Some Expansions for the Multivariable Gimel-Function in Series Involving Jacobi Polynomials

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ABSTRACT
Shrivastava et al [5] have studied the expansions for the multivariable H-function in series involving Jacobi polynomials. In this paper a few integrals involving product of Jacobi polynomials and the multivariable Gimel-function defined here of general arguments have been evaluated. These integrals have been utilized to establish the expansion formulae for the multivariable Gimel-function in series involving Jacobi polynomials.

Keywords : multivariable Gimel-function, Jacobi polynomials, expansion formula.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We define a generalized transcendental function of several complex variables noted $\gimel$.

$$\gimel(z_1, \cdots, z_r) = \gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array} \right|$$

$$[(a_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(a_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots; \quad \ldots$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]: \; [(c^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{1,q_r}]: \; [(d^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left. \begin{array}{l} ;\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}] \\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}] \end{array} \right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha_{2ji_2}^{(k)}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta_{2ji2}^{(k)}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,\cdots,0\leqslant m_r,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)} \in \mathbb{C}; (j = 1, \cdots, n^{(k)}); (k = 1, \cdots, r); d_j^{(k)} \in \mathbb{C}; (j = 1, \cdots, m^{(k)}); (k = 1, \cdots, r).$

$a_{kji_k} \in \mathbb{C}; (j = n_k + 1, \cdots, p_{i_k}); (k = 2, \cdots, r).$

$b_{kji_k} \in \mathbb{C}; (j = 1, \cdots, q_{i_k}); (k = 2, \cdots, r).$

$d_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour L_k is in the $s_k (k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha_{2j}^{(k)} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha_{3j}^{(k)} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha_{rj}^{(i)}\right) (j = 1, \cdots, n_r), \Gamma^{C_j^{(k)}}\left(1 - c_j^{(k)} + \gamma_j^{(k)} s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$to the right of the contour L_k and the poles of $\Gamma^{D_j^{(k)}}\left(d_j^{(k)} - \delta_j^{(k)} s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A_i^{(k)} \pi$ where

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)} \gamma_j^{(k)} - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)} \delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)} \gamma_{ji^{(k)}}^{(k)} \right) +$$

$$-\tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha_{2ji_2}^{(k)} + \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta_{2ji_2}^{(k)} \right) - \cdots - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha_{rji_r}^{(k)} + \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta_{rji_r}^{(k)} \right) \tag{1.4}$$

Following the lines of Braaksma ([3] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$ where $i = 1, \cdots, r :$

$\alpha_i = \min_{1 \leqslant j \leqslant m^{(i)}} Re \left[D_j^{(i)} \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right]$ and $\beta_i = \max_{1 \leqslant j \leqslant n^{(i)}} Re \left[C_j^{(i)} \left(\frac{c_j^{(i)} - 1}{\gamma_j^{(i)}} \right) \right]$

Remark 1.
If $n_2 = \cdots = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ $A_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [2].

Remark 2.
If $n_2 = \cdots = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [7].

Remark 3.
If$A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [6].

Remark 4.

If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and panda [9,10].

In your investigation, we shall use the following notations.

$$\mathbb{A} =[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A=[(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B}=[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B}= [\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B}=[(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U=0,n_2;0,n_3;\cdots;0,n_{r-1};V=m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X=p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

2 Required results.

The following results [1, page 164 (1.6), (1.7)] will be utilized in the present discussion :

Lemma 1.

$$\int_{-1}^{1} x^{\lambda}(1-x)^{\rho}(1+x)^{\beta}\,P_n^{(\alpha,\beta)}(x)\,\mathrm{d}x=(-1)^n 2^{\rho+\beta+1}\frac{\Gamma(\rho+1)\Gamma(n+\beta+1)\Gamma(\rho-\alpha+1)}{n!\Gamma(\rho-\alpha-n+1)\Gamma(\rho+\beta+n+2)}$$

$${}_3F_2\left(\begin{array}{c} -\lambda,\rho-\alpha+1,1 \\ \cdot \\ \rho-\alpha-n+1,\rho+\beta+n+2\end{array};2\right) \tag{2.1}$$

Lemma 2.

