Some Integrals Involving Multivariable Gimel-Function

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Abstract
The object of the present paper is to evaluate two integrals involving multivariable Gimel-function defined here. The integrals are quite general character. A number of integrals can be evaluated by appropriately reducing the multivariable Gimel-function involved into simpler functions.

Keywords :Multivariable Gimel-function, integrals contour, double integrals.

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1. Introduction and preliminaries.

The object of this document is to evaluate two finite double integrals involving the multivariable Gimel-function.

Throughout this paper, let $\mathbb{C},\mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

We define a generalized transcendental function of several complex variables noted $\gimel$.

$$\gimel(z_1,\cdots,z_r)=\gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2}:R_2;p_{i_3},q_{i_3},\tau_{i_3}:R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|}\mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\end{array}\right.$$

$$[(\mathrm{a}_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots;\quad\cdots$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{1,q_r}]:\ [(\mathrm{d}^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}]\\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha^{(k)}_{2ji_2}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,\cdots,0\leqslant m_r,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n^{(k)});(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$a_{kji_k} \in \mathbb{C}; (j = n_k + 1, \cdots, p_{i_k}); (k = 2, \cdots, r).$

$b_{kji_k} \in \mathbb{C}; (j = 1, \cdots, q_{i_k}); (k = 2, \cdots, r).$

$d^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$\gamma^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour L_k is in the $s_k(k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha^{(k)}_{2j} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha^{(k)}_{3j} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha^{(i)}_{rj}\right) (j = 1, \cdots, n_r), \Gamma^{C^{(k)}_j}\left(1 - c^{(k)}_j + \gamma^{(k)}_j s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$to the right of the contour L_k and the poles of $\Gamma^{D^{(k)}_j}\left(d^{(k)}_j - \delta^{(k)}_j s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A^{(k)}_i \pi$ where

$$A^{(k)}_i = \sum_{j=1}^{m^{(k)}} D^{(k)}_j \delta^{(k)}_j + \sum_{j=1}^{n^{(k)}} C^{(k)}_j \gamma^{(k)}_j - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q^{(k)}_i} D^{(k)}_{ji^{(k)}} \delta^{(k)}_{ji^{(k)}} + \sum_{j=n^{(k)}+1}^{p^{(k)}_i} C^{(k)}_{ji^{(k)}} \gamma^{(k)}_{ji^{(k)}} \right) +$$

$$-\tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha^{(k)}_{2ji_2} + \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta^{(k)}_{2ji_2} \right) - \cdots - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha^{(k)}_{rji_r} + \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta^{(k)}_{rji_r} \right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$ where $i = 1, \cdots, r$:

$\alpha_i = \min_{1 \leqslant j \leqslant m^{(i)}} Re \left[D^{(i)}_j \left(\frac{d^{(i)}_j}{\delta^{(i)}_j} \right) \right]$ and $\beta_i = \max_{1 \leqslant j \leqslant n^{(i)}} Re \left[C^{(i)}_j \left(\frac{c^{(i)}_j - 1}{\gamma^{(i)}_j} \right) \right]$

Remark 1.
If $n_2 = \cdots = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ $A_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [1].

Remark 2.
If $n_2 = \cdots = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [8].

Remark 3.
If$A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [7].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and panda [9,10].

In your investigation, we shall use the following notations.

$$\mathbb{A} =[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A = [(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B} = [\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B} = [\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B} = [(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U = 0,n_2;0,n_3;\cdots;0,n_{r-1};V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

2. Required results.

In this section, we give three finite integrals. These results will utilized in the following section.

Lemma 1. ([4] Mathai and Saxena, 1973)

$$\int_0^{\frac{\pi}{2}} e^{\omega(a+b)\theta}(\sin\theta)^{a-1}(\cos\theta)^{b-1}{}_2F_1(c,d,b;e^{\omega\theta}\cos\theta)\mathrm{d}\theta = e^{\frac{1}{2}\omega\pi a}\frac{\Gamma(a)\Gamma(b)\Gamma(a+b-c-d)}{\Gamma(a+b-c)\Gamma(a+b-d)} \tag{2.1}$$

provided $\min\{Re(a),Re(b),Re(a+b-c-d)\} > 0$

Lemma 2. ([3] MacRobert, 1960-1961)

$$\int_0^1 \frac{x^{p-1}(1-x)^{q-1}}{[a'x+b'(1-x)]^{p+q}}{}_2F_1\left(e,f;p;\frac{a'x}{a'x+b'(1-x)}\right)\mathrm{d}x = \frac{\Gamma(p)\Gamma(q)\Gamma(p+q-e-f)}{a'^pb'^q\Gamma(p+q-e)\Gamma(p+q-f)} \tag{2.2}$$

provided $Re(p) > 0, Re(q) > 0, Re(p+q-e-f) > 0, |a'x+b'(1-x)] \neq 0, 0 \leqslant x \leqslant 1.$

Lemma 3. ([5] Munot and Mathur, 1983).

