),(1-\rho-\sum_{j=1}^{s}K_ju_j;\eta_1,\cdots,\eta_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\frac{e^{\eta_r\frac{\pi}{2}\omega}}{(4(1+\mu))^{\theta_r}} & \mathbb{B};(\frac{1+a}{2}-c;\theta_1,\cdots,\theta_r;1),(1+\frac{a}{2}-b-c;\theta_1,\cdots,\theta_r;1),\mathbf{B}, \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-}2\sigma-2\sum_{j=1}^{s}K_jv_j;2\zeta_1,\cdots,2\zeta_r;1),\mathbf{A},(1+a-b-c;\theta_1,\cdots,\theta_r;1):A \\ \cdot \\ \cdot \\ (1\text{-}\rho-2\sigma-\sum_{j=1}^{s}(u_j+2v_j)K_j;\eta_1+2\zeta_1,\cdots,\eta_r+2\zeta_r;1):B \end{array}\right) \tag{3.1}$$

provided

$\mu>-1, Re(1-2b)>0,\theta_i,\eta_i,\zeta_i>0;(i=1,\cdots,r)\min\{Re(u_j),Re(v_j)\}\geqslant 0,(j=1,\cdots,s).$

$$Re(c)+\sum_{i=1}^{r}\theta_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re(\rho)+\sum_{i=1}^{r}\eta_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re(\sigma)+\sum_{i=1}^{r}\zeta_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re(a-2c+1)-\sum_{i=1}^{r}\theta_i\max_{\substack{1\leqslant k\leqslant n_i\\1\leqslant j\leqslant n^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+C_k^{(i)}\frac{c_k^{(i)}-1}{\gamma_k^{(i)}}\right)<0 \text{ and}$$

$$\left|arg\left(z_iy^{\eta_1}(1-y^2)^{\zeta_i}\left[\sqrt{1-y^2}+\omega y\right]^{\eta_i+2\zeta_i}\left\{\frac{x(1+\mu x)}{(1-x)^2}\right\}^{\theta_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi \text{ where } A_i^{(k)} \text{ is defined by (1.4).}$$

Proof

To establish the integral (2.1), we first use the series representation of the multivariable polynomial with the help of (1.13) and express the generalized multivariable Gimel-function as Mellin-Barnes multiple integrals contour with the the help of (1.1), interchanging the order of summation and integration which is justified under the conditions mentioned above, we have (say I)

$$I=\sum_{K_1,\cdots,K_s=0}^{M_1K_1+\cdots+M_sK_s\leqslant n}A\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}$$

$$\left[\int_0^1 x^{c+\sum_{i=1}^{r}\theta_is_i-1}(1-x)^{a-2c-2\sum_{i=1}^{r}\theta_is_i}(1+\mu x)^{c+\sum_{i=1}^{r}\theta_is_i-a-1}(1-y^2)^{\sigma-1}{}_2F_1\left[a,b;1+a-b;\frac{(1+\mu)x}{1+\mu x}\right]\mathrm{d}x\right]$$

$$\left[\int_0^1 y^{\rho+\sum_{j=1}^{s}K_ju_j+\sum_{i=1}^{r}\eta_is_i-1}(1-y^2)^{\sigma+\sum_{j=1}^{s}K_jv_j+\sum_{i=1}^{r}\zeta_is_i-1}\left[\sqrt{1-y^2}+\omega y\right]^{\rho+2\sigma+\sum_{j=1}^{s}K_j(u_j+2v_j)+\sum_{i=1}^{r}(\eta+2\zeta_is_i)}\mathrm{d}y\right]$$

$$\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.2}$$

Evaluating the x and y-integrals with the help of Lemmae 4 and 2 respectively and Interpreting the resulting expression with the help of (1.1), we obtain the desired theorem 1.

