Contour Integrals Involving Generalized Associated Function and Generalzed Multivariable Gimel-Function

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ABSTRACT
The aim of the paper is to evaluate some contour integrals involving generalized Legendre associated functions and the generalized multivariable Gimel-function where the integration is performed with respect to parameters of generalized Legendre associated functions.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, contour integral, generalized Legendre functions.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C},\mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

We define a generalized transcendental function of several complex variables.

$$\gimel(z_1,\cdots,z_r)=\gimel^{m_2,n_2;m_3,n_3;\cdots;m_r,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c} \mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\end{array}\right|$$

$$[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$
$$[(\mathrm{b}_{2j};\beta_{2j}^{(1)},\beta_{2j}^{(2)};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta_{3j}^{(1)},\beta_{3j}^{(2)},\beta_{3j}^{(3)};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{rj};\beta_{rj}^{(1)},\cdots,\beta_{rj}^{(r)};B_{rj})_{1,m_r}],$$

$$[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{n_r+1,p_r}]:\ [(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}]$$
$$[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{m_r+1,q_r}]:[(\mathrm{d}_j^{(1)}),\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}]$$

$$\left.\begin{array}{l};\cdots;[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}]\\ ;\cdots;[(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,n^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{n^{(r)}+1,q_i^{(r)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{m_2}\Gamma^{B_{2j}}(b_{2j}-\sum_{k=1}^{2}\beta_{2j}^{(k)}s_k)\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha_{2ji_2}^{(k)}s_k)\prod_{j=m_2+1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta_{2ji2}^{(k)}s_k)]}$$

$$\frac{\prod_{j=1}^{m_3}\Gamma^{B_{3j}}(b_{3j}-\sum_{k=1}^{3}\beta_{3j}^{(k)}s_k)\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=m_3+1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{m_r}\Gamma^{B_{rj}}(b_{rj}-\sum_{k=1}^{r}\beta_{rj}^{(k)}s_k)\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=m_r+1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)})]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $m_2,n_2,\cdots,m_r,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2\leqslant q_{i_2},0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant m_r\leqslant q_{i_r},0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kj}^{(l)},B_{kj}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{R}^+; (i=1,\cdots,R^{(k)}); (j=n^{(k)}+1,\cdots,p_{i^{(k)}}); (k=1,\cdots,r).$

5) $c_j^{(k)} \in \mathbb{C}; (j=1,\cdots,n_k); (k=1,\cdots,r); d_j^{(k)} \in \mathbb{C}; (j=1,\cdots,m_k); (k=1,\cdots,r).$

$a_{kji_k} \in \mathbb{C}; (j=n_k+1,\cdots,p_{i_k}); (k=2,\cdots,r).$

$b_{kji_k} \in \mathbb{C}; (j=m_k+1,\cdots,q_{i_k}); (k=2,\cdots,r).$

$d_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i=1,\cdots,R^{(k)}); (j=m^{(k)}+1,\cdots,q_{i^{(k)}}); (k=1,\cdots,r).$

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i=1,\cdots,R^{(k)}); (j=n^{(k)}+1,\cdots,p_{i^{(k)}}); (k=1,\cdots,r).$

The contour L_k is in the $s_k(k=1,\cdots,r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k\right)(j=1,\cdots,n_2), \Gamma^{A_{3j}}\left(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k\right)$ $(j=1,\cdots,n_3),\cdots,\Gamma^{A_{rj}}\left(1-a_{rj}+\sum_{i=1}^{r}\alpha_{rj}^{(i)}\right)(j=1,\cdots,n_r)$, $\Gamma^{C_j^{(k)}}\left(1-c_j^{(k)}+\gamma_j^{(k)}s_k\right)(j=1,\cdots,n^{(k)})(k=1,\cdots,r)$to the right of the contour L_k and the poles of $\Gamma^{B_{2j}}\left(b_{2j}-\sum_{k=1}^{2}\beta_{2j}^{(k)}s_k\right)(j=1,\cdots,m_2), \Gamma^{B_{3j}}\left(b_{3j}-\sum_{k=1}^{3}\beta_{3j}^{(k)}s_k\right)(j=1,\cdots,m_3)$ $,\cdots,\Gamma^{B_{rj}}\left(b_{rj}-\sum_{i=1}^{r}\beta_{rj}^{(i)}\right)(j=1,\cdots,m_r)$, $\Gamma^{D_j^{(k)}}\left(d_j^{(k)}-\delta_j^{(k)}s_k\right)(j=1,\cdots,m^{(k)})(k=1,\cdots,r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2}A_i^{(k)}\pi$ where

