Certain Transformation Formulae for the Multivariable Gimel Function

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ABSTRACT
In this paper three transfotmation of double infinite series involving the multivariable Gimel-function defined here. These transformations have further been used to obtain double summation formulae for the said function. Our results are quite general in character and a number of transformation formulae and summation formulae can be deduced as particular cases.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, transformation formulae,.

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1. Introduction and preliminaries.

In this paper we establish four summation formulae for the multiple series involving the multivariable Gimel-function defined here. Our results are quite general in character and , on specialization of parameters, a number of summation formulae can be deduced as particular cases.

Throughout this paper, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We define a generalized transcendental function of several complex variables noted $\gimel$.

$$\gimel(z_1,\cdots,z_r) = \gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\right.$$

$$[(a_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(a_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots;\quad\cdots$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(c^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{1,q_r}]:\ [(d^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}] \\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}]\end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha^{(k)}_{2ji_2}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \quad (1.2)$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \quad (1.3)$$

1) $[(c_j^{(1)};\gamma_j^{(1)})]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,\cdots,0\leqslant m_r,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n^{(k)});(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r)$.

$b_{kji_k} \in \mathbb{C}; (j = 1, \cdots, q_{i_k}); (k = 2, \cdots, r).$

$d^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$\gamma^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour L_k is in the $s_k(k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha^{(k)}_{2j} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha^{(k)}_{3j} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha^{(i)}_{rj}\right) (j = 1, \cdots, n_r), \Gamma^{C^{(k)}_j}\left(1 - c^{(k)}_j + \gamma^{(k)}_j s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$ to the right of the contour L_k and the poles of $\Gamma^{D^{(k)}_j}\left(d^{(k)}_j - \delta^{(k)}_j s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A^{(k)}_i \pi$ where

$$A^{(k)}_i = \sum_{j=1}^{m^{(k)}} D^{(k)}_j \delta^{(k)}_j + \sum_{j=1}^{n^{(k)}} C^{(k)}_j \gamma^{(k)}_j - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q^{(k)}_i} D^{(k)}_{ji^{(k)}} \delta^{(k)}_{ji^{(k)}} + \sum_{j=n^{(k)}+1}^{p^{(k)}_i} C^{(k)}_{ji^{(k)}} \gamma^{(k)}_{ji^{(k)}} \right) +$$

$$-\tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha^{(k)}_{2ji_2} + \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta^{(k)}_{2ji_2} \right) - \cdots - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha^{(k)}_{rji_r} + \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta^{(k)}_{rji_r} \right) \quad (1.4)$$

Following the lines of Braaksma ([3] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \to 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \to \infty$ where $i = 1, \cdots, r$:

$\alpha_i = \min_{1 \leqslant j \leqslant m^{(i)}} Re \left[D^{(i)}_j \left(\frac{d^{(i)}_j}{\delta^{(i)}_j} \right) \right]$ and $\beta_i = \max_{1 \leqslant j \leqslant n^{(i)}} Re \left[C^{(i)}_j \left(\frac{c^{(i)}_j - 1}{\gamma^{(i)}_j} \right) \right]$

Remark 1.
If $n_2 = \cdots = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ $A_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [2].

Remark 2.
If $n_2 = \cdots = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [8].

Remark 3.
If $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [7].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and panda [10,11].

In your investigation, we shall use the following notations.

$$\mathbb{A} = [(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A = [(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B} = [\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B} = [\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B} = [(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U = 0,n_2;0,n_3;\cdots;0,n_{r-1};V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

2. Main results.

In this section, we shall establish the following these transformations formulae for the multivariable Gimel-function :

Theorem 1.

$$\sum_{m,n=0}^{\infty}\frac{x^m y^n}{m!n!}\gimel_{X;p_{i_r}+4,q_{i_r}+1,\tau_{i_r}:R_r:Y}^{U;0,n_r+4:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};(1-a-m;a_1,\cdots,a_r;1),(1-b-m;b_1,\ldots,b_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\mathbf{B},(1-c-m-n;c_1,\cdots,c_r;1) \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-c+a-n};\mathrm{c}_1-a_1,\cdots,c_r-a_r;1),(1-c+b-n;c_1-b_1,\ldots,c_r-b_r;1),\mathbf{A}:A \\ \cdot \\ \cdot \\ :\mathrm{B} \end{array}\right)$$

$$= \sum_{s=0}^{\infty} \frac{(1-y)^{a+b-c}(x+y-xy)^s}{s!} \gimel^{U;0,n_r+4:V}_{X;p_{i_r}+4,q_{i_r}+1,\tau_{i_r}:R_r:Y} \left(\begin{array}{c|c} z_1(1-y)^{u_1+v_1-w_1} & \mathbb{A}; (1-a-s; a_1,\cdots,a_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r(1-y)^{u_r+v_r-w_r} & \mathbb{B}; \mathbf{B}, (1-c-s; c_1,\cdots,c_r;1) \end{array} \right.$$

