# Certain Class of Eulerian Integrals of Multivariable Gimel-Function

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ABSTRACT
In this present paper we evaluate a general class of Eulerian integrals involving the generalized multivariable Gimel-function. . The results proved here provide closes-form expression for numerous other potentially useful integrals. At the end , we shall establish several particular cases.

KEYWORDS : Generalized multivariable Gimel-function, multiple integral contours, Eulerian integral.

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## 1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We define a generalized transcendental function of several complex variables.

$$\gimel(z_1,\cdots,z_r) = \gimel^{m_2,n_2;m_3,n_3;\cdots;m_r,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{array}\right|$$

$$[(\mathrm{a}_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[(\mathrm{b}_{2j};\beta^{(1)}_{2j},\beta^{(2)}_{2j};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta^{(1)}_{3j},\beta^{(2)}_{3j},\beta^{(3)}_{3j};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj};B_{rj})_{1,m_r}],$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{m_r+1,q_r}]:[(\mathrm{d}^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}] \\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}]\end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{m_2}\Gamma^{B_{2j}}(b_{2j}-\sum_{k=1}^{2}\beta^{(k)}_{2j}s_k)\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha^{(k)}_{2ji_2}s_k)\prod_{j=m_2+1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{m_3}\Gamma^{B_{3j}}(b_{3j}-\sum_{k=1}^{3}\beta_{3j}^{(k)}s_k)\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=m_3+1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{m_r}\Gamma^{B_{rj}}(b_{rj}-\sum_{k=1}^{r}\beta_{rj}^{(k)}s_k)\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=m_r+1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)})]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $m_2,n_2,\cdots,m_r,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2\leqslant q_{i_2},0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant m_r\leqslant q_{i_r},0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kj}^{(l)},B_{kj}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n_k);(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$a_{kji_k} \in \mathbb{C}; (j = n_k+1,\cdots,p_{i_k}); (k = 2,\cdots,r).$

$b_{kji_k} \in \mathbb{C}; (j = m_k+1,\cdots,q_{i_k}); (k = 2,\cdots,r).$

$d^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1,\cdots,R^{(k)}); (j = m^{(k)}+1,\cdots,q_{i^{(k)}}); (k = 1,\cdots,r).$

$\gamma^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1,\cdots,R^{(k)}); (j = n^{(k)}+1,\cdots,p_{i^{(k)}}); (k = 1,\cdots,r).$

The contour $L_k$ is in the $s_k(k=1,\cdots,r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where $\sigma$ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k\right)(j=1,\cdots,n_2),\Gamma^{A_{3j}}\left(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k\right)$ $(j=1,\cdots,n_3),\cdots,\Gamma^{A_{rj}}\left(1-a_{rj}+\sum_{i=1}^{r}\alpha_{rj}^{(i)}\right)(j=1,\cdots,n_r)$, $\Gamma^{C_j^{(k)}}\left(1-c_j^{(k)}+\gamma_j^{(k)}s_k\right)(j=1,\cdots,n^{(k)})(k=1,\cdots,r)$to the right of the contour $L_k$ and the poles of $\Gamma^{B_{2j}}\left(b_{2j}-\sum_{k=1}^{2}\beta_{2j}^{(k)}s_k\right)(j=1,\cdots,m_2),\Gamma^{B_{3j}}\left(b_{3j}-\sum_{k=1}^{3}\beta_{3j}^{(k)}s_k\right)(j=1,\cdots,m_3)$ $,\cdots,\Gamma^{B_{rj}}\left(b_{rj}-\sum_{i=1}^{r}\beta_{rj}^{(i)}\right)(j=1,\cdots,m_r)$, $\Gamma^{D_j^{(k)}}\left(d_j^{(k)}-\delta_j^{(k)}s_k\right)(j=1,\cdots,m^{(k)})(k=1,\cdots,r)$ lie to the left of the contour $L_k$. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2}A_i^{(k)}\pi$ where

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)}\delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)}\gamma_j^{(k)} - \tau_{i^{(k)}}\left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)}\delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)}\gamma_{ji^{(k)}}^{(k)}\right)+$$

$$\sum_{j=1}^{n_2} A_{2j}\alpha_{2j}^{(k)} + \sum_{j=1}^{m_2} B_{2j}\beta_{2j}^{(k)} - \tau_{i_2}\left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2}\alpha_{2ji_2}^{(k)} + \sum_{j=m_2+1}^{q_{i_2}} B_{2ji_2}\beta_{2ji_2}^{(k)}\right)+\cdots+$$

$$\sum_{j=1}^{n_r} A_{rj}\alpha_{rj}^{(k)} + \sum_{j=1}^{m_r} B_{rj}\beta_{rj}^{(k)} - \tau_{i_r}\left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r}\alpha_{rji_r}^{(k)} + \sum_{j=m_r+1}^{q_{i_r}} B_{rji_r}\beta_{rji_r}^{(k)}\right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,), max(\,|z_1|,\cdots,|z_r|\,) \to 0$

