Fractional Matrix and Spectral Analysis on Real and Complex Operators

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We propose a new matrix called ***Fractional - Matrix****. Suitable matrix, real operators and complex operators have been considered for practical use of it . Essentially this matrix is a cross-checking device on spectral analysis.*

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I. INTRODUCTION

Matrix method has an important application in literature of eigenvalue calculation[1]. In fact the eigenvalues obtained in a matrix can be cross checked. through inverse matrix. Let A is a square matrix(nxn) having eigenvalues $\lambda_1; \lambda_2; \lambda_3; \lambda_4 \dots \dots \dots \lambda_n$. If $|A| \neq = 0$ then eigenvalues of A^{-1} can be written as $\frac{1}{\lambda_1}; \frac{1}{\lambda_2}; \frac{1}{\lambda_3}; \frac{1}{\lambda_4}; \dots \frac{1}{\lambda_n}$. This method is valid provided $\lambda_1 \neq 0$. Now question arises as if λ_1=0 , then how one will cross check the result? . To the best of my knowledge, no literature in mathematics nor applied mathematics deal with it. Keeping this in mind, I introduce a new matrix called Fractional - matrix to the literature on matix analysis and perform different types of calculation pertaining to mathematics as well as physics as discussed below.

II. FRACTIONAL MATRIX IN EIGENVALUE CALCULATION

Let A be a matrix having eigenvalue relation.

$$A|\Psi> = \lambda|\Psi> \tag{1}$$

Now define a matrix F as

$$F = \frac{A}{A+L} \tag{2}$$

here L is a numerical constant . Let the eigenvalues of F be β then

$$\lambda = \frac{L\beta}{1-\beta} \tag{3}$$

more explicitly

$$\lambda_n = \frac{L\beta_n}{1-\beta_n} \tag{4}$$

In order to give an explicit example .we consider simple (3x3) matrix as

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \tag{5}$$

having eigenvalues $\lambda_1 = 4; \lambda_2 = 0; \lambda_3 = 0$ and $|A| = 0$ Now define the matrix F as for different values of L as *F_L=I,2.3*

The explicit expression for F is

$$F_1 = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{5} \end{bmatrix} \tag{6}$$

$$F_2 = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} \tag{7}$$

$$F_3 = \begin{bmatrix} \frac{1}{7} & \frac{2}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{2}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{2}{7} & \frac{1}{7} \end{bmatrix} \tag{8}$$

in table-I we compare the eigenvalues of F_L and A .

Table -1 : Eigenvalues of A and F_L=I,2,3.

.L	β_n	Direct Calculation λ_n	Eq(4) $\lambda_n = \frac{L\beta_n}{1-\beta_n}$
1	**0** **0** $\mathbf{\frac{4}{5}}$	0 0 4	0 0 4
2	**0** **0** $\frac{2}{3}$	0 0 4	0 0 4
3	0 0 $\frac{4}{7}$	0 0 4	0 0 4

III. QUANTUM OPERATORS AND MATNX ANALYSIS

All quantum operators consisting of co-ordinate(x) and *momentum(p)* satisfies the commutation relation[2,3]

$$[x, p] = i \tag{9}$$

where $i = \sqrt{-1}$. However while applying matrix method to operators involving x and p ,we need a suitable base function. In this analysis, we consider wave functions of harmonic oscillator[2] as a suitable base function as follows.

$$H_0|\Psi_n> = [p^2 + x^2]|\Psi_n> = (2n+1)|\Psi_n> \tag{10}$$

$$\Psi_n = N_n H_n(x) e^{-x^2/2} \tag{11}$$

where $H_n(x)$ is Hemite polynomial and N_n is the corresponding normalization constant . In short Ψ_n can be written as $|n>$ Matrix elements of x and p are as follows[2,3]

$$< n|x|n+1 >=< n+1|x|n >= \frac{\sqrt{n+1}}{\sqrt{2}} \tag{12}$$

$$< n|p|n+1 >= -i\,\frac{\sqrt{n+1}}{\sqrt{2}} \tag{13}$$

$$< n+1|p|n >= i\,\frac{\sqrt{n+1}}{\sqrt{2}} \tag{14}$$

Further if the operator is even then it is easy to show that

$$< n|x^{2m}|n >=< n|p^{2m}|n > \tag{15}$$

IV. DIRECT EIGENVALUE CALCULATION USING MATRIX METHOD : REAL MATRIX

In order to calculate eigenvalues we wse matix diagonalisation method [4-9] by solving the eigenvalue relation as

$$H|\Psi > = E|\Psi > \tag{16}$$

where

$$|\Psi> = \sum_m A_m |m > \tag{17}$$

In fact solving the eigenvalue using matrix analysis depends on the selection H. For example consider the a model Hamiltonian as

$$H = p^2 + x^6 - 3x^2 \tag{18}$$

Where $|m>$ is the harmonic oscillator as described above. In this case we get

$$P_m A_{m-6} + Q_m A_{m-4} + R_m A_{m-2} + S_m A_m + T_m A_{m+2} + U_m A_{m+4} + V_m A_{m+6} = 0 \tag{19}$$

where

$$P_m =< m - 6|H|m > \tag{20}$$
$$Q_m =< m - 4|H|m > \tag{21}$$
$$R_m =< m - 2|H|m > \tag{22}$$
$$S_m =< m|H|m > -E \tag{23}$$
$$T_m = R_{m+2} \tag{24}$$
$$U_m = Q_{m+4} \tag{25}$$
$$V_m = P_{m+6} \tag{26}$$