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{2\alpha-1}\theta\cos^{2\beta-1}\theta}{(a\sin^2\theta+b\cos^2\theta)^{\alpha+\beta}} {}_2F_1\left(\gamma,\delta;\beta;\frac{b\cos^2\theta}{a\sin^2\theta+b\cos^2\theta}\right)\mathrm{d}\theta = \frac{1}{2}a^{-\alpha}b^{-\beta}\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta-\gamma-\delta)}{\Gamma(\alpha+\beta-\gamma)\Gamma(\alpha+\beta-\delta)} \tag{2.3}$$

provided $a, b, Re(\alpha), Re(\beta), Re(\alpha+\beta-\gamma-\delta) > 0$

3. Main integrals.

In this section,we evaluate two finite double integrals with general arguments.

Theorem 1.

$$\int_0^1\int_0^{\frac{\pi}{2}} e^{\omega(a+b)\theta)}(\sin\theta)^{a-1}(\cos\theta)^{b-1}{}_2F_1(c,d;b;e^{i\theta}\cos\theta)x^{p-1}(1-x)^{q-1}(a'x+b'(1-x))^{-p-q}$$

$${}_2F_1\left(e,f;p;\frac{a'x}{a'x+b'(1-x)}\right)\gimel\left(\frac{z_1e^{\omega a_1\theta}(\sin\theta)^{a_1}(1-x)^{b_1}}{[a'x+b'(1-x)]^{b_1}},\cdots,\frac{z_re^{\omega a_r\theta}(\sin\theta)^{a_r}(1-x)^{b_r}}{[a'x+b'(1-x)]^{b_r}}\right)\mathrm{d}\theta\mathrm{d}x$$

$$\frac{e^{\omega\pi\frac{a}{2}}\Gamma(b)\Gamma(p)}{a'^pb'^q}\gimel^{U;0,n_r+4:V}_{X;p_{i_r}+4,q_{i_r}+4,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1e^{\omega\pi\frac{a_1}{2}}b'^{-b_1} & \mathbb{A};(1-a;a_1,\cdots,a_r;1),(1-a-b+c+d;a_1,\cdots,a_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_re^{\omega\pi\frac{a_r}{2}}b'^{-b_r} & \mathbb{B};\mathbf{B},(1-a-b+c;a_1,\cdots,a_r;1),(1-a-b+d;a_1,\cdots,a_r;1), \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-q};\mathrm{b}_1,\cdots,b_r;1),(1-p-q+e+f;b_1,\cdots,b_r;1),\mathbf{A}:A \\ \cdot \\ \cdot \\ (1\text{-p-q+e};\mathrm{b}_1,\cdots,b_r;1),(1-p-q+f;b_1,\cdots,b_r;1):B \end{array}\right) \tag{3.1}$$

provided

$a_i,b_i>0(i=1,\cdots,r),[a'x+b'(1-x)]\neq 0,\ Re(a)+\sum_{i=1}^r a_i\min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0.$

$Re(a)+\sum_{i=1}^r a_i\min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0.$

$Re(a+b-c-d)+\sum_{i=1}^r a_i\min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0.$

$Re(q)+\sum_{i=1}^r b_i\min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0.\ Re(p+q-e-f)+\sum_{i=1}^r b_i\min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0$ and

$\left|arg\left(z_i\frac{e^{\omega a_i\theta}\sin^{a_i}\theta(1-x)^{b_i}}{[a'x+b'(1-x)]^{b_i}}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof
To establish the theorem 1, expressing the multivariable Gimel-function in the Mellin-Barnes multiple integrals contour with the help of (1.1) and interchanging the order of integrations, which is justified under the conditions mentioned above, we get (say I)

$$I=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}$$

$$\left[\int_0^{\frac{\pi}{2}}e^{\omega(a+b+\sum_{i=1}^r a_is_i)\theta}(\sin\theta)^{a+\sum_{i=1}^r a_is_i-1}(\cos\theta)^{b-1}{}_2F_1[c,d;b;e^{\omega\theta}\cos\theta]\mathrm{d}\theta\right]$$

$$\left[\int_0^1 x^{p-1}(1-x)^{q+\sum_{i=1}^r b_is_i-1}[a'x+b'(1-x)]^{-p-q-\sum_{i=1}^r b_is_i}\mathrm{d}x\right]\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.2}$$

Now Evaluating the inner integrals with the help of lemmae 1and 2 and interpreting the resulting expression with the help of (1.1), we obtain the desired theorem 1.

