Laceability Properties in the Cartesian Product of Brick Product Graphs

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Abstract

A connected graph G is termed hamiltonian laceable if there exists in it a hamiltonian path between every pair of distinct vertices at an odd distance. The brick product of even cycles C(2n,m,r) was introduced by Alspach et.al. in [1] to study hamiltonian laceability properties. In this paper, we prove that the triple cartesian product of the brick product graph C(2n,1,3) with cycle graph of order3 is hamiltonian laceable.

Keywords — *Hamiltonian path, hamiltonian laceable graph, brick product graph, cartesian product*

I. INTRODUCTION

Let G is a finite, simple, connected and undirected graph. Let u and v be two vertices in G. The distance between u and v denoted by $d(u,v)$ is the length of a shortest path in G. G is called a hamiltonian laceable graph if there exists a hamiltonian path between every pair of vertices at an odd distance in it. This definition was generalized in [4], where hamiltonian-t-laceable graphs were introduced. G is said to be a hamiltonian-t-laceable graph if there exists a hamiltonian path between every pair of vertices u and v in it that satisfy the property $d(u,v)$=t, where $1 \leq t \leq diamG$. By definition, G is hamiltonian connected if it is hamiltonian-t-laceable for all t. Laceability properties of brick product graphs associated with even cycles was studied by Alspach et.al. in [1]. Inspired by this work, in this paper, we explore the laceability properties of the triple cartesian product of the two brick product graph $C(2n,1,3)$ with cycle graph of order 3. We refer [3] for the standard definitions not provided in this paper.

II. BRICK PRODUCT

Let m, n and r be a positive integers. Let $C_{2n}=v_0,v_1,...,v_{2n-1},v_0$ denote a cycle of order 2n ($v_0,v_1,...,v_{2n-1}$ are called cycle vertices). The $(m.r)$-brick-product of C_{2n}, denoted by $C(2n,m,r)$ is defined in two cases as follows.
Case (i): For $m=1$, we require that r be odd and greater than 1. Then, $C(2n,m,r)$ is obtained from C_{2n} by adding chords (v_{2k}, v_{2k+r}), $k=1,2,...,n$, where the computation is performed modulo 2n.
Case (ii): For $m>1$, we require that $m+r$ be even. Then, $C(2n,m,r)$ is obtained by first taking the disjoint union of m copies of C_{2n}, namely $C_{2n}(1), C_{2n}(2)$ . . . . . $,C_{2n}(m)$, where for each $i=1,2,...,m$, $C_{2n}(i)=(v_i, v_0)(v_i, v_1)$.......$(v_i, v_{2n})$. Next, for each odd $i=1,2,...,m-1$ and for each even $k=1,2,...,2n-2$, an edge (called a brick edge) is drawn to join (v_i, v_k) to (v_{i+1}, v_k), whereas, for each even $i=1,2,...,m-1$ and for each odd $k=1,2,...,2n-1$, an edge (also called a brick edge) is drawn to join (v_i, v_k) to (v_{i+1}, v_k). Finally, for each odd $k=1,2,...,2n-1$, an edge (called a hooking edge) is drawn to join (v_1, v_k) to (v_m, v_{k+r}). An edge in $C(2n,m,r)$ which is neither a brick edge nor a hooking edge is called a flat edge.

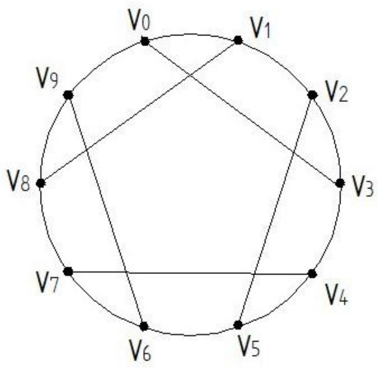

Figure 1. Brick product *C(10,1,3)*

Definition 2.1: Let P be a path between the vertices v_i to v_j in a graph G and let P' be a path between the vertices v_i and v_k. Then, the path P is the path obtained by extending the path $P \cup P'$ from v_i to v_j, to v_k through the common vertex v_i (i.e. if $P : v_i \ldots . v_j$ and $P' : v_j \ldots . v_k$ then $P \cup P' : v_i \ldots v_j \ldots v_k$).

Definition 2.2: The cartesian product $G_1 X G_2$ of the graphs G_1 and G_2 is a graph such that the vertex set of $G_1 X G_2$ is the cartesian product $V(G_1) \times V(G_2)$, and, any two vertices (u, u') and (v, v') are adjacent in $G_1 X G_2$ if and only if either $u = v$ and u' is adjacent to v' in G_1, or $u' = v'$ and u is adjacent to v in G_2.

