Persistence of Topological Properties through Layer Map

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Abstract

In Topology, two spaces are said to be homeomorphic if there exist a homeomorphism between them. It is difficult to consider all functions from one space to other for verifying they are homeomorphic or not. Triangulation of spaces in to simplicial complex is useful in determining the topological properties of spaces. In this paper Layer Topology and Layer Maps on Simplices are used as the tools to prove the homeomorhism between simplices.

Keywords - *Barycentric Map, Layer Topology, Layer Map*

I. INTRODUCTION

In many of the cases ordinary functions are unable to discuss homeomorphism between two spaces, especially if they are of different dimensions. By Ref[1], We defined layers on simplices and followed layer Topology on it. Here, it is an attempt to construct a map from one simplex to another to compare the topological properties of simplices having different dimension in the light of layer Topology.

II. PRELIMINARIES

A. Barycentric Maps and Layer Toplogy

Let $\{a_0, a_1, \dots . a_n\}$ be geometrically independent set in $\boldsymbol{R}^N$. We define the **n-simplex** σ spanned by $a_0, a_1, \dots . a_n$ be the set of all points x of $\boldsymbol{R}^N$ such that

$$x = \sum_{i=0}^{n} t_i a_i \text{ , where } \sum_{i=0}^{n} t_i = 1$$

and $t_i \geq 0$ for all i. The numbers t_i are uniquely determined by x; they are called the **barycentric coordinates** of the point x of σ with respect to $a_0, a_1, \dots . a_n$.

If x is a point of the polyhedron $|K|$, then x is point of one simplex K whose vertices are $a_0, a_1, \dots . a_n$.then

$$x = \sum_{i=0}^{n} t_i a_i \text{ , where } \sum_{i=0}^{n} t_i = 1$$

and $t_i \geq 0$ for all i. If v is an arbitrary vertex of K, then the barycentric coordinates $t_v(x)$ of with respect to v is $t_v(x) = 0 \; if \; v \neq a_i$ and $t_v(x) = t_i, if \; v = a_i$.

By Ref [1], Let $v = a_i$ be any vertex of a simplex σ of dimension $n \geq 1$ with vertices $a_0, a_1, \dots . a_n$ then $t_v(a_j) = 1 \; for \; i = j$ and $t_v(a_j) = 0 \; for \; i \neq j$.

Let $v = a_0$ be a vertex of a simplex σ of dimension $n \geq 1$ with vertices $a_0, a_1, \dots . a_n$, then the map $t_v : \sigma \rightarrow [0,1]$ is called **barycentric map** with respect to v.

Let $v = a_0$ be a vertex of a simplex σ of dimension $n \geq 1$ with vertices $a_0, a_1, \dots . a_n$ and let $t \in [0,1]$, then the set $l_t(\sigma) = \{x \in \sigma : t_v(x) = t\}$ is called the $\boldsymbol{t}^{\text{th}}$ **layer** of σ.

Let U be a subset of I = [0, 1], then $l_U = \{l_t : t \in U\}$. Whenever U is open we can call l_U as **open layer.**
The collection $\mathcal{L}$ of open layers l_U form a basis for a topology on σ

Let $v = a_0$ be a vertex of a simplex σ of dimension $n \geq 1$ with vertices $a_0, a_1, \dots . a_n$. Then the topology on σ in which open layers form a basis is called **layer topology.**

III. LAYER MAP ON SIMPLICES

Definition: Let σ and σ' are two simplices with dimension ≥ 1 and let $\sigma = \bigcup_{t\in I} l_t(\sigma)$ and $\sigma' = \bigcup_{t\in I} l_t(\sigma')$ where $l_t(\sigma)$ and $l_t(\sigma')$ are t[th] layers of σ and σ'. Then the Layer Map is the map $f: \sigma \to \sigma'$ defined by $f(l_t(\sigma)) = l_t(\sigma')$.