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,), min(\,|z_1|,\cdots,|z_r|\,) \to \infty$ where $i = 1,\cdots,r$ :

$\alpha_i = \min_{\substack{1\leqslant k\leqslant m_i\\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)$ and $\beta_i = \max_{\substack{1\leqslant k\leqslant n_i\\ 1\leqslant j\leqslant n^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}} + C_k^{(i)}\frac{c_k^{(i)}-1}{\gamma_k^{(i)}}\right)$

**Remark 1.**

If $m_2 = n_2 = \cdots = m_{r-1} = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the generalized multivariable Gimel-function reduces in the generalized multivariable Aleph- function ( extension of multivariable Aleph-function defined by Ayant [1]).

**Remark 2.**

If $m_2 = n_2 = \cdots = m_r = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [6]).

**Remark 3.**

If $A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [5].

**Remark 4.**
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and Panda [9,10].

In your investigation, we shall use the following notations.

$$\mathbb{A} = [(\mathrm{a}_{2j}; \alpha_{2j}^{(1)}, \alpha_{2j}^{(2)}; A_{2j})]_{1,n_2}, [\tau_{i_2}(a_{2ji_2}; \alpha_{2ji_2}^{(1)}, \alpha_{2ji_2}^{(2)}; A_{2ji_2})]_{n_2+1,p_{i_2}}, [(a_{3j}; \alpha_{3j}^{(1)}, \alpha_{3j}^{(2)}, \alpha_{3j}^{(3)}; A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3}; \alpha_{3ji_3}^{(1)}, \alpha_{3ji_3}^{(2)}, \alpha_{3ji_3}^{(3)}; A_{3ji_3})]_{n_3+1,p_{i_3}}; \cdots ; [(\mathrm{a}_{(r-1)j}; \alpha_{(r-1)j}^{(1)}, \cdots, \alpha_{(r-1)j}^{(r-1)}; A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}}; \alpha_{(r-1)ji_{r-1}}^{(1)}, \cdots, \alpha_{(r-1)ji_{r-1}}^{(r-1)}; A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A} = [(\mathrm{a}_{rj}; \alpha_{rj}^{(1)}, \cdots, \alpha_{rj}^{(r)}; A_{rj})_{1,n_r}], [\tau_{i_r}(a_{rji_r}; \alpha_{rji_r}^{(1)}, \cdots, \alpha_{rji_r}^{(r)}; A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A = [(\mathrm{c}_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,n^{(1)}}], [\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)}; C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}]; \cdots ;$$

$$[(c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1,m^{(r)}}], [\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)}; C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B} = [(b_{2j}; \beta_{2j}^{(1)}, \beta_{2j}^{(2)}; B_{2j})]_{1,m_2}, [\tau_{i_2}(b_{2ji_2}; \beta_{2ji_2}^{(1)}, \beta_{2ji_2}^{(2)}; B_{2ji_2})]_{m_2+1,q_{i_2}}, [(b_{3j}; \beta_{3j}^{(1)}, \beta_{3j}^{(2)}, \beta_{3j}^{(3)}; B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(b_{3ji_3}; \beta_{3ji_3}^{(1)}, \beta_{3ji_3}^{(2)}, \beta_{3ji_3}^{(3)}; B_{3ji_3})]_{m_3+1,q_{i_3}}; \cdots ; [(\mathrm{b}_{(r-1)j}; \beta_{(r-1)j}^{(1)}, \cdots, \beta_{(r-1)j}^{(r-1)}; B_{(r-1)j})_{1,m_{r-1}}],$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}}; \beta_{(r-1)ji_{r-1}}^{(1)}, \cdots, \beta_{(r-1)ji_{r-1}}^{(r-1)}; B_{(r-1)ji_{r-1}})_{m_{r-1}+1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B} = [(\mathrm{b}_{rj}; \beta_{rj}^{(1)}, \cdots, \beta_{rj}^{(r)}; B_{rj})_{1,m_r}], [\tau_{i_r}(b_{rji_r}; \beta_{rji_r}^{(1)}, \cdots, \beta_{rji_r}^{(r)}; B_{rji_r})_{m_r+1,q_{i_r}}] \tag{1.9}$$