Theorem 2.

$$\int_0^1\int_0^1 x^{c-1}y^{2\rho-1}(1-x)^{a-2c}(1+uy^2)^{-\rho-\sigma}(1+\mu x)^{c-a-1}(1-y^2)^{\sigma-1}{}_2F_1\left[a,b;1+a-b;\frac{(1+\mu)x}{1+\mu x}\right]$$

$$S_N^{M_1,\cdots,M_s}\left[t_1y^{2u_1}(1-y^2)^{v_1}(1+uy^2)^{-u_1-v_1},\cdots,t_sy^{2u_s}(1-y^2)^{v_s}(1+uy^2)^{-u_s-v_s}\right]$$

$$\daleth\left(z_1y^{2\eta_1}(1-y^2)^{\zeta_1}(1+uy^2)^{\eta_1+\zeta_1}\left\{\frac{x(1+\mu x)}{(1-x)^2}\right\}^{\theta_1},\cdots,z_ry^{2\eta_r}(1-y^2)^{\zeta_r}(1+uy^2)^{\eta_r+\zeta_r}\left\{\frac{x(1+\mu x)}{(1-x)^2}\right\}^{\theta_r}\right)$$

$$\mathrm{d}x\mathrm{d}y=\frac{2^{a-2c-1}(1+\mu)^{-c}\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)(1+u)^{-\rho}}{\sqrt{\pi}\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)}\sum_{K_1,\cdots,K_s=0}^{M_1K_1+\cdots+M_sK_s\leqslant n}A$$

$$\prod_{j=1}^{s}\left[\frac{t_j^{K_j}(1+u)^{,-u_iK_i}}{K_i!}\right]\daleth^{U;m_r+2,n_r+3:V}_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|l} \mathrm{z}_1\frac{(1+u)^{\eta_1}}{(4(1+\mu))^{\theta_1}} & \mathbb{A};(1-c;\theta_1,\cdots,\theta_r;1),(1-\rho-\sum_{j=1}^{s}K_ju_j;\eta_1,\cdots,\eta_r;1),\\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r\frac{(1+u)^{\eta_r}}{(4(1+\mu))^{\theta_r}} & \mathbb{B};\left(\frac{1+a}{2}-c;\theta_1,\cdots,\theta_r;1\right),\left(1+\frac{a}{2}-b-c;\theta_1,\cdots,\theta_r;1\right),\mathbf{B}, \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-}\sigma-\sum_{j=1}^{s}K_jv_j;\zeta_1,\cdots,\zeta_r;1),\mathbf{A},(1+a-b-c;\theta_1,\cdots,\theta_r;1):A \\ \cdot \\ \cdot \\ (1\text{-}\rho-\sigma-\sum_{j=1}^{s}(u_j+v_j)K_j;\eta_1+\zeta_1,\cdots,\eta_r+\zeta_r;1):B \end{array}\right) \tag{3.3}$$

Provided

$u>-1,Re(1-2b)>0,\theta_i,\eta_i,\zeta_i>0;(i=1,\cdots,r)\min\{Re(u_j),Re(v_j)\}\geqslant 0,(j=1,\cdots,s).$

$$Re(c)+\sum_{i=1}^{r}\theta_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re(\rho)+\sum_{i=1}^{r}\eta_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re(\sigma)+\sum_{i=1}^{r}\zeta_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re(a-2c+1)-\sum_{i=1}^{r}\theta_i\max_{\substack{1\leqslant k\leqslant n_i\\1\leqslant j\leqslant n^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+C_k^{(i)}\frac{c_k^{(i)}-1}{\gamma_k^{(i)}}\right)<0 \text{ and}$$

$$\left|arg\left(z_iy^{2\eta_i}(1-y^2)^{\zeta_i}(1+uy^2)^{\eta_i+\zeta_i}\left\{\frac{x(1+\mu x)}{(1-x)^2}\right\}^{\theta_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi \text{ where } A_i^{(k)} \text{ is defined by (1.4).}$$

Theorem 3.

$$\int_0^1\int_0^1 x^{c-1}y^{\rho-1}(1-x)^{a-2c}(1+\mu x)^{c-a-1}(1-y^2)^{\sigma-1}\left[\sqrt{1-y^2}+\omega y\right]^{\rho+2\sigma}{}_2F_1\left[a,b;1+a-b;\frac{(1+\mu)x}{1+\mu x}\right]$$

$${}_2F_1\left[\alpha,\beta;\rho+\sigma;\sqrt{1-y^2}[\sqrt{1-y^2}+\omega y\right]S_N^{M_1,\cdots,M_s}\left[t_1y^{u_1}\left[\sqrt{1-y^2}+\omega y\right]^{u_1},\cdots,t_sy^{u_s}\left[\sqrt{(1-y^2}+\omega y\right]^{u_s}\right]$$

$$\daleth\left(z_1y^{u_1}\left[\sqrt{1-y^2}+\omega y\right]^{u_1},\cdots,z_ry^{u_r}\left[\sqrt{1-y^2}+\omega y\right]^{u_r}\right)\mathrm{d}x\mathrm{d}y=$$