$$\int_{-1}^{1} x^{\lambda}(1-x)^{\alpha}(1+x)^{\sigma}\, P_n^{(\alpha,\beta)}(x)\,\mathrm{d}x = (-1)^n 2^{\sigma+\alpha+1}\frac{\Gamma(\sigma+1)\Gamma(n+\alpha+1)\Gamma(\sigma-\beta+1)}{n!\Gamma(\sigma-\beta-n+1)\Gamma(\sigma+\alpha+n+2)}$$

$${}_3F_2\left(\begin{array}{c} -\lambda, \sigma-\beta+1, 1 \\ . \\ \sigma-\beta-n+1, \sigma+\alpha+n+2 \end{array} ; 2\right) \tag{2.2}$$

3. Main integrals

The integral to be evaluate are :

Theorem 1.

$$\int_{-1}^{1} x^{\lambda}(1-x)^{\rho}(1+x)^{\beta}P_v^{(\alpha,\beta)}(x)\, \daleth\left(\begin{array}{c} \mathrm{z}_1 x^{h_1}(1-x)^{\mu_1} \\ \cdots \\ \cdots \\ \mathrm{z}_r x^{h_r}(1-x)^{\mu_r} \end{array}\right)\mathrm{d}x =$$

$$\frac{(-)^v 2^{\rho+\beta+1}\Gamma(\beta+v+1)}{v!}\sum_{k=0}^{\infty}\frac{(-)^k 2^k}{k!}\,\daleth^{U;0,n_r+3:V}_{X;p_{i_r}+3,q_{i_r}+3,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c} \mathrm{z}_1 2^{\mu_1} \\ \cdots \\ \cdots \\ \mathrm{z}_r 2^{\mu_r} \end{array}\left|\begin{array}{c} \mathbb{A}\ ; (\alpha-\rho-k;\mu_1,\cdots,\mu_r;1),(-\rho-k;\mu_1,\cdots,\mu_r;1),(-\lambda;h_1,\cdots,h_r;1),\mathbf{A}:A \\ . \\ . \\ \mathbb{B};\,\mathbb{B},\,(\alpha-\rho+v-k;\mu_1,\cdots,\mu_r;1),(-\rho-\beta-v-k-1;\mu_1,\cdots,\mu_r;1),(k-\lambda;h_1,\cdots,h_r;1)/B \end{array}\right.\right) \tag{3.1}$$

Provided that

$\Gamma(1+\beta), h_i > 0, \mu_i > 0, i = 1,\cdots,r,\ Re(1+\rho)+\sum_{i=1}^{r}\mu_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0$

$Re(1+\lambda)+\sum_{i=1}^{r}h_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0$

$\left|arg\left(z_i(1-x)^{\mu_i}x^{h_i}\right)\right| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 2.

$$\int_{-1}^{1} x^{\lambda}(1-x)^{\alpha}(1+x)^{\sigma}P_v^{(\alpha,\beta)}(x)\ I\left(\begin{array}{c} \mathrm{z}_1 x^{h_1}(1+x)^{\mu_1} \\ \cdots \\ \cdots \\ \mathrm{z}_r x^{h_r}(1+x)^{\mu_r} \end{array}\right)\mathrm{d}x$$

$$= \frac{(-)^v 2^{\sigma+\alpha+1}\Gamma(\alpha+v+1)}{v!}\sum_{k=0}^{\infty}\frac{(-)^k 2^k}{k!}\,\daleth^{U;0,n_r+3:V}_{X;p_{i_r}+3,q_{i_r}+3,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} z_1 2^{\mu_1} & \mathbb{A}\,;(\beta-\sigma-k;\mu_1,\cdots,\mu_r;1),(-\sigma-k;\mu_1,\cdots,\mu_r;1),(-\lambda;h_1,\cdots,h_r;1),\mathbf{A}:A \\ \cdots & \cdot \\ \cdots & \cdot \\ z_r 2^{\mu_r} & \mathbb{B};\mathbf{B},(\beta-\sigma+v-k;\mu_1,\cdots,\mu_r;1),(-\sigma-\alpha-v-k-1;\mu_1,\cdots,\mu_r;1),(k-\lambda;h_1,\cdots,h_r;1):B \end{array}\right) \quad (3.2)$$

provided that

$$\Gamma(1+\alpha), h_i>0, \mu_i>0, i=1,\cdots,r,\ Re(1+\sigma)+\sum_{i=1}^{r}\mu_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$$Re(1+\lambda)+\sum_{i=1}^{r}h_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$$

$\left|arg\left(z_i(1+x)^{\mu_i}x^{h_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Provided that

$h_i>0,\mu_i>0,i=1,\cdots,r\,;|argz_i|<\frac{1}{2}\Omega_i\pi$, where Ω_i is defined by (1.3).