$$B = [(\mathrm{d}_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1,m^{(1)}}], [\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)}; D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}]; \cdots ;$$

$$[(\mathrm{d}_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{1,m^{(r)}}], [\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)}; D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U = m_2, n_2; m_3, n_3; \cdots ; m_{r-1}, n_{r-1}; V = m^{(1)}, n^{(1)}; m^{(2)}, n^{(2)}; \cdots ; m^{(r)}, n^{(r)} \tag{1.11}$$

$$X = p_{i_2}, q_{i_2}, \tau_{i_2}; R_2; \cdots ; p_{i_{r-1}}, q_{i_{r-1}}, \tau_{i_{r-1}} : R_{r-1}; Y = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}; \cdots ; p_{i^{(r)}}, q_{i^{(r)}}; \tau_{i^{(r)}}; R^{(r)} \tag{1.12}$$

## 2. Required result.

The following result have been evaluated by ([7], p.301 (2.2.6)).

**Lemma 1.**

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}\mathrm{d}t = (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}B(\alpha,\beta)\,{}_2F_1\left(\alpha, -\gamma; \alpha+\beta; -\frac{(b-a)u}{au+v}\right) \tag{2.1}$$

provided $Re(\alpha), Re(\beta) > 0; \left|arg\left(\frac{bu+v}{au+v}\right)\right| < \pi, a \neq b$

We consider the binomial expansion.

**Lemma 2.**

$$(ut+v)^{\gamma} = (au+v)^{\gamma}\sum_{m=0}^{\infty}\frac{(-\gamma)_m\left(-\frac{(t-a)u}{au+v}\right)^m}{m!} \tag{2.2}$$

provided $\left|\frac{(t-a)u}{au+v}\right| < 1, t \in [a,b]$

## 3. Main integral.

In this section we shall deal with problem of clossed-form evaluation of following general Eulerian integral involving the multivariable gimel-function :

In this section, let $g(t) = \frac{(t-a)^{\delta}(b-t)^{\eta}(ut+v)^{1-\delta-\eta}}{D(ut+v)+(C-D)(t-a)}$ and

$X_i = D^{-v_i}(b-a)^{(\delta+\eta)v_i}(au+v)^{-(\delta+\eta)v_i}; (i=1,\cdots,r)$

**Theorem.**

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}\gimel\left[z_1(g(t))^{a_1},\cdots,z_r(g(t))^{a_r}\right]\mathrm{d}t = (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}$$

$$\sum_{l,m=0}^{\infty}\frac{(B-\frac{A}{B})^m\left(\frac{b-a}{au+v}\right)^m\left(-\frac{(b-a)^u}{au+v}\right)^l}{l!m!}\gimel^{U;0,n_r+4:V}_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1X_1 & \mathbb{A};(1-m;a_1,\cdots,a_r;1), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_1X_r & \mathbb{B};\mathbf{B},\ (1;v_1,\cdots,v_r;1) \end{array}\right.$$

$$\left.\begin{array}{c}(1+\gamma-l-m;(\delta+\eta)a_1,\cdots,(\delta+\eta)a_r;1),(1-\beta;\eta a_1,\cdots,\eta a_r;1),(1-\alpha-l-m;\delta a_1,\cdots,\delta c=a_r;1),\mathbf{A}:A \\ \cdot \\ \cdot \\ (1\text{-}\alpha-\beta-l-m;(\delta+\eta)a_1,\cdots,(\delta+\eta)a_r;1),(1+\gamma-m;(\delta+\eta)a_1,\cdots,(\delta+\eta)a_r;1):B\end{array}\right) \tag{3.1}$$

provided

$a_i > 0; i=1,\cdots,r; min\{Re(\alpha),Re(\beta)\} > 0; b \neq a\ ; max\left\{\left|\frac{(D-C)(t-a)}{B(tu+v)}\right|,\left|\frac{(b-a)u}{au+v}\right|\right\} < 1, t \in [a,b]$

$$Re\,(\alpha)+\sum_{i=1}^{r}a_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0\ ;\ Re\,(\beta)+\sum_{i=1}^{r}a_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0$$

$|arg\,(z_i(g(t))^{a_i})\,| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof

To prove the theorem, expressing the generalized multivariable Gimel-function with the help of (1.1), we get

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}$$

$$\left[\frac{t-a)^{\delta}(b-t)^{\eta}(ut+v)^{1-\delta-\eta}}{D(ut+v)+(C-D)(t-a)}\right]^{\sum_{i=1}^{r}a_is_i}\mathrm{d}s_1\cdots\mathrm{d}s_r\,\mathrm{d}t \tag{3.2}$$

interchanging the order of integration and summation in (3.2) which is justifiable due to the absolute convergence of the integrals involved in the process, we get