$$\frac{2^{a-2c}(1+\mu)^{-c}\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)\Gamma(2\sigma)e^{\rho\omega\frac{\pi}{2}}}{\sqrt{\pi}\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)}\sum_{K_1,\cdots,K_s=0}^{M_1K_1+\cdots+M_sK_s\leqslant n}A\prod_{j=1}^{s}\left[\frac{t_j^{K_j}e^{\frac{\pi}{2}\omega u_iK_i}}{K_i!}\right]$$

$$\daleth^{U;m_r+2,n_r+3:V}_{X;p_{i_r}+4,q_{i_r}+4,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c}z_1\frac{e^{\eta_1\frac{\pi}{2}\omega}}{(4(1+\mu))^{\theta_1}} & \mathbb{A};(1-c;\theta_1,\cdots,\theta_r;1),(1-\rho-\sum_{j=1}^{s}K_ju_j;\eta_1,\cdots,\eta_r;1),\\ \cdot & \cdot\\ \cdot & \cdot\\ z_r\frac{e^{\eta_r\frac{\pi}{2}\omega}}{(4(1+\mu))^{\theta_r}} & \mathbb{B};(\frac{1+a}{2}-c;\theta_1,\cdots,\theta_r;1),\left(1+\frac{a}{2}-b-c;\theta_1,\cdots,\theta_r;1\right),\mathbf{B},\end{array}\right.$$

$$\left.\begin{array}{c}(1\text{-}\rho-\sigma-2\sum_{j=1}^{s}K_jv_j+\alpha+\beta;\eta_1,\cdots,\eta_r;1),\mathbf{A},(1+a-b-c;\theta_1,\cdots,\theta_r;1):A\\ \cdot\\ \cdot\\ (1\text{-}\rho-\sigma-\sum_{j=1}^{s}u_jK_j;+\alpha;\eta_1,\cdots,\eta_r;1),(1-\rho-\sigma-\sum_{j=1}^{s}u_jK_j;+\beta;\eta_1,\cdots,\eta_r;1):B\end{array}\right) \tag{3.4}$$

provided

$\mu>-1,Re(\rho+2\sigma-a-b)>0,\theta_i,\eta_i,\zeta_i>0;(i=1,\cdots,r)\min\{Re(u_j),Re(v_j)\}\geqslant 0,(j=1,\cdots,s).$

$$Re(c)+\sum_{i=1}^{r}\theta_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re(\rho)+\sum_{i=1}^{r}\eta_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re(\sigma)+\sum_{i=1}^{r}\zeta_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re(a-2c+1)-\sum_{i=1}^{r}\theta_i\max_{\substack{1\leqslant k\leqslant n_i\\1\leqslant j\leqslant n^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+C_k^{(i)}\frac{c_k^{(i)}-1}{\gamma_k^{(i)}}\right)<0 \text{ and}$$

$\left|arg\left(z_iy^{u_i}\left[\sqrt{1-y^2}+\omega y\right]^{u_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

The proofs of theorems 2 and 3 are similar to that theorem 1 with the difference that here we make use of (2.2) and (2.4), (2.3) and (2.4) respectively instead of (1.3) and (1.6).

4. Special cases.

Taking $M_j=0,t_j=1(j=2,\cdots,s)$ and replacing $A(N;K_1,\cdots,K_s)$ by $A(N;K)$ therein,ollowing we arrive at the following general integrals

Corollary 1.

$$\int_0^1\int_0^1x^{c-1}y^{\rho-1}(1-x)^{a-2c}(1+\mu x)^{c-a-1}(1-y^2)^{\sigma-1}\left[\sqrt{1-y^2}+\omega y\right]^{\rho+2\sigma}{}_2F_1\left[a,b;1+a-b;\frac{(1+\mu)x}{1+\mu x}\right]$$

$${}_2F_1\left[a,b;1+a-b;\frac{(1+\mu)x}{1+\mu x}\right]S_N^M\left[ty^u(1-y^2)^v\left[\sqrt{1-y^2}+\omega y\right]^{u+2v}\right]$$

$$\gimel\left(z_1y^{\eta_1}(1-y^2)^{\zeta_1}\left[\sqrt{1-y^2}+\omega y\right]^{\eta_1+2\zeta_1}\left\{\frac{x(1+\mu x)}{(1-x)^2}\right\}^{\theta_1},\cdots,z_ry^{\eta_r}(1-x^2)^{\zeta_r}\left[\sqrt{1-y^2}+\omega y\right]^{\eta_r+2\zeta_r}\left\{\frac{x(1+\mu x)}{(1-x)^2}\right\}^{\theta_r}\right)$$