$$\left. \begin{array}{c} (1\text{-b-s}; v_1,\cdots,v_r;1), (1-c+a; c_1-a_1,\cdots,c_r-a_r;1), (1-c+b; b_1,\ldots,b_r;1), \mathbf{A} : A \\ \cdot \\ \cdot \\ : \mathrm{B} \end{array} \right) \tag{2.1}$$

provided

$a_i, b_i, c_i > 0, c_i - a_i > 0, c_i - b_i > 0 (i = 1,\cdots,r)$. $|arg\,(z_i)| < \frac{1}{2}(A_i^{(k)} - c_i)\pi$. $max\{|x|,|y|\} < 1$, either $|x+y-xy| < 1$ or $x = 1$ with $Re(c-a-b) > 0$.

Proof

To prove the theorem 1, expressing the multivariable Gimel-function with the help of (1.1) and changing the order of integration and summation, we obtain (say I)

$$I = \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1,\cdots,s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \frac{\Gamma(a+\sum_{i=1}^r a_i s_i)\Gamma(b+\sum_{i=1}^r b_i s_i)\Gamma(c-a+\sum_{i=1}^r (c_i-a_i)s_i)}{\Gamma(c+\sum_{i=1}^r c_i s_i)}$$

$$\Gamma\left(c-b+\sum_{i=1}^r (c_i-b_i)s_i\right)$$

$$F_2\left[a+\sum_{i=1}^r a_i s_i, c-a+\sum_{i=1}^r (c_i-a_i)s_i, b+\sum_{i=1}^r b_i s_i, c-b+\sum_{i=1}^r (c_i-b_i)s_i; c+\sum_{i=1}^r b_i s_i\right] \mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.2}$$

Now we use the result ([5], p. 238, Eq. (14))

$$F_2[a, c-a, c-b; c; x, y) = (1-y)^{a+b-c}{}_2F_1[a, b; c; x+y-xy] \tag{2.3}$$

Now using the above equation, we get

$$I = \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1,\cdots,s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \frac{\Gamma(a+\sum_{i=1}^r a_i s_i)\Gamma(b+\sum_{i=1}^r b_i s_i)\Gamma(c-a+\sum_{i=1}^r (c_i-a_i)s_i)}{\Gamma(c+\sum_{i=1}^r c_i s_i)}$$

$$\Gamma\left(c-b+\sum_{i=1}^r (c_i-b_i)s_i\right)(1-y)^{a+b+c+\sum_{i=1}^r (a_i+b_i-c_i)s_i}$$

$$\left[{}_2F_1\left[a+\sum_{i=1}^r a_i s_i, 1; b+\sum_{i=1}^r b_i s_i; c+\sum_{i=1}^r c_i s_i; x+y-xy\right] \mathrm{d}s_1\cdots\mathrm{d}s_r \right. \tag{2.4}$$

Now expressing the Gauss hypergeometric function in terms of their series and changing the order of integration and summation, and interpreting the resulting expression with the help of (1.1), we obtain the desired theorem1.

Theorem 2.

$$\sum_{m,n=0}^{\infty} \frac{x^m y^n}{m!n!} \gimel^{U;0,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y} \left(\begin{array}{c|c} z_1 & \mathbb{A}; (1-a-m; a_1,\cdots,a_r;1), (1-c+a-n; b_1-a_1,\ldots,b_r-a_r;1), \mathbf{A} : A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B}; \mathbf{B}, (1-c-m-n; b_1,\cdots,b_r;1) : B \end{array} \right)$$

$$= e^y \sum_{s=0}^{\infty} \frac{(x-y)^{-a}}{s!} \gimel_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}^{U;m_r+2,0:V}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A}; (1-a-s; a_1, \cdots, a_r; 1), (1-c+a; b_1-a_1, \ldots, b_r-a_r; 1), \mathbf{A}: A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B}; \mathbf{B}, (1-c-s; b_1, \cdots, b_r; 1): B \end{array}\right) \tag{2.5}$$

provided

$a_i, b_i > 0, b_i - a_i > 0, (i = 1, \cdots, r). |arg\,(z_i)| < \frac{1}{2}(A_i^{(k)} - b_i)\pi.$

Theorem 3.