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}B^{-\sum_{i=1}^{r}a_is_i}$$

$$\left[\int_a^b(t-a)^{\alpha+\delta\sum_{i=1}^{r}a_is_i-1}(b-t)^{\beta+\eta\sum_{i=1}^{r}a_is_i-1}(ut+v)^{\gamma-(\delta+\eta)\sum_{i=1}^{r}a_is_i}\left\{1-\frac{(D-C)(t-a)}{D(ut+v)}\right\}^{-\sum_{i=1}^{r}a_is_i}\mathrm{d}t\right]$$

$$\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.3}$$

Using the lemma 2, we get

$$\sum_{m=0}^{\infty}\frac{\left(D-\frac{C}{D}\right)^m}{m!}\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\frac{\Gamma\left(m+\sum_{i=1}^{r}a_is_i\right)}{\Gamma\left(\sum_{i=1}^{r}a_is_i\right)}$$

$$\left[\int_a^b(t-a)^{\alpha+m+\delta\sum_{i=1}^{r}a_is_i-1}(b-t)^{\beta+\eta\sum_{i=1}^{r}a_is_i-1}(ut+v)^{\gamma-m-(\delta+\eta)\sum_{i=1}^{r}a_is_i}\mathrm{d}t\right]\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.4}$$

Now using tthe lemma 1 and and interpreting the resulting expression with the help of (1.1), we obtain the desired theorem .

## 4. Special cases.

In this section, we shall give several partcular cases.

In (2.1), taking $\gamma=-\alpha-\beta, u=-\lambda-\mu, v=(1+\mu)b-(1+\lambda)a, f(t)=ut+v=(b-a)+\lambda(t-a)+\mu(b-t)$

$g(t)=\dfrac{(t-a)^{\delta}(b-t)^{\eta}(f(t))^{1-\delta-\eta}}{D(b-a)+(t-a)(C-D+D\lambda)+\mu D(b-t)}$ and finally replacing $\alpha,\beta$ by $(\alpha+1),(\beta+1)$ respectively, we

get (taking above substtuting the lemma 1would yield ([4], p. 287)

$$\int_a^b\frac{(t-a)^{\alpha-1}(b-t)^{\beta-1}}{[(b-a)+\lambda(t-a)+\mu(b-t)]^{\alpha+\beta}}\mathrm{d}t=\frac{(1+\lambda)^{-\alpha}(1+u)^{-\beta}}{(b-a)}B(\alpha,\beta)$$

provided $Re(\alpha),Re(\beta)>0; b-a+\lambda(t-a)+\mu(b-t)\neq 0, t\in[a,b,(a\neq b)$, We obtain the following integral

**Corollary 1.**

$$\int_a^b\frac{(t-a)^{\alpha}(b-t)}{[f(t)]^{\alpha+\beta+2}}\daleth(z_1(g(t))^{a_1},\cdots,z_r(g(t))^{a_r})\mathrm{d}t=(b-a)^{-1}(1+\lambda)^{-1-\alpha}(1+\mu)^{-1-\beta}\sum_{m=0}^{\infty}\frac{\left(D-\frac{C}{D}(1+\lambda)\right)^m}{m!}$$

$$\daleth_{X;p_{i_r}+3,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;0,n_r+3:V}\left(\begin{array}{c|c}z_1D^{-a_1}(1+\lambda)^{-\delta a_1}(1+\mu)^{-\eta a_1} & \mathbb{A};(1-m;a_1,\cdots,a_r;1),\\ \cdot & \cdot\\ \cdot & \cdot\\ z_1D^{-a_r}(1+\lambda)^{-\delta a_r}(1+\mu)^{-\eta a_r} & \mathbb{B};\mathbf{B},\ (1;\mathrm{v}_1,\cdots,v_r;1),\end{array}\right)$$

$$\left.\begin{matrix} (-\alpha - m; \delta a_1, \cdots, \delta a_r; 1), (-\beta; \eta a_1, \cdots, \eta a_r; 1), \mathbf{A} : A \\ \cdot \\ \cdot \\ (1-\alpha - \beta - m; (\delta+\eta)a_1, \cdots, (\delta+\eta)a_r; 1) : B \end{matrix}\right) \tag{4.1}$$

In the above corollary, taking $\lambda = \mu = 0, z_i = (b-a)^{\delta+\eta-1} v_i; (i = 1, \cdots, r)$ and $\delta = \eta = 1, \alpha = \beta = -\frac{1}{2}, A \to A, B \to B^2$, using the relation ([3], p. 101, Eq. (2.8) (6)) and the duplication formula, we get