$$\mathrm{d}x\mathrm{d}y=\frac{2^{a-2c}(1+\mu)^{-c}\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)e^{\rho\omega\frac{\pi}{2}}}{\sqrt{\pi}\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)}\sum_{K=0}^{M/N]}(-N)_{MK}A(N;K)\frac{T^K}{K!}$$

$$e^{\frac{\pi}{2}\omega uK}\gimel^{U;m_r+2,n_r+3:V}_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c}\mathrm{z}_1\frac{e^{\eta_1\frac{\pi}{2}\omega}}{(4(1+\mu))^{\theta_1}} & \mathbb{A};(1-c;\theta_1,\cdots,\theta_r;1),(1-\rho-KU;\eta_1,\cdots,\eta_r;1),\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r\frac{e^{\eta_r\frac{\pi}{2}\omega}}{(4(1+\mu))^{\theta_r}} & \mathbb{B};\left(\frac{1+a}{2}-c;\theta_1,\cdots,\theta_r;1\right),\left(1+\frac{a}{2}-b-c;\theta_1,\cdots,\theta_r;1\right),\mathbf{B},\end{array}\right.$$

$$\left.\begin{array}{c}(1\text{-}2\sigma-2Kv;\zeta_1,\cdots,2\zeta_r;1),\mathbf{A},(1+a-b-c;\theta_1,\cdots,\theta_r;1):A\\ \cdot\\ \cdot\\ (1\text{-}\rho-2\sigma-(u+2v)K;\eta_1+2\zeta_1,\cdots,\eta_r+2\zeta_r;1):B\end{array}\right) \quad (4.1)$$

Taking $\theta=0$ in the (3.1), and evaluate the x-integral, we obtain

Corollary 2.

$$\int_0^1 y^{\rho-1}(1-y^2)^{\sigma-1}\left[\sqrt{1-y^2}+\omega y\right]^{\rho+2\sigma}$$

$$S_N^{M_1,\cdots,M_s}\left[t_1y^{u_1}(1-y^2)^{v_1}\left[\sqrt{1-y^2}+\omega y\right]^{u_1+2v_1},\cdots,t_sy^{u_s}(1-y^2)^{v_s}\left[\sqrt{(1-y^2}+\omega y\right]^{u_s+2v_s}\right]$$

$$\gimel\left(z_1y^{\eta_1}(1-y^2)^{\zeta_1}\left[\sqrt{1-y^2}+\omega y\right]^{\eta_1+2\zeta_1},\cdots,z_ry^{\eta_r}(1-x^2)^{\zeta_r}\left[\sqrt{1-y^2}+\omega y\right]^{\eta_r+2\zeta_r}\right)\mathrm{d}y=$$

$$e^{\rho\omega\frac{\pi}{2}}\sum_{K_1,\cdots,K_s=0}^{M_1K_1+\cdots+M_sK_s\leqslant n}A\prod_{j=1}^{s}\left[\frac{t_j^{K_j}e^{\frac{\pi}{2}\omega u_iK_i}}{K_i!}\right]\gimel^{U;m_r+2,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c}\mathrm{z}_1e^{\eta_1\frac{\pi}{2}\omega} & \mathbb{A};(1-\rho-\sum_{j=1}^s K_ju_j;\eta_1,\cdots,\eta_r;1),\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_re^{\eta_r\frac{\pi}{2}\omega} & \mathbb{B};\mathbf{B},\end{array}\right.$$

$$\left.\begin{array}{c}(1\text{-}2\sigma-2\sum_{j=1}^s K_jv_j;2\zeta_1,\cdots,2\zeta_r;1),\mathbf{A}:A\\ \cdot\\ \cdot\\ (1\text{-}\rho-2\sigma-\sum_{j=1}^s(u_j+2v_j)K_j;\eta_1+2\zeta_1,\cdots,\eta_r+2\zeta_r;1):B\end{array}\right) \quad (4.2)$$

5. Conclusion.

Similar type of integrals would follow from (2.2) and (2.3). The integrals (3.1), (3.2) and (3.3) are also quite general in nature. By suitably specializing the arbitrary coefficients in the general class of polynomials and the parameters of the generalized multivariable gimel-function, a large number of integrals can be evaluated.

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