$$\sum_{m,n=0}^{\infty} \frac{(16x)^m (1-x)^n}{m!n!} \gimel_{X;p_{i_r}+5,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;0,n_r+5:V} \left(\begin{array}{c|c} z_1 & \mathbb{A}; (1-a-m-n; a_1, \cdots, a_r; 1), (1-b-m-n; b_1, \ldots, b_r; 1), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B}; \mathbf{B}, (\frac{1}{2}-a-b-2m; a_1+b_1, \cdots, a_r+b_r; 1), \end{array}\right.$$

$$\left.\begin{array}{c} (\frac{1}{2}-a-m; a_1, \cdots, a_r; 1), (\frac{1}{2}-b-m; b_1, \cdots, b_r), (\frac{3}{2}-c-2m; c_1, \cdots, c_r; 1)\, \mathbf{A}: A \\ \cdot \\ \cdot \\ (2\text{-}2c\text{-}2m; 2c_1, \cdots, 2c_r; 1): B \end{array}\right) = 4^{1-c}\pi \sum_{s=0}^{\infty} \frac{x^s}{s!}$$

$$\gimel_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}^{U;0,n_r+2:V} \left(\begin{array}{c|c} z_1 4^{-c_1} & \mathbb{A}; (1-a-s; a_1, \cdots, a_r; 1), (1-b-s; b_1, \ldots, b_r; 1), \mathbf{A}: A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r 4^{-c_r} & \mathbb{B}; \mathbf{B}, (1-c-s; w_1, \cdots, w_r; 1): B \end{array}\right) \tag{2.6}$$

provided
$a_i, b_i, c_i > 0 (i = 1, \cdots, r). |arg\,(z_i)| < \frac{1}{2}(A_i^{(k)} - a_i - b_i - c_i)\pi. |x| < 1$ or $x = 1$ with $Re(c-a-b) > 0.$

To prove the theorems 2 and 3, we use the similar lines to the formula (2.1) by using the following formulae ([4], p. 124, Eq. 64) see also ([9], p. 322, Eq. 181) and ([3], p. 339, Eq. 12) respectively, instead of result ([6], p. 238, Eq. (14)).

3. Summation formulae.

If we take $x = 1$ in (2.1) and make use the gauss's summation theorem, we get

Corollary 1.

$$\sum_{m,n=0}^{\infty} \frac{y^n}{m!n!} \gimel_{X;p_{i_r}+4,q_{i_r}+1,\tau_{i_r}:R_r:Y}^{U;0,n_r+4:V} \left(\begin{array}{c|c} z_1 & \mathbb{A}; (1-a-m; a_1, \cdots, a_r; 1), (1-c+a-n; c_1-a_1, \ldots, c_r-a_r; 1), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B}; \mathbf{B}, (1-c-m-n; c_1, \cdots, c_r; 1) \end{array}\right.$$

$$(1\text{-b-n};b_1,\cdots,b_r;1),(1-c+b-n;c_1-b_1,\ldots,c_r-b_r;1),\mathbf{A}:A \; \vdots \; :B \Big)$$

$$=\sum_{s=0}^{\infty}\frac{(1-y)^{a+b-c}(x+y-xy)^s}{s!}\gimel^{U;0,n_r+4:V}_{X;p_{i_r}+4,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1(1-y)^{u_1+v_1-w_1} & \mathbb{A};(1-a-s;a_1,\cdots,a_r;1), \\ \vdots & \vdots \\ z_r(1-y)^{u_r+v_r-w_r} & \mathbb{B};\mathbf{B},(1-c-s;c_1,\cdots,c_r;1) \end{array}\right.$$

$$(1\text{-b-s};v_1,\cdots,v_r;1),(1-c+a;c_1-a_1,\cdots,c_r-a_r;1),(1-c+b;b_1,\ldots,b_r;1),\mathbf{A}:A \; \vdots \; :B \Big) \tag{3.1}$$

provided

$a_i,b_i,c_i>0,c_i-a_i>0,c_i-b_i>0(i=1,\cdots,r).\ |arg\,(z_i)|<\frac{1}{2}(A_i^{(k)}-c_i)\pi.\ |y|<1,$

Consider the above formula, taking $y\to 0$

Corollary 2.

$$\sum_{m=0}^{\infty}\frac{1}{m!}\gimel^{U;0,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbb{A};(1-a-m;a_1,\cdots,a_r;1),(1-b-m;b_1,\ldots,b_r;1) \\ \vdots & \vdots \\ z_r & \mathbb{B};\mathbf{B},(1-c-m;c_1,\cdots,c_r;1) \end{array}\right)$$

$$\gimel^{U;0,n_r+3:V}_{X;p_{i_r}+3,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbb{A};(1-a;a_1,\cdots,a_r;1), \\ \vdots & \vdots \\ z_r & \mathbb{B};\mathbf{B},(1-c+a;c_1-a_1,\cdots,c_r-a_r;1) \end{array}\right.$$

$$(1\text{-b};v_1,\cdots,v_r;1),(1-c+a+b;c_1-a_1-b_1,\cdots,c_r-a_r-b_r;1)\mathbf{A}:A \; \vdots \; :B \Big) \tag{3.2}$$