The triple cartesian product $G_1 X G_2 X G_3$ is defined analogously.

III. MAIN RESULTS

In this section, we prove that the triple cartesian product of the two brick product graph $C(2n,1,3)$ with a cycle graph of order 3, which is a graph with $8n^3$ vertices and has diameter = 4, is both hamiltonian-1-laceable and hamiltonian-3-laceable i.e., H is hamiltonian laceable. If G_1, G_2 are two copies of $C(2n,1,3)$ and $G_3=C_3$ is a cycle graph of order 3, we denote by H the graph $G_1 X G_2 X G_3$.

Theorem 3.1: For n ≥ 3, the graph H is hamiltonian-1-laceable.

Proof: Let $v_{i,j}$, $v'_{i,j}$ and $v''_{i,j}$ be the cycle vertices of G'_1, G'_2 and G'_3 are the first, second and third copies of $G_1 X G_2$ respectively, where $i=0,1,2,\ldots,2n-1$.

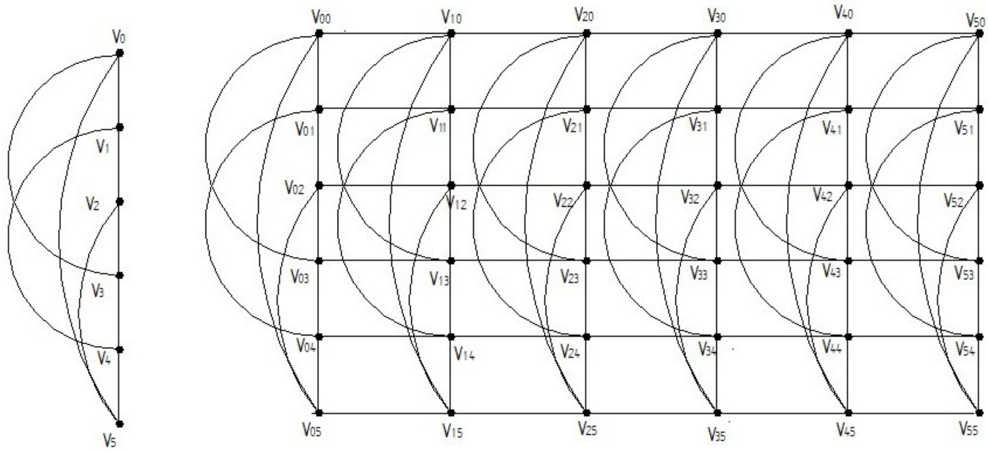

Figure 2: Graph $G_1 = G_2 = C(6,1,3)$ and its Cartesian product of $G_1 X G_2$

Claim 1: There exists a hamiltonian path between $v_{i,j}$ and $v_{i+1,j}$.

Let P_1 be the path between $v_{i,j}$ and $v_{i+2,j}$ traversing on the cycle G'_1 in anticlockwise direction that covers all the vertices except $v_{i+1,j}$ and let $e=P_2$ be the edge $(v_{i+2,j}, v'_{i+3,j})$. Since G'_2 is hamiltonian laceable, there exists a hamiltonian path P_3 between the vertices $v'_{i+2,j}$ and $v'_{i+3,j}$ in G'_2.

Now, let $e'=P_4$ be the edge $(v'_{i+2,j}, v''_{i+3,j})$. Again since G'_3 is hamiltonian laceable, there exists a path P_5 between the vertices $v''_{i+2,j}$ and $v''_{i+3,j}$.

Then, clearly, the path $P = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6$, where $e''=P_6=(v_{i+1}, v''_{i+2})$ is a hamiltonian path between the vertices v_i and v_{i+1} in H.

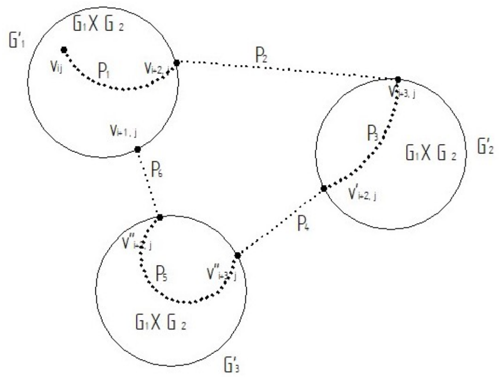

Figure 3. Hamiltonian path between $v_{i,j}$ and $v_{i+1,j}$ in H

Claim 2: There exists a hamiltonian path between $v_{2k,j}$ and $v_{2k+3,j}$, where $k=1,2,3,...,n$.

