A General Solution of Pell's Equation

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Abstract
Pell's equation is a quadratic Diophantine equation with two unknowns. I provide a general solution of this equation.

Keywords: *Pell's Equation ; General Solution*

I. INTRODUCTION

Pell's equation is anwell known Diophantine equation of the form [1]

$$x^2 - dy^2 = 1$$

where $x, y \in \mathbb{Z}$ and d is a given natural number, which is not a square.

This type of equation was also found in ancient Indian Mathematics as named 'Vargaprakriti'. The great Indian Mathematician Baudhayana noted that x = 577 and y = 408 is a solution of the equation $x^2 - 2y^2 = 1$ in 4^{th} century and later Brahmagupta in 7^{th} century and Bhaskaracharya in 12^{th}century also studied on it.

II. EXISTING SOLUTIONS

If one is able to find a specific solutions x_1 and y_1, then using it one can obtain infinite number of solutions through the following formulae :

$$x_n = \frac{z^n + \overline{z}^n}{2} \quad ; \quad y_n = \frac{z^n - \overline{z}^n}{2\sqrt{d}}$$

where

$$z = (x_1 + y_1\sqrt{d}) \quad and \quad \overline{z} = (x_1 - y_1\sqrt{d}) \quad for\ n \geq 1.$$

But we have to find x_1 and $y_1$analytically.

III. NEW ANALYTICAL METHOD OF THE SOLUTION

Let us consider the Pell's equation

$$x^2 - dy^2 = 1,$$

$$or,\ 4x^2 - 4dy^2 = 4,$$

$$or,\ 4x^2 - (y^2 + d)^2 + (y^2 - d)^2 = 4,$$

$$or,\ (2x)^2 + (y^2 - d)^2 = (y^2 + d)^2 + 2^2.$$

This equation is of the form

$$a^2 + b^2 = r^2 + s^2,$$

whose general solutions are given by

$$a = (mn + pq) \quad , \quad b = (mp - nq),$$

$$r = (mp + nq) \quad , \quad s = (mn - pq).$$

So putting these values, we get

$$x = \frac{1}{2}(mn + pq) \quad , \quad y^2 = mp$$

$$d = nq \quad and \quad 2 = (mn - pq).$$

We have two known quantities dand2, so one can easily get

$$x = (mn - 1) \quad , \quad y = \sqrt{\frac{m^2n^2 - 2mn}{d}}.$$

Setting mn = t , we get the general one parameter solution as

$$x = t - 1 \quad and \quad y = \sqrt{\frac{t^2 - 2t}{d}},$$

where $t > 2 \ \& \ d \mid t(t-2)$ and the quotient must be a perfect square of an integer.

IV.EXAMPLES

Now we provide some examples:

Example - 1: Solve for positive integers $x^2 - 2y^2 = 1$.

Answer: Here the general solutions are x = t – 1 and $y = \sqrt{\frac{t^2-2t}{2}}$ for suitable choice of t. If one chooses t = 4, then one obtain x = 3 and y = 2.

Similarly, for the other values of t say, t= 18 and t=100, one will find x= 17 &y= 12 and x = 99 &y=70 respectively and so on.

In the ancient time (4th century) Baudhayana obtained the solutions of the above equation as x= 577 and y = 408. Here just we have to choose t = 578.

Example - 2: Solve for positive integers $x^2 - 3y^2 = 1.$

Answer : Again we have the general solutions x= t – 1 and $y = \sqrt{\frac{t^2-2t}{3}}$. If we choose t= 3, 8, 7 respectively, then we can obtainx= 2 & y= 1; x= 7 & y= 4; x= 26 & y=15 respectively.

Example – 3 : Solve for positive integers $x^2 - 5y^2 = 1.$

Answer : If we choose t= 10, 62 then we can obtain x= 9 & y = 4 ; x= 161 &y= 72 respectively.

Example – 4 : Solve for positive integers $x^2 - 92y^2 = 1.$

Answer : In the seventh century Brahmagupta obtained the solutions x=1151 and y=120. If we choose t = 1152 then we can obtain the same solution.

Example – 5 : Solve for positive integers $x^2 - 61y^2 = 1.$

Answer: In the twelfth century Bhaskara found the solutions x=1766319049 and y=226153980. If we choose t = 1766319050 then we can obtain the same solution.

V. CONCLUDING REMARKS

The choice of t depends on the value of d.

As $\sqrt{\frac{t^2-2t}{d}}$ be an integer, so we just have to choose such a t, for which $d \mid t(t-2)$and the quotient be a perfect square of an integer.

Also I can provide some general form of t for some specific form of d

If d is of the form (k^2+1), then t is of the form $2(k^2+1)$.
If d is of the form (k^2-1), then t is of the form $(k+1)$.
If d is of the form (k^2+2), then t is of the form (k^2+2) .
If d is of the form (k^2-2), then t is of the form k^2 .
[here, k is a positive integer]

These are some specific forms of t. In a future work, one can try to find a general form t as a function of d.

ACKNOWLEDGEMENTS

I would like to thank my father Prof. Dr. FarookRahaman for several illuminating discussions.

REFERENCES

[1] An Introduction to Diophantine Equations, TituAndreescu, DorinAndrica and Ion Cucurezeanu, Birkh¨auser (2010).