Theorem 2.

$$\int_0^1\int_0^{\frac{\pi}{2}}\frac{\sin^{2\alpha-1}\theta\cos^{2\beta-1}\theta}{[a\sin^2\theta+b\cos^2\theta]^{\alpha+\beta}}{}_2F_1\left(\gamma,\delta;\beta;\frac{b\cos^2\theta}{a\sin^2\theta+b\cos^2\theta}\right)$$

$$x^{p-1}(1-x)^{q-1}[a'x+b'(1-x)]^{-p-q}{}_2F_1\left(c,d;p;\frac{a'x}{a'x+b'(1-x)}\right)$$

$$\gimel\left(\frac{z_1(\sin\theta)^{2a_1}(1-x)^{b_1}}{(a\sin^2\theta+b\cos^2\theta)^{a_1}[a'x+b'(1-x)]^{b_1}},\cdots,\frac{z_r(\sin\theta)^{2a_r}(1-x)^{b_r}}{(a\sin^2\theta+b\cos^2\theta)^{a_r}[a'x+b'(1-x)]^{b_r}}\right)\mathrm{d}\theta\mathrm{d}x$$

$$\frac{\Gamma(\beta)\Gamma(p)}{2a^\alpha a'^p b^\beta b'^q}\gimel^{U;0,n_r+4:V}_{X;p_{i_r}+4,q_{i_r}+4,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1a^{-a_1}b'^{-b_1} & \mathbb{A};(1\text{-}\alpha;a_1,\cdots,a_r;1),(1-\alpha-\beta+\gamma+\delta;a_1,\cdots,a_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_ra^{-a_r}b'^{-b_r} & \mathbb{B};\mathbf{B},(1\text{-}\alpha-\beta+\gamma;a_1,\cdots,a_r;1),(1-\alpha-\beta+\delta;a_1,\cdots,a_r;1), \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-q};\mathrm{b}_1,\cdots,b_r;1),(1-p-q+c+d;b_1,\cdots,b_r;1),\mathbf{A}:A \\ \cdot \\ \cdot \\ (1\text{-p-q+c};\mathrm{b}_1,\cdots,b_r;1),(1-p-q+d;b_1,\cdots,b_r;1):B \end{array}\right) \tag{3.3}$$

provided

$$\alpha,\beta,a_i,b_i>0(i=1,\cdots,r),[a'x+b'(1-x)]\neq0,\ Re(\alpha)+\sum_{i=1}^r a_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0.$$

$$Re(\alpha+\beta-\gamma-\delta)+\sum_{i=1}^r a_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0.$$

$$Re(q)+\sum_{i=1}^r b_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0.\ Re(p+q-c-d)+\sum_{i=1}^r b_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0.$$

$$\left|arg\left(z_i\frac{\sin^{2a_i}\theta(1-x)^{b_i}}{(a\sin^2\theta+b\cos^2\theta)^{a_i}[a'x+b'(1-x)]^{b_i}}\right)\right|<\frac{1}{2}A_i^{(k)}\pi \text{ where } A_i^{(k)} \text{ is defined by (1.4).}$$

Proof

To establish the theorem 2, expressing the multivariable Gimel-function in the Mellin-Barnes multiple integrals contour with the help of (1.1) and interchanging the order of integrations, which is justified under the conditions mentioned above, we get (say I)

$$I=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}$$

$$\left[\int_0^{\frac{\pi}{2}}\frac{\sin^{2\alpha+2\sum_{i=1}^r a_is_i-1}\cos^{2\beta-1}\theta}{[a\sin^2\theta+b\cos^2\theta]^{\alpha+\beta+\sum_{i=1}^r a_is_i}}{}_2F_1\left(c,d;p;\frac{a'x}{a'x+b'(1-x)}\right)\mathrm{d}\theta\right]$$

$$\left[\int_0^1 x^{p-1}(1-x)^{q+\sum_{i=1}^r b_is_i-1}[a'x+b'(1-x)]^{-p-q-\sum_{i=1}^r b_is_i}\mathrm{d}x\right]\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.4}$$

Now Evaluating the inner integrals with the help of lemmae 3and 2 and interpreting the resulting expression with the help of (1.1), we obtain the desired theorem 2.

Remark 5.
If $m_2=n_2=\cdots=m_{r-1}=n_{r-1}=p_{i_2}=q_{i_2}=\cdots=p_{i_{r-1}}=q_{i_{r-1}}=0$ and $A_{2j}=B_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=B_{rj}=A_{rji_r}=B_{rji_r}=1$, then we can obtain the same double finite integrals in the generalized multivariable Aleph- function (extension of multivariable Aleph-function defined by Ayant [1]).

Remark 5.
If $m_2=n_2=\cdots=m_r=n_r=p_{i_2}=q_{i_2}=\cdots=p_{i_r}=q_{i_r}=0$ and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2==\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then we can obtain the same double finite integrals in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [8]).

Remark 7.
If$A_{2j}=B_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=B_{rj}=A_{rji_r}=B_{rji_r}=1$and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then we can obtain the same double finite integrals in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [7]).

Remark 8.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and Panda [9,10] and then we can obtain the same double finite integrals, see Mathur and Tulsiani for more details [5].

4. Conclusion.

The importance of our all the results lies in their manifold generality. Firstly, in view of general arguments utilized in these double integrals, we can obtain a large simpler double or single finite integrals, Secondly by specialising the various parameters as well as variables in the generalized multivariable Gimel-function, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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