Finite Summation Formulae for the Multivariable Gimel-Function

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ABSTRACT
In this paper, we have obtained some interesting double finite summations for the multivariable gimel-function defined here. At the end, we shall see special cases.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, tmultiple summation formulae,.

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1. Introduction and preliminaries.

In the present paper we establish some finite summation relations for the multivariable Gimel-function using the results due to Carlitz (1,2,3). If we provide a few particular values for the parameters used in the summation formulae we get some new results.

Throughout this paper, let $\mathbb{C},\mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. We define a generalized transcendental function of several complex variables.

$$\gimel(z_1,\cdots,z_r)=\gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|}z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{array}\right.$$

$$[(a_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(a_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;\quad\cdots$$

$$[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{n_r+1,p_r}]:\ [(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}]$$
$$[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{1,q_r}]:\ [(d_j^{(1)}),\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}]$$

$$\left.\begin{array}{l};\cdots;[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}]\\ ;\cdots;[(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,n^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{n^{(r)}+1,q_i^{(r)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha_{2ji_2}^{(k)}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta_{2ji2}^{(k)}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant m_r,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n_k);(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r)$.

$b_{kji_k}\in\mathbb{C};(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r)$.

$d_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour L_k is in the $s_k (k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}} \left(1 - a_{2j} + \sum_{k=1}^{2} \alpha_{2j}^{(k)} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}} \left(1 - a_{3j} + \sum_{k=1}^{3} \alpha_{3j}^{(k)} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}} \left(1 - a_{rj} + \sum_{i=1}^{r} \alpha_{rj}^{(i)}\right) (j = 1, \cdots, n_r), \Gamma^{C_j^{(k)}} \left(1 - c_j^{(k)} + \gamma_j^{(k)} s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$to the right of the contour L_k and the poles of $\Gamma^{D_j^{(k)}} \left(d_j^{(k)} - \delta_j^{(k)} s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A_i^{(k)} \pi$ where

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)} \gamma_j^{(k)} - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)} \delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)} \gamma_{ji^{(k)}}^{(k)} \right) +$$

$$-\tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha_{2ji_2}^{(k)} + \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta_{2ji_2}^{(k)} \right) - \cdots - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha_{rji_r}^{(k)} + \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta_{rji_r}^{(k)} \right) \quad (1.4)$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$ where $i = 1, \cdots, r$:

$\alpha_i = \min_{1 \leqslant j \leqslant m^{(i)}} Re \left[D_j^{(i)} \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right]$ and $\beta_i = \max_{1 \leqslant j \leqslant n^{(i)}} Re \left[C_j^{(i)} \left(\frac{c_j^{(i)} - 1}{\gamma_j^{(i)}} \right) \right]$

Remark 1.
If $n_2 = \cdots = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ $A_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [1].

Remark 2.
If $n_2 = \cdots = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [7].

Remark 3.
If$A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)}$ $= \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [6].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and Panda [8,9].

In your investigation, we shall use the following notations.

$\mathbb{A} = [(\mathrm{a}_{2j}; \alpha_{2j}^{(1)}, \alpha_{2j}^{(2)}; A_{2j})]_{1,n_2}, [\tau_{i_2}(a_{2ji_2}; \alpha_{2ji_2}^{(1)}, \alpha_{2ji_2}^{(2)}; A_{2ji_2})]_{n_2+1,p_{i_2}}, [(a_{3j}; \alpha_{3j}^{(1)}, \alpha_{3j}^{(2)}, \alpha_{3j}^{(3)}; A_{3j})]_{1,n_3},$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(a_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A}=[(a_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A=[(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B}=[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B}=[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{1,q_{i_r}}] \tag{1.9}$$

$$B=[(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U=0,n_2;0,n_3;\cdots;0,n_{r-1};V=m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X=p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

2. Main formulae.

In this section we give three double summation formulae for the multivariable Gimel-function.