Proof

Let $M\{\}=\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\{\}$

To establish (2.1), express the multivariable gimel-function on the left-hand side as contour integral with the help of (1.1) and interchange the order of integration which is justifiable due to absolute convergence of the integral involved in the process, we get.

$$M\left\{\prod_{i=1}^{r}z^{s_i}\int_{-1}^{1}x^{\lambda+\sum_{i=1}^{r}h_is_i}(1-x)^{\rho+\sum_{i=1}^{r}\mu_is_i}(1+x)^{\beta}P_v^{(\alpha,\beta)}(x)\mathrm{d}x\right\}\mathrm{d}s_1\cdots\mathrm{d}s_r \quad (3.3)$$

evaluating the inner integral with the help of (1.10), we obtain.

$$M\left\{\prod_{i=1}^{r}z^{s_i}\frac{2^{\rho+\beta+1+\sum_{i=1}^{r}\mu_is_i}\Gamma(\beta+v+1)}{v!\Gamma(\rho+1-\alpha-v+\sum_{i=1}^{r}\mu_is_i)}\frac{\Gamma(\rho-\alpha+1+\sum_{i=1}^{r}\mu_is_i)\Gamma(\rho+1+\sum_{i=1}^{r}\mu_is_i)}{\Gamma(\rho+\beta+v+2+\sum_{i=1}^{r}\mu_is_i)}\right.$$

$$\left.{}_3F_2\left(\begin{array}{c}-(\lambda+\sum_{i=1}^{r}h_is_i),\rho-\alpha+1+\sum_{i=1}^{r}\mu_is_i,\rho+1+\sum_{i=1}^{r}\mu_is_i \\ \cdot \\ \cdot \\ \rho-\alpha-v+1+\sum_{i=1}^{r}\mu_is_i,\rho+\beta+v+2+\sum_{i=1}^{r}\mu_is_i\end{array};2\right)\right\}\mathrm{d}s_1\cdots\mathrm{d}s_r \quad (3.4)$$

Now expressing the hypergeometric function as series, changing the order of summation and integration in view of [4, page 176 (75)] which is permissible under the conditions given in (2.1) and applying the definition of multivariable Gimel-function defined in the section I , we obtain the desired result.

By using (1.2), we obtain the formula (2.2) by similar methods.

3. Expansion.

In this section, we establish two expansion formulae for the multivariable I-function in series involving the Jacobi polynomials. These expansions hold good provided

$h_i > 0, \mu_i > 0, i = 1, \cdots, r$; $|argz_i| < \frac{1}{2}\Omega_i\pi$, where Ω_i is defined by (1.3).

The expansions to be established are :

Theorem 3.

$$x^{\lambda}(1-x)^{\rho}\,\beth\left(\begin{array}{c} z_1 x^{h_1}(1-x)^{\mu_1} \\ \cdots \\ \cdots \\ z_r x^{h_r}(1-x)^{\mu_r} \end{array}\right) = 2^{\rho}\sum_{s,k=0}^{\infty}\frac{(-)^{s+k}2^k\Gamma(\alpha+\beta+s+1)(\alpha+\beta+2s+1)}{\Gamma(\alpha+s+1)k!}P_s^{(\alpha,\beta)}(x)$$

$$\beth_{X;p_{i_r}+3,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;0,n_r+3:V}\left(\begin{array}{c} z_1 2^{\mu_1} \\ \cdot \\ \cdot \\ z_r 2^{\mu_r} \end{array}\right|$$

$$\left.\begin{array}{c} \mathbb{A};\ (\text{-}\alpha-\rho-k;\mu_1,\cdots,\mu_r;1),(-\rho-k;\mu_1,\cdots,\mu_r;1),(-\lambda;h_1,\cdots,h_r;1),\mathbf{A}:A \\ \cdot \\ \cdot \\ \mathbb{B};\ \mathbf{B}\ ,\ (\text{-}\rho+s-k;\mu_1,\cdots,\mu_r;1),(-\rho-\alpha-\beta-s-k-1;\mu_1,\cdots,\mu_r;1),(k-\lambda;h_1,\cdots,h_r;1):B \end{array}\right) \tag{4.1}$$

under the conditions mentioned in (3.1).