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)}\delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)}\gamma_j^{(k)} - \tau_{i^{(k)}}\left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)}\delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)}\gamma_{ji^{(k)}}^{(k)}\right) +$$

$$\sum_{j=1}^{n_2} A_{2j}\alpha_{2j}^{(k)} + \sum_{j=1}^{m_2} B_{2j}\beta_{2j}^{(k)} - \tau_{i_2}\left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2}\alpha_{2ji_2}^{(k)} + \sum_{j=m_2+1}^{q_{i_2}} B_{2ji_2}\beta_{2ji_2}^{(k)}\right) + \cdots +$$

$$\sum_{j=1}^{n_r} A_{rj}\alpha_{rj}^{(k)} + \sum_{j=1}^{m_r} B_{rj}\beta_{rj}^{(k)} - \tau_{i_r}\left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r}\alpha_{rji_r}^{(k)} + \sum_{j=m_r+1}^{q_{i_r}} B_{rji_r}\beta_{rji_r}^{(k)}\right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,), max(\,|z_1|,\cdots,|z_r|\,) \rightarrow 0$

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,), min(\,|z_1|,\cdots,|z_r|\,) \rightarrow \infty$ where $i=1,\cdots,r$:

$\alpha_i = \min\limits_{\substack{1\leqslant j\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)$ and $\beta_i = \max\limits_{\substack{1\leqslant j\leqslant n_i \\ 1\leqslant j\leqslant n^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}} + C_j^{(i)}\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)$

Remark 1.
If $m_2=n_2=\cdots=m_{r-1}=n_{r-1}=p_{i_2}=q_{i_2}=\cdots=p_{i_{r-1}}=q_{i_{r-1}}=0$ and $A_{2j}=B_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=B_{rj}=A_{rji_r}=B_{rji_r}=1$, then the generalized multivariable Gimel-function reduces in the generalized multivariable Aleph- function (extension of multivariable Aleph-function defined by Ayant [1]).

Remark 2.

If $m_2 = n_2 = \cdots = m_r = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [6]).

Remark 3.

If $A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [5].

Remark 4.

If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and Panda [8,9]

In your investigation, we shall use the following notations.

$$\mathbb{A} = [(a_{2j}; \alpha_{2j}^{(1)}, \alpha_{2j}^{(2)}; A_{2j})]_{1,n_2}, [\tau_{i_2}(a_{2ji_2}; \alpha_{2ji_2}^{(1)}, \alpha_{2ji_2}^{(2)}; A_{2ji_2})]_{n_2+1,p_{i_2}}, [(a_{3j}; \alpha_{3j}^{(1)}, \alpha_{3j}^{(2)}, \alpha_{3j}^{(3)}; A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3}; \alpha_{3ji_3}^{(1)}, \alpha_{3ji_3}^{(2)}, \alpha_{3ji_3}^{(3)}; A_{3ji_3})]_{n_3+1,p_{i_3}}; \cdots ; [(a_{(r-1)j}; \alpha_{(r-1)j}^{(1)}, \cdots, \alpha_{(r-1)j}^{(r-1)}; A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}}; \alpha_{(r-1)ji_{r-1}}^{(1)}, \cdots, \alpha_{(r-1)ji_{r-1}}^{(r-1)}; A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A} = [(a_{rj}; \alpha_{rj}^{(1)}, \cdots, \alpha_{rj}^{(r)}; A_{rj})_{1,n_r}], [\tau_{i_r}(a_{rji_r}; \alpha_{rji_r}^{(1)}, \cdots, \alpha_{rji_r}^{(r)}; A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A = [(c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,n^{(1)}}], [\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)}; C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}]; \cdots ;$$

$$[(c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1,m^{(r)}}], [\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)}; C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B} = [(b_{2j}; \beta_{2j}^{(1)}, \beta_{2j}^{(2)}; B_{2j})]_{1,m_2}, [\tau_{i_2}(b_{2ji_2}; \beta_{2ji_2}^{(1)}, \beta_{2ji_2}^{(2)}; B_{2ji_2})]_{m_2+1,q_{i_2}}, [(b_{3j}; \beta_{3j}^{(1)}, \beta_{3j}^{(2)}, \beta_{3j}^{(3)}; B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(b_{3ji_3}; \beta_{3ji_3}^{(1)}, \beta_{3ji_3}^{(2)}, \beta_{3ji_3}^{(3)}; B_{3ji_3})]_{m_3+1,q_{i_3}}; \cdots ; [(b_{(r-1)j}; \beta_{(r-1)j}^{(1)}, \cdots, \beta_{(r-1)j}^{(r-1)}; B_{(r-1)j})_{1,m_{r-1}}],$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}}; \beta_{(r-1)ji_{r-1}}^{(1)}, \cdots, \beta_{(r-1)ji_{r-1}}^{(r-1)}; B_{(r-1)ji_{r-1}})_{m_{r-1}+1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B} = [(b_{rj}; \beta_{rj}^{(1)}, \cdots, \beta_{rj}^{(r)}; B_{rj})_{1,m_r}], [\tau_{i_r}(b_{rji_r}; \beta_{rji_r}^{(1)}, \cdots, \beta_{rji_r}^{(r)}; B_{rji_r})_{m_r+1,q_{i_r}}] \tag{1.9}$$