***Proposition 1**: The Layer Map is bijective.*
Proof:
Suppose $f\left(l_{t_1}(\sigma)\right) = f\left(l_{t_2}(\sigma)\right)$. Then $l_{t_1}(\sigma') = l_{t_2}(\sigma')$.
Therefore $t_1{}^{th}$ layer and $t_2{}^{th}$ layer of σ' are same.
Since the layers are distinct for distinct t's in I. Therefore $t_1 = t_2$ and hence $l_{t_1}(\sigma)$=$l_{t_2}(\sigma)$.
i.e Layer Map is injective.
For each layer $l_t(\sigma')$ in σ', there exist exactly one $t \in I$. Corresponding to that particular t, there exists one layer $l_t(\sigma)$ in σ such that $f(l_t(\sigma)) = l_t(\sigma')$. Therefore Layer Map is surjective and hence bijective. ∎

Theorem 2: The layer Map is continuous relative to the Layer Topology on σ and σ'
Proof:
Let V be open in $\sigma' = b_0 b_1 \dots b_m$ relative to Layer Topology and let $U = f^{-1}(V)$. We have to show that U is open in $\sigma = a_0 a_1 \dots a_n$.
Let $x \in U$. Then $x \in l_t(\sigma)$ for some $t \in I$, since $\sigma = \bigcup_{t\in I} l_t(\sigma)$.
Therefore $t_{a_0}(x) = t$. Where t_{a_0} is the barycentric Map on σ.
Also $f(x) \in f(l_t(\sigma))$. ie $f(x) \in l_t(\sigma')$. Hence $t_{b_0}(f(x)) = t$. Where t_{b_0} is the barycentric Map on σ'.
More over $f(x) \in V$.
Since V is open, there exist $l'_G \in \mathcal{L}'$ such that $t \in G$, G is open in I and $l'_G \subset V$. Where $\mathcal{L}'$ is the base for the Layer Topology of σ'.
Since $t \in G$ and G is open in I we have $x \in l_G$ where $l_G \in \mathcal{L}$. Here $\mathcal{L}$ is the base for the Layer Topology of σ.
We claim that $l_G \subset U$.
Suppose not, there exist $t_1 \in G$ such that $l_{t_1}(\sigma) \notin U$.
Therefore $f\left(l_{t_1}(\sigma)\right) \notin V$. i.e $l_{t_1}(\sigma') \notin V$. Which is a contradiction to $l'_G \subset V$.
we have $l_G \in \mathcal{L}$ such that $x \in l_G \subset U$. i.e U is open. ∎

Corollary 3*:* The layer Map is open relative to the Layer Topology on σ and σ'.

From the above three results, we reach at the conclusion that the Layer Map is a homeomorphism from one simplex to the other, both having dimension $n \geq 1$.

As a result, any property of the simplex σ that is entirely expressed in terms of the Layer Topology yields, via the correspondence of the Layer Map, the corresponding property of the simplex σ'.

Next we have to think about the simplex of dimension zero. As we know simplex of dimension zero is a one point set a_0, at first we have to define layers of a_0 to define layer Topology and layer Map.

Definition: Let $\sigma = a_0$ be a one simplex. The t[th] layer of σ is defined to be a_0 for all $t \in I$.
Here the corresponding Barycentric Map will be a Multivalued function.

Lemma4: The Layer Toplogy on $\sigma = a_0$ is the unique topology which is discrete (and indiscrete).
Proof: If U is nonempty subset of I, then there exist at least one $t \in I$ such that $t \in U$.
According to the definition, t[th] layer of σ is a_0 for all $t \in I$. Hence we can conclude that $a_0 \in l_U$ for all non empty open set U in I.

If U is empty, then the empty set $\varphi \in l_U$.
Hence the open layers of $\sigma = a_0$ are either $\{a_0\}$ or φ.
The topology on $\sigma = a_0$ with $\{a_0\}$ and φ are bases is the discrete topology. ∎

Since 0- simplex contains only one layer, the layer map defined from 0- simplex to higher dimensional simplex are continuous multi valued functions.

CONCLUSION

Layer map will help us to understand the persistence of topological properties on simplices which we can define with the help of Layer Topology. As simpices and simplicial complexes are the triangulized versions of topological spaces, we can extend those understanding to general topological spaces in a greater extend.

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