Taking $x=y$ in (2.5), we obtain the following formula

Corollary 3.

$$\sum_{m,n=0}^{\infty}\frac{x^{m+n}}{m!n!}\gimel^{U;0,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbb{A};(1-a-m;a_1,\cdots,a_r;1),(1-c+a-n;b_1-a_1,\ldots,b_r-a_r;1),\mathbf{A}:A \\ \vdots & \vdots \\ z_r & \mathbb{B};\mathbf{B},(1-c-m-n;b_1,\cdots,b_r;1):B \end{array}\right)$$

$$= e^{x} \gimel^{U;m_r+2,0:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};(1-a;a_1,\cdots,a_r;1),(1-c+a;b_1-a_1,\ldots,b_r-a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\mathbf{B},(1-c;b_1,\cdots,b_r;1):B \end{array}\right) \tag{3.3}$$

Taking $x = 1$ in (2.5), we get the following summation formula

Corollary 4.

$$\sum_{m,n=0}^{\infty} \frac{1}{(2m)!} \gimel^{U;0,n_r+3:V}_{X;p_{i_r}+3,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};(1-2a-2m;2a_1,\cdots,2a_r;1),(1-2b-2m;2b_1,\ldots,2b_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\mathbf{B},(\frac{1}{2}-a-b+2m;a_1+b_1,\cdots,a_r+b_r;1), \end{array}\right.$$

$$\left.\begin{array}{c} (\frac{3}{2}-c-2m;c_1,\cdots,c_r;1)\mathbf{A}:A \\ \cdot \\ \cdot \\ (2+2c-2m;2c_1,\cdots,2c_r;1):B \end{array}\right) = 4^{a+b-c} \gimel^{U;0,n_r+23V}_{X;p_{i_r}+3,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} \mathrm{z}_1 4^{a_1+b_1-c_1} & \mathbb{A};(1-2a;2a_1,\cdots,2a_r;1),(1-2b;2b_1,\ldots,2b_r;1),\left(\frac{3}{2}-c;c_1,\ldots,c_r;1\right)\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r 4^{a_r+b_r-c_r} & \mathbb{B};\mathbf{B},\left(\frac{1}{2}-a-b;a_1+b_{11},\cdots,a_r+b_r;1\right),(2+2c;2c_1,\cdots,2c_r;1):B \end{array}\right) \tag{3.4}$$

Remark 6.
If $m_2 = n_2 = \cdots = m_{r-1} = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$, then we can obtain the same summation formulae in the generalized multivariable Aleph- function (extension of multivariable Aleph-function defined by Ayant [2]).

Remark 7.
If $m_2 = n_2 = \cdots = m_r = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then we can obtain the same summation formulae in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [8]).

Remark 8.
If$A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then we can obtain the same summation formulae in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [7]).

Remark 9.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and Panda [10,11] and then we can obtain the the same summation formulae, see Audich [1] for more details.

5. Conclusion.

The importance of our all the results lies in their manifold generality. Firstly, in view of general arguments utilized in these transformation of double infinite series formulae we can obtain a large single, double or multiple summations. Secondly by specialising the various parameters as well as variables in the generalized multivariable Gimel-function, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

REFERENCES.

[1] S. Audich, Certain transformation formulas for the H-function of several variables, Pure and Applied Mathematika sciences, 26(1-2) (1987), 65-74.

[2] F. Ayant, An integral associated with the Aleph-functions of several variables. International Journal of Mathematics Trends and Technology (IJMTT), 31(3) (2016), 142-154.

[3] B.L.J. Braaksma, Asymptotics expansions and analytic continuations for a class of Barnes-integrals, Compositio Math. 15 (1962-1964), 239-341.

[4] J.L. Burchnall and T.W. Chaundy, Expansions of Appell's double hypergeometric function (II), Quart. J. Math. (Oxford), 12(1941), 112-128.

[5] A. Erdelyi, Higher transcentental function, Vol I, McGraw-Hill, New York (1953).

[6] H. Exton, A note on a hypergeometric transformation, Bull. Calcutta. Math. Soc. 71(1979), 337-340.

[7] Y.N. Prasad, Multivariable I-function , Vijnana Parishad Anusandhan Patrika 29 (1986) , 231-237.

[8] J. Prathima, V. Nambisan and S.K. Kurumujji, A Study of I-function of Several Complex Variables, International Journal of Engineering Mathematics Vol (2014), 1-12.

[9] H.M. Srivastava and P.W. Karlson, Multiplz Gaussian hypergeometric series, Halsted Press (Ellis Horwood limited, Chichester), John Wiley and Sons, New York (1985).

[10] H.M. Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables. Comment. Math. Univ. St. Paul. 24 (1975),119-137.

[11] H.M. Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables II. Comment. Math. Univ. St. Paul. 25 (1976), 167-197.