Let P_1 be the path between $v_{2k,j}$ and $v_{2k+4,j}$ traversing on the cycle G'_1 in anticlockwise direction that covers all the vertices except $v_{2k+3,j}$ and let $e=P_2$ be the edge $(v_{2k+2,j}, v'_{m,j})$. Since G'_2 is hamiltonian laceable, there exists a hamiltonian path P_3 between the vertices $v'_{m,j}$ and $v'_{m+1,j}$ in G'_2.

Now, let $e'=P_4$ be the edge $(v'_{m+1,j}, v''_{l,j})$. Again since G'_3 is hamiltonian laceable, there exists a path P_5 between the vertices $v''_{l,j}$ and $v''_{l+1,j}$.

Then, clearly, the path $P = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6$, where $e''=P_6=(v_{2k+3,j}, v''_{l+1,j})$ is a hamiltonian path between the vertices $v_{2k,j}$ and $v_{2k+3,j}$ in H.

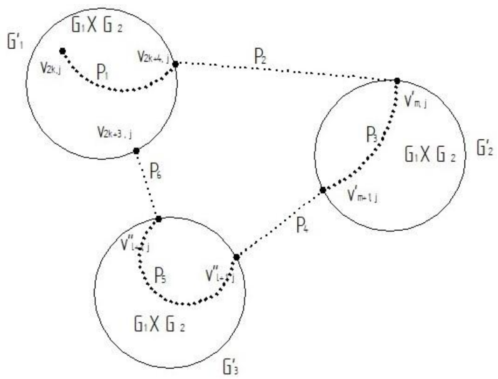

Figure 4. Hamiltonian path between $v_{2k,j}$ and $v_{2k+3,j}$ in H

Theorem 3.2: For $n \geq 4$ the graph H is a hamiltonian-3-laceable.
Proof: The vertices of H are as in theorem 3.1.
Claim: There exists a hamiltonian path between $v_{i,j}$ and $v_{i+5,j}$.

Let P_1 be the path between $v_{i,j}$ and $v_{i+4,j}$ traversing on the cycle G'_1 in anticlockwise direction that covers all the vertices except $v_{i+5,j}$ and let $e=P_2$ be the edge $(v_{i+4,j}, v'_{m,j})$. Since G'_2 is hamiltonian laceable, there exists a hamiltonian path P_3 between the vertices $v'_{m,j}$ and $v'_{m+1,j}$ in G'_2.

Now, let $e'=P_4$ be the edge $(v'_{m+1,j}, v''_{l,j})$. Again since G'_3 is hamiltonian laceable, there exists a path P_5 between the vertices $v''_{l,j}$ and $v''_{l+1,j}$.

Then, clearly, the path $P = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \cup P_6$, where $e''=P_6=(v_{i+5,j}, v''_{l+1,j})$ is a hamiltonian path between the vertices $v_{i,j}$ and $v_{i+5,j}$ in H.

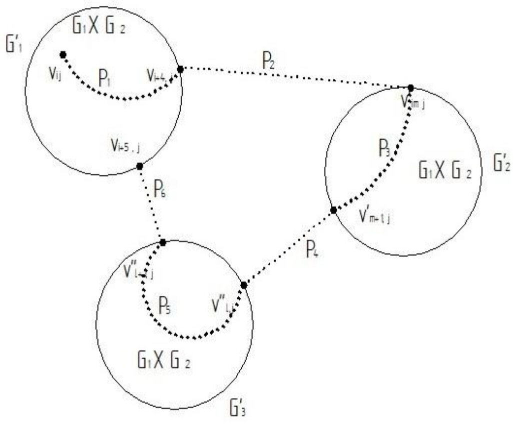

Figure 4. Hamiltonian path between $v_{i,j}$ and $v_{i+5,j}$ in H $\left(d\left(v_{i,j}, v_{i+5,j}\right) = 3\right)$

IV. CONCLUSIONS

In this paper, we show that the triple cartesian product of the two brick product graph $C(2n,1,3)$ with cycle graph of order 3 (for all $n \geq 3$) is hamiltonian laceable. The significance of hamiltonicity and hamiltonian laceability has been established in computer networks. This concludes that the existence of a hamiltonian path in such networks suffice to solve data communication problems.

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