Theorem 4.

$$x^{\lambda}(1+x)^{\sigma}\,\beth\left(\begin{array}{c} z_1 x^{h_1}(1+x)^{\mu_1} \\ \cdots \\ \cdots \\ z_r x^{h_r}(1+x)^{\mu_r} \end{array}\right) = 2^{\rho}\sum_{s,k=0}^{\infty}\frac{(-)^{s+k}2^k\Gamma(\alpha+\beta+s+1)(\alpha+\beta+2s+1)}{\Gamma(\beta+s+1)k!}P_s^{(\alpha,\beta)}(x)$$

$$\beth_{X;p_{i_r}+3,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;0,n_r+3:V}\left(\begin{array}{c} z_1 2^{\mu_1} \\ \cdot \\ \cdot \\ z_r 2^{\mu_r} \end{array}\right|$$

$$\left.\begin{array}{c} \mathbb{A}\ ;\ (\text{-}\beta-\sigma-k;\mu_1,\cdots,\mu_r;1),(-\sigma-k;\mu_1,\cdots,\mu_r;1),(-\lambda;h_1,\cdots,h_r;1),\mathbf{A}:A \\ \cdot \\ \cdot \\ \mathbb{B};\ \mathbf{B},\ (\text{-}\sigma+s-k;\mu_1,\cdots,\mu_r;1),(-\sigma-\alpha-\beta-s-k-1;\mu_1,\cdots,\mu_r;1),(k-\lambda;h_1,\cdots,h_r;1):B \end{array}\right) \tag{4.2}$$

under the conditions mentioned in (3.2).

Proof

To obtain (3.1), let

$$x^{\lambda}(1-x)^{\rho}\,\gimel\left(\begin{array}{c} z_1 x^{h_1}(1-x)^{\mu_1} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ z_r x^{h_r}(1-x)^{\mu_r} \end{array}\right) = \sum_{s=0}^{\infty} M_s P_s^{(\alpha,\beta)}(x) \tag{4.3}$$

The equation is valid since the expression on the left-hand side is continuous and is of bounded variation in the open interval $(-1,1)$. Multiplying both sides of (3.3) by $(1-x)^{\alpha}(1+x)^{\beta}P_v^{(\alpha,\beta)}(x)$, integrating with respect to x between the limit -1 to 1, on the right-hand side using the orthogonality property for Jacobi polynomial [5 page 285 (5 and 9)] and on the left-hand side using (3.1), we get.

$$M_s = \frac{(-)^s 2^{\rho}\Gamma(\alpha+\beta+s+1)(\alpha+\beta+2s+1)}{\Gamma(\alpha+s+1)k!}\sum_{k=0}^{\infty}\frac{(-)^k 2^k}{k!}\,\gimel^{U;0,n_r+3:V}_{X;p_{i_r}+3,q_{i_r}+3,\tau_{i_r}:R_r:Y}\left(\begin{array}{c} z_1 2^{\mu_1} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ z_r 2^{\mu_r} \end{array}\middle|\right.$$

$$\left.\begin{array}{c} \mathbb{A};\, (-\alpha-\rho-k;\mu_1,\cdots,\mu_r;1),(-\rho-k;\mu_1,\cdots,\mu_r;1),(-\lambda;h_1,\cdots,h_r;1),\mathbf{A}:A \\ \cdot \\ \cdot \\ \mathbb{B};\,\mathbf{B}\,,(-\rho+s-k;\mu_1,\cdots,\mu_r;1),(-\rho-\alpha-\beta-s-k-1;\mu_1,\cdots,\mu_r;1),(k-\lambda;h_1,\cdots,h_r;1):B \end{array}\right) \tag{4.4}$$

Substituting the value of M_s from (4.4) in (4.3), we obtain the desired result. By using (3.2), we obtain (4.2) by similar methods.

Remarks :

We obtain the same integrals and expansions of series of Jacobi polynomials with the multivariable Aleph-function defined by Ayant [2], the multivariable I-function defined by Prasad [6] and Pratima [7], the multivariable H-function defined by Srivastava and Panda [9,10]. Concerning the last function, see Shrivastava and Nigam [8] for more details.

3. Conclusion

The multivariable Gimel-function presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions o several variables such as multivariable I-function ,multivariable Fox's H-function, Fox's H-function , Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, MacRobert's E-function, generalized hypergeometric function, Bessel function of first kind, modied Bessel function, Whittaker function, exponential function , binomial function etc. as its special cases, and therefore, various unified integral presentations and expansion in series of Jacobi polynomials can be obtained as special cases of our results.

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