For the benefit of interested reader we give the explicit expression for S_m as

$S_m = 2.5m^3 + 3.75m^2 + 3m + 0.875 - E$ (27)

The explicit expression for a (5x5) matrix of H is given below.

$$H_{matrix} == \begin{vmatrix} 0.875 & 0 & 5.12652 & 0 & 9.18558 \\ 0 & 10.125 & 0 & 27.25057 & 0 \\ 5.12652 & 0 & 41.875 & 0 & 77.50927 \\ 0 & 27.25057 & 0 & 111.125 & 0 \\ 91.85586 & 0 & 77.50927 & 0 & 232.875 \end{vmatrix} \tag{28}$$

Further we also notice that the equivalent operator [8]

$$h = x^2 + p^6 - 3p^2 \tag{29}$$

also yield the same eigenvalue. For the interest of the reader we present (5x5) matrix of h as given below.

$$h_{matrix} == \begin{vmatrix} 0.875 & 0 & 5.12652 & 0 & 9.18558 \\ 0 & 10.125 & 0 & 27.25057 & 0 \\ 5.12652 & 0 & 41.875 & 0 & 77.50927 \\ 0 & 27.25057 & 0 & 111.125 & 0 \\ 91.85586 & 0 & 77.50927 & 0 & 232.875 \end{vmatrix} \tag{30}$$

One will see that in both the cases diagonal elements remains the same. The first five eigenvalues are tabulated in table-2.

Table -2 : Eigenvalues of *H* and *h*.

n	Eigenvalues of H	Eigenvalues of h	Previous [7]
0	0	0	0
1	1.935 482	1.935 482	1.935 482
2	6.298 495	6.298495	6.298495
3	11.680 970	11.680 970	11.680 970
4	18.042 634	18.042 634	

IVB. Direct Eigenvalue Calculation using Matrix Method : Complex Matrix

Here we consider the complex cubic oscillator characterized by the Hamiltonian [10]

$$H = p^2 + ix^3 \tag{31}$$

Following the above method we get a seven term recurrence relation satisfied by A_m as follows

$$P_m A_{m-3} + Q_m A_{m-2} + R_m A_{m-1} + S_m A_m + T_m A_{m+3} + U_m A_{m+2} + V_m A_{m+1} = 0 \tag{32}$$

Here

$$P_m =< m-3|H|m > \tag{33}$$

$$Q_m =< m-2|H|m > \tag{34}$$

$$R_m =< m-1|H|m > \tag{35}$$

$$S_m =< m|H|m > -E \tag{36}$$

$$T_m = R_{m+1} \tag{37}$$

$$U_m = Q_{m+2} \tag{38}$$

$$V_m = P_{m+3} \tag{39}$$

For the benefit of interested reader we give the explicit expression for S_m as

$$S_m = m + 0.5 - E \tag{40}$$

In this case also we notice its equivalent operato h

$$h = x^2 + ip^3 \tag{41}$$

also yields the same eigenvalue. Interested reader can check the iso-spectra.
The first five eigenvalues using direct matrix method are given below in table-3.

Table -3 : Eigenvalues of Complex Cubic Oscillator.

n	Eigenvalues of H	Eigenvalues of h	Previous[10.5]
0	1.156 2	1.156 2	1.156 2
1	4.109 2	4.109 2	4.109 2
2	7.562 1	7.562 1	7.562 1
3	11.314 3	11.314 3	11.314 3
4	15.291 6	15.291 6	15.291 6

V. Fractional Matrix Eigenvalue Method(FMEM)

Here we use FMEM to compute eigenvalues of matrix H_{matrix} by defining F as

$$F = \frac{H}{H+L} \tag{42}$$

The eigenvalues are calculated using the method reported earlier[4-6] **In** particular we consider L=1 and compute the values of β_n as shown in table.

Table -3 : Eigenvalues of *E*, *H* and *h*.

Hamitonian(H or h)	β_n	$E_n = \frac{\beta_n}{(1-\beta_n)}$	Remarks
$H = p^2 + x^2$	$\frac{1}{2}$ $\frac{3}{4}$ $\frac{5}{6}$ $\frac{7}{8}$ $\frac{9}{10}$	1 3 5 7 9	no difference in eigenvalues
$H = p^2 + x^6 - 3x^2$ $h = x^2 + p^6 - 3p^2$	0 $\frac{60}{291}$ $\frac{422}{489}$ $\frac{549}{596}$ $\frac{848}{895}$	0 1.935 4 6.298 5 11.507 85 18.043 55	no difference in eigenvalues
$H = p^2 + ix^3$ $h = x^2 + ip^3$	$\frac{2123}{3974}$ $\frac{480}{608}$ $\frac{2125}{2406}$ $\frac{396}{431}$ $\frac{367}{391}$	1.156 2 4.109 2 7.562 2 11.314 2 15.291 6	no difference in eigenvalues

VI. CONCLUSION

We have proposed a new matrix to the literature on matrix. The present application to standard matrix and quantum operator will motivate the reader for further study. In fact FMEM is a self checked method on eigenvalue. We believe this paper will be of interest to students, researchers in physics as well as in mathematics.

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