$$B = [(d_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1,m^{(1)}}], [\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)}; D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}]; \cdots ;$$

$$[(d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{1,m^{(r)}}], [\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)}; D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U = m_2, n_2; m_3, n_3; \cdots ; m_{r-1}, n_{r-1}; V = m^{(1)}, n^{(1)}; m^{(2)}, n^{(2)}; \cdots ; m^{(r)}, n^{(r)} \tag{1.11}$$

$$X = p_{i_2}, q_{i_2}, \tau_{i_2}; R_2; \cdots ; p_{i_{r-1}}, q_{i_{r-1}}, \tau_{i_{r-1}} : R_{r-1}; Y = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}; \cdots ; p_{i^{(r)}}, q_{i^{(r)}}; \tau_{i^{(r)}}; R^{(r)} \tag{1.12}$$

2. Required results.

In the first times, we evaluate an integral concerning the generalized Legendre associated function.

Lemma 1. (Braaksma and Meulenbeld [3], p. 277(8.2))

$$\int_{-1}^{1}(1-x)^{p}(1+x)^{k-p-1}P_k^{n,m}(t)\mathrm{d}t=2^{k+\frac{m-n}{2}}\frac{\Gamma\left(k-p+\frac{m}{2}\right)\Gamma\left(k-p-\frac{m}{2}\right)\Gamma\left(p-\frac{n}{2}+1\right)}{\Gamma\left(k+\frac{m-n}{2}+1\right)\Gamma\left(k-\frac{m+n}{2}+1\right)\Gamma\left(p-\frac{n}{2}\right)} \tag{2.1}$$

provided $Re\left(p-\frac{n}{2}\right)>-1, Re(k-p)>\frac{1}{2}|Re(m)|$

Lemma 2.(Braaksma and Meulenbeld [3])

Let n_1, be a real number and

$n_1<min[Re(2k+2-m, Re(-2k-m)]$; $\phi(t)$ is a function such that for all, $-1<a<1:$

$\phi(t)=(1+t)^{-\frac{1}{2}-\frac{1}{2}|Re(m)|}\in L(-1,a), if Re(m)\neq 0$

$\phi(t)=(1+t)^{-\frac{1}{2}}log(1+t)\ inL(-1,a), if Re(m)=0$

$\phi(t)=(1-t)^{-\frac{n_1}{2}-1}\in L(a,1).$

Let further be of bounded variation in a neighbourhood of $t=x(-1<x<1)$. Then $\phi(t)$ satsfies the relations

$$\frac{1}{2\omega\pi}\int_{n_1-\omega\infty}^{n_1+\omega\infty}ndn\Gamma(\lambda+1)\Gamma(-\mu)P_k^{m,n}(-x)\int_{-1}^{1}\phi(t)P_k^{m,n}(t)\frac{\mathrm{d}t}{1-t}=-[\phi(x-0)+\phi(x+0)] \tag{2.2}$$

and

$$\frac{1}{2\omega\pi}\int_{n_1-\omega\infty}^{n_1+\omega\infty}ndn\Gamma(\lambda+1)\Gamma(-\mu)P_k^{m,n}(-x)\int_{-1}^{1}\phi(t)P_k^{m,n}(-t)\frac{\mathrm{d}t}{1-t}=-[\phi(x-0)+\phi(x+0)] \tag{2.3}$$

where $\lambda=k-\frac{m+n}{2},\mu=k+\frac{m+n}{2}$

3. Main integral.

Theorem 1.