**Corollary 2.**

$$\int_a^b [(t-a)(b-t)]^{-\frac{1}{2}} \gimel \left( \left[ \frac{(t-a)(b-t)}{C^2(t-a)+D^2(b-t)} \right]^{a_1}, \cdots, \left[ \frac{(t-a)(b-t)}{C^2(t-a)+D^2(b-t)} \right]^{a_r} \right) \mathrm{d}t = \sqrt{\pi}$$

$$\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+1,\tau_{i_r}:R_r:Y} \left( \begin{matrix} \left(\frac{(b-a)}{C+D)^2}\right)^{a_1} \\ \cdot \\ \cdot \\ \left(\frac{(b-a)}{C+D)^2}\right)^{a_r} \end{matrix} \middle| \begin{matrix} \mathbb{A}; \left(\frac{1}{2}; a_1, \cdots, a_r; 1\right), \mathbf{A}; A \\ \cdot \\ \cdot \\ \mathbb{B}; \mathbf{B}, (0; \mathrm{v}_1, \cdots, v_r; 1) : B \end{matrix} \right) \tag{4.2}$$

In the corollary 1, taking $\lambda = \mu = 0, z_i = (b-a)^{\delta+\eta-1)a_i}, \delta = \eta = \frac{1}{2}, \beta = -\alpha - 2, a_i \to 2a_i (i = 1, \cdots, r)$, using the duplication formula an binomial expansion ; finally $a = 0, b = 1, a_i = 1 (i = 1, \cdots, r)$, we get the relation

**Corollary 3.**

$$\int_a^b t^{\alpha}(1-t)^{-\alpha-2} \gimel \left( \left[ \frac{t(1-t)}{Ct+D(1-t)} \right]^2, \cdots, \left[ \frac{t(1-t)}{Ct+D(1-t)} \right]^2 \right) \mathrm{d}t = 2\sqrt{\pi} \left(\frac{D}{C}\right)^{\alpha+1}$$

$$\gimel^{U;0,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y} \left( \begin{matrix} (4\mathrm{CD})^{-1} \\ \cdot \\ \cdot \\ (4\mathrm{CD})^{-1} \end{matrix} \middle| \begin{matrix} \mathbb{A}; (-\alpha; a_1, \cdots, a_r; 1), (\alpha+2; a_1, \cdots, a_r; 1), \mathbf{A}; A \\ \cdot \\ \cdot \\ \mathbb{B}; \mathbf{B}, (1;1, \cdots, 1; 1), \left(\frac{1}{2}; 1, \cdots, 1; 1\right) : B \end{matrix} \right) \tag{4.3}$$

in corollary 2, taking $a = 0, b = 1, a_i = 1 (i = 1, \cdots, r)$ , we obtain the following integral

**Corollary 4.**

$$\int_a^b t^{\alpha}(1-t)^{-\frac{1}{2}} \gimel \left( \left[ \frac{t(1-t)}{C^2t+D^2(1-t)} \right], \cdots, \left[ \frac{t(1-t)}{C^2t+D^2(1-t)} \right] \right) \mathrm{d}t = \sqrt{\pi}$$

$$\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+1,\tau_{i_r}:R_r:Y} \left( \begin{matrix} (\mathrm{C+D})^{-2} \\ \cdot \\ \cdot \\ (\mathrm{C+D})^{-2} \end{matrix} \middle| \begin{matrix} \mathbb{A}; \left(\frac{1}{2}; 1, \cdots, 1; 1\right), \mathbf{A}; A \\ \cdot \\ \cdot \\ \mathbb{B}; \mathbf{B}, (0;1, \cdots, 1; 1) : B \end{matrix} \right) \tag{4.4}$$

**Remark 6.**
If $m_2 = n_2 = \cdots = m_{r-1} = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$, then we can obtain the same Eulerian integrals in the generalized multivariable Aleph- function ( extension of multivariable Aleph-function defined by Ayant [1])

**Remark 7.**
If $m_2 = n_2 = \cdots = m_r = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 =$

$= \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then we can obtain the same Eulerian integrals in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [6]).

**Remark 8.**
If $A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then we can obtain the same Eulerian integrals in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [5]).

**Remark 9.**
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and Panda [9,10] and then we can obtain the same Eulerian integrals, see Shrivastava [8] for more details.

## 5. Conclusion.

The Eulerian integrals possesses double fold generality in term of variables. By specializing the various parameters and variables involved, these formulae can suitably be applied to derive the corresponding results involving wide variety of useful functions (or product of several such functions) which can be expressed in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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