Theorem 1.

$$\sum_{g=0}^{m}\sum_{k=0}^{n}\frac{(-m)_g(-n)_k(b)_{g+k}}{g!k!(b)_g(b)_k}\gimel_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}^{U;0,n_r+2:V}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};(1-d_1-g;a_1,\cdots,a_r;1),(1-d_2-k;b_1,\cdots,b_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\mathbf{B},(1-d_1-d_2-g-k;a_1+b_1,\cdots,a_r+b_r;1):B \end{array}\right)=\frac{(b)_{m+n}}{(b)_m(b)_n}$$

$$\gimel_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}^{U;0,n_r+2:V}\left(\begin{array}{c|c} z_1 & \mathbb{A};(1-d_1-n;a_1,\cdots,a_r;1),(1-d_2-m;b_1,\cdots,b_r1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\mathbf{B},(1-d_1-d_2-m-n;a_1+b_1,\cdots,a_r+b_r;1):B \end{array}\right) \tag{2.1}$$

provided

$a_i, b_i, > 0 (i = 1, \cdots, r)$, $|arg\,(z_i)| < \frac{1}{2} A_i^{(k)} \pi$

Theorem 2.

$$\sum_{g=0}^{m} \sum_{k=0}^{n} \frac{(-m)_g (-n)_k (b-e-n+1)_g (b)_k}{g! k! (2-e-m-n)_g (e)_k} \gimel^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\gimel^{U;0,n_r+3:V}_{X;p_{i_r}+3,q_{i_r}+1,\tau_{i_r}:R_r:Y} \left(\begin{array}{c|c} z_1 & \mathbb{A}; (1-c-g-k; a_1, \cdots, a_r; 1), \mathbf{A} : A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B}; \mathbf{B}, (1-c-g; a_1, \cdots, a_r; 1), (1-c-k; a_1, \cdots, a_r; 1) : B \end{array} \right) = \frac{(b)_m (e-b)_n}{(e)_n (e+n-1)_m}$$

$$\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+2,\tau_{i_r}:R_r:Y} \left(\begin{array}{c|c} z_1 & \mathbb{A}; (1-c-m-n; a_1, \cdots, a_r; 1), \mathbf{A} : A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B}; \mathbf{B}, (1-c-m; a_1, \cdots, a_r; 1), (1-c-n; a_1, \cdots, a_r; 1) : B \end{array} \right) \tag{2.2}$$

provided

$a_i, b_i, > 0 (i = 1, \cdots, r), , |arg\,(z_i)| < \frac{1}{2} A_i^{(k)} \pi$, $a-b$ is not an integer, $a = b - e + -n + 1 \text{and} b = a - d - m + 1$

Theorem 3.

$$\sum_{g=0}^{m} \sum_{k=0}^{n} \frac{(-m)_g (-n)_k}{g! k! (b-d-m+1)_g (b-c-n+1)_k} \gimel^{U;0,n_r+3:V}_{X;p_{i_r}+3,q_{i_r}+1,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A}; (1-c-g; a_1, \cdots, a_r; 1), (1-d-k; a_1, \cdots, a_r; 1), (1-b-g-k; a_1, \cdots, a_r; 1), \mathbf{A} : A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B}; \mathbf{B}, (1-c-d-g-k; 2a_1, \cdots, 2a_r; 1) B \end{array} \right)$$

$$= \frac{1}{(d-b)_m (c-b)_n} \gimel^{U;0,n_r+3:V}_{X;p_{i_r}+3,q_{i_r}+1,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A}; (1-c-n; a_1, \cdots, a_r; 1), (1-d-m; a_1, \cdots, a_r; 1), (1+b-c-d-m-n; a_1, \cdots, a_r; 1), \mathbf{A} : A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B}; \mathbf{B}, (1-c-d-m-n; 2a_1, \cdots, 2a_r; 1 : B \end{array} \right) \tag{2.3}$$

provided

$a_i, b_i, > 0 (i = 1, \cdots, r), , |arg\,(z_i)| < \frac{1}{2} A_i^{(k)} \pi$

Proof

To establish the theorem 1, expressing the multivariable Gimel-function occurring on the left-hand side in the Mellin-Barnes multiple integrals contour with the help of (1.1) and interchanging the order of integration and summation, which is justified under the conditions mentioned above and using the following result due to carlitz [3],

$$\sum_{g=0}^{m} \sum_{k=0}^{n} \frac{(-m)_g (-n)_k (b)_{g+k} (c)_g (d)_k}{g! k! (b)_g (b)_k (c+d)_{g+k}} = \frac{(b)_{m+n} (d)_m (c)_n}{(b)_m (b)_n (c+d)_{m+n}} \tag{2.4}$$

Now applying the above equation and interpreting the resulting expression with the help of (1.1), we obtain the desired theorem 1.