$$\int_{-1}^{1}(1-x)^{p}(1+x)^{k-p-1}P_k^{n,m}(t)\daleth\left(z_1\left(\frac{1-x}{1+x}\right)^{a_1},\cdots,z_r\left(\frac{1-x}{1+x}\right)^{a_r}\right)\mathrm{d}x=$$

$$2^{k+\frac{m-n}{2}}\left[\Gamma\left(k+\frac{m-n}{2}+1\right)\Gamma\left(k-\frac{m+n}{2}+1\right)\right]^{-1}$$

$$\daleth_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;m_r+2,n_r+1:V}\left(\begin{array}{c|c}\mathrm{z}_1 & \mathbb{A};\left(\frac{n}{2}-p;a_1,\cdots,a_r;1\right),\mathbf{A},\left(-\frac{n}{2}-p;a_1,\cdots,a_r;1\right):A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r & \mathbb{B};\ \left(k-p+\frac{m}{2};a_1,\cdots,a_r:1\right),\left(k-p-\frac{m}{2};a_1,\cdots,a_r:1\right),\mathbf{B}:B\end{array}\right) \tag{3.1}$$

provided

$a_i>0(i=1,\cdots,r)$, $Re(k-p)>\frac{1}{2}|Re(m)|,\ Re\left(p-\frac{n}{2}\right)>0$

$$Re(p)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-1$$

$$Re(k-p)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant j\leqslant n_i\\1\leqslant j\leqslant n^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+C_j^{(i)}\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)>\frac{1}{2}|Re(m)|$$

$\left|arg\left(z_1\left(\frac{1-x}{1+x}\right)^{a_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof

To prove (3.1), on the left hand side of (2.1), expressing the generalized multivariable Gimel-function as Mellin-Barnes multiple integrals contour with the the help of (1.1), interchanging the order of integrations which is justified under the conditions mentioned above, we get (say I)

$$I=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}$$

$$\left[\int_0^1(1-x)^{p+\sum_{i=1}^r a_is_i}(1+x)^{k-p-\sum_{i=1}^r a_is_i-1}P_k^{m,n}(x)\mathrm{d}x\right] \tag{3.2}$$

Now, we evaluate the inner integral with the help of the lemma 1.

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\ 2^{k+\frac{m-n}{2}}$$

$$\frac{\Gamma\left(k-p-\sum_{i=1}^r a_is_i+\frac{m}{2}\right)\Gamma\left(k-p-\sum_{i=1}^r a_is_i-\frac{m}{2}\right)\Gamma\left(p+\sum_{i=1}^r a_is_i-\frac{n}{2}+1\right)}{\Gamma\left(k+\frac{m-n}{2}+1\right)\Gamma\left(k-\frac{m+n}{2}+1\right)\Gamma\left(p+\sum_{i=1}^r a_is_i-\frac{n}{2}\right)}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.3.}$$

and finally interpreting the Mellin-Barnes multiple integrals contour in terms of the generalized multivariable Gimel-function, we get the theorem 1.

4. Integrals contour.

In this section, we evaluate some contour integrals.

Theorem 2.

$$\frac{1}{2\pi\omega}\int_{n_1-\omega\infty}^{n_1+\omega\infty}2^{-\frac{n}{2}}n\Gamma\left(-k-\frac{m+n}{2}\right)\left[\Gamma\left(k+\frac{m-n}{2}+1\right)\right]^{-1}P_k^{m,n}(x)$$

$$\gimel_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;m_r+2,n_r+1:V}\left(\begin{array}{c|c}z_1 & \mathbb{A};\left(\frac{n}{2}-p;a_1,\cdots,a_r;1\right),\mathbf{A},\left(-\frac{n}{2}-p;a_1,\cdots,a_r;1\right):A\\ \cdot & \cdot\\ \cdot & \cdot\\ z_r & \mathbb{B};\left(k-p+\frac{m}{2};a_1,\cdots,a_r:1\right),\left(k-p-\frac{m}{2};a_1,\cdots,a_r:1\right),\mathbf{B}:B\end{array}\right)\mathrm{d}n=$$

$$-2^{1-k-\frac{m}{2}}(1+x)^{p+1}(1-x)^{k-p-1}\gimel\left(z_1\left(\frac{1+x}{1-x}\right)^{a_1},\cdots,z_r\left(\frac{1+x}{1-x}\right)^{a_r}\right) \tag{4.1}$$

provided

$$a_i>0(i=1,\cdots,r)\ min\left[2+Re(p)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right),-Re(2k+m)\right]>n_1$$

$$Re(k-p)-\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant j\leqslant n_i\\1\leqslant j\leqslant n^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+C_j^{(i)}\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)>\frac{1}{4};-1<x<1$$

$\left|arg\left(z_1\left(\frac{1+x}{1-x}\right)^{a_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Proof