The theorem (2.2) is obtained in a similar manner if we use following result due to carlitz [4],

$$\sum_{g=0}^{m}\sum_{k=0}^{n}\frac{(-m)_g(-n)_k(a)_g(b)_k(c)_{g+k}}{g!k!(d)_g(e)_k(c)_g(c)_k}=\frac{(b)_m(a)_n(c)_{m+n}}{(b-a)_m(a-b)_n(c)_m(c)_n} \tag{2.5}$$

($a-b$ is not integer , $a=b-e-n+1, b=a-d-m+1$)

The theorem 3 can be established in a similar manner if we use following result due to carlitz [5],

$$\sum_{g=0}^{m}\sum_{k=0}^{n}\frac{(-m)_g(-n)_k(b)_{g+k}(c)_g(d)_k}{g!k!(c+d)_{g+k}(b-c-n+1)_k(b-d-m+1)_g}=\frac{(c+d-b)_{m+n}(d)_m(c)_n}{(c+d)_m(d-b)_m(c-b)_n} \tag{2.6}$$

3. Special cases.

If we take $n=0$ in the theorem 1, the double finite series reduces to the single finite series for the multivariable Gimel-function, we obtain

Corollary 1.

$$\sum_{g=0}^{m}\frac{(-m)_g}{g!}\gimel^{U;0,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbb{A};(1-d_1-g;a_1,\cdots,a_r;1),(1-d_2;b_1,\cdots,b_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\mathbf{B},(1-d_1-d_2-g;a_1+b_1,\cdots,a_r+b_r;1):B \end{array}\right)$$

$$\gimel^{U;0,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbb{A};(1-d_1;a_1,\cdots,a_r;1),(1-d_2-m;b_1,\cdots,b_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\mathbf{B},(1-d_1-d_2-m;a_1+b_1,\cdots,a_r+b_r;1):B \end{array}\right) \tag{3.1}$$

provided

$a_i, b_i, > 0(i=1,\cdots,r)$, $|arg\,(z_i)|<\frac{1}{2}A_i^{(k)}\pi$

Taking $n=0$ in the theorem 2 the double finite series reduces to the single finite series for the multivariable Gimel-function, we obtain

Corollary 2.

$$\sum_{g=0}^{m}\frac{(-m)_g}{g!}\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbb{A};(1-c-g;a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\mathbf{B},(1-c-g;a_1,\cdots,a_r;1),(1-c;a_1,\cdots,a_r;1):B \end{array}\right)=\frac{(b)_m}{(e-1)_m}$$

$$\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbb{A};(1-c-m;a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\mathbf{B},(1-c-m;a_1,\cdots,a_r;1),(1-c;a_1,\cdots,a_r;1):B \end{array}\right) \tag{3.2}$$

provided

$a_i, b_i, > 0(i = 1, \cdots, r), , |arg\,(z_i)| < \frac{1}{2}A_i^{(k)}\pi$, $a - b$ is not an integer, $a = b - e + -n + 1$and$b = a - d - m + 1$
Taking $n = 0$ in the theorem 3 the double finite series reduces to the single finite series for the multivariable Gimel-function, we obtain

Corollary 3.

$$\sum_{g=0}^{m} \frac{(-m)_g}{g!(b-d-m+1)_g} \gimel^{U;0,n_r+3:V}_{X;p_{i_r}+3,q_{i_r}+1,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};(1-c-g;a_1,\cdots,a_r;1),(1-d;a_1,\cdots,a_r;1),(1-b-g;a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\mathbf{B},(1-c-d-g;2a_1,\cdots,2a_r;1):B \end{array}\right)$$

$$= \frac{1}{(d-b)_m} \gimel^{U;0,n_r+3:V}_{X;p_{i_r}+3,q_{i_r}+1,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};(1-c;a_1,\cdots,a_r;1),(1-d-m;a_1,\cdots,a_r;1),(1+b-c-d-m;a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\mathbf{B},(1-c-d-m;2a_1,\cdots,2a_r;1)B \end{array}\right) \quad (3.3)$$

provided

$$a_i, b_i, > 0(i = 1, \cdots, r), , |arg\,(z_i)| < \frac{1}{2}A_i^{(k)}\pi$$

We can taking $m = 0$ in the above theorems, we obtain the single finite series .

Remark 7.
If $m_2 = n_2 = \cdots = m_r = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then we can obtain the same double finite series formulae in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [7]).

Remark 8.
If$A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then we can obtain the same t double finite series formulae in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [6]).

Remark 9.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and Panda [8,9] and then we can obtain the same double finite series formulae.

5. Conclusion.

The importance of our all the results lies in their manifold generality. Firstly, in view of general double finite series formulae utilized, we can obtain a large single, double summations. Secondly by specialising the various parameters as well as variables in the generalized multivariable Gimel-function, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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