In the lemma 2 (2.2) , let

$\phi(t)=-2^{1-k-\frac{m}{2}}(1+x)^{p+1}(1-x)^{k-p-1}\gimel\left(z_1\left(\frac{1-x}{1+x}\right)^{a_1},\cdots,z_r\left(\frac{1-x}{1+x}\right)^{a_r}\right)$ and using the theorem 1, we get (4.1)

Theorem 3.

$$\frac{1}{2\pi\omega}\int_{n_1-\omega\infty}^{n_1+\omega\infty}2^{-\frac{n}{2}}n\Gamma\left(-k-\frac{m-n}{2}\right)\left[\Gamma\left(k+\frac{m-n}{2}+1\right)\right]^{-1}P_k^{m,n}(x)$$

$$\gimel_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;m_r+2,n_r+1:V}\left(\begin{array}{c|c}z_1 & \mathbb{A};\left(p-k+\frac{m}{2}+1;a_1,\cdots,a_r;1\right),\left(p-k+\frac{m}{2}+1;a_1,\cdots,a_r;1\right),\mathbf{A}:A\\ \cdot & \cdot\\ \cdot & \cdot\\ z_r & \mathbb{B};\ \left(p+\frac{n}{2}+1;a_1,\cdots,a_r:1\right),\mathbf{B},\left(p+\frac{n}{2}+1;a_1,\cdots,a_r:1\right):B\end{array}\right)\mathrm{d}n=$$

$$-2^{1-k-\frac{m}{2}}(1+x)^{p+1}(1-x)^{k-p-1}\gimel\left(z_1\left(\frac{1-x}{1+x}\right)^{a_1},\cdots,z_r\left(\frac{1-x}{1+x}\right)^{a_r}\right)\tag{4.2}$$

provided

$$a_i>0(i=1,\cdots,r),\ Re(k-p)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)<2$$

$$min\left[2+Re(p)-2\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant j\leqslant n_i\\1\leqslant j\leqslant n^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+C_j^{(i)}\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right),-Re(2k+m)\right]>n_1$$

$$Re(k-p)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{a_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)<\frac{1}{4};-1<x<1$$

$\left|arg\left(z_1\left(\frac{1-x}{1+x}\right)^{a_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

In (7.1), using the algebraic property concerning the multivariable Gimel-function (similar property about H-function, see (Srivastava et al. [7], p. 14, Eq. (2.2.16)), we obtain the above result.

Theorem 4.

$$\frac{1}{2\pi\omega}\int_{n_1-\omega\infty}^{n_1+\omega\infty}(2k+1)\Gamma\left(k-\frac{m-n}{2}\right)\left[\Gamma\left(k+\frac{m-n}{2}+1\right)\right]^{-1}P_k^{m,n}(x)$$

$$\daleth^{U;m_r+2,n_r+1:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbb{A};\left(-k-p-\frac{1}{2};a_1,\cdots,a_r;1\right),\mathbf{A},\left(k-p+\frac{1}{2};a_1,\cdots,a_r;1\right):A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\left(-p-\frac{m+n}{2}-\frac{1}{2};a_1,\cdots,a_r:1\right),\left(-p-\frac{m-n}{2}-\frac{1}{2};a_1,\cdots,a_r:1\right),\mathbf{B}:B \end{array}\right)\mathrm{d}k=$$

$$2^{p+\frac{m-n+3}{2}}(1-x)^{p-\frac{n}{2}-\frac{3}{2}}(x+1)^{\frac{n}{2}}\daleth\left(z_1 2^{a_1}(x-1)^{-a_1},\cdots,z_r 2^{a_r}(x-1)^{-a_r}\right) \tag{4.3}$$

provided

$a_i > 0(i=1,\cdots,r)$

$$Re(n+2p)-\sum_{i=1}^{r}a_i \min_{\substack{1\leqslant j\leqslant n_i\\ 1\leqslant j\leqslant n^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+C_j^{(i)}\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)<-\frac{3}{2}$$

$$Max\left[Re\left(-\frac{3}{2}+p\right)+\sum_{i=1}^{r}a_i \min_{\substack{1\leqslant j\leqslant m_i\\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{a_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right),\frac{1}{2}Re(m-n)-1\right]<k_1$$

$|arg\left(z_1\left(x-1\right)\right)^{a_i})|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 5.

$$\frac{1}{2\pi\omega}\int_{n_1-\omega\infty}^{n_1+\omega\infty}(2k+1)\Gamma\left(k-\frac{m-n}{2}+1\right)\left[\Gamma\left(k+\frac{m-n}{2}+1\right)\right]^{-1}P_k^{m,n}(x)$$

$$\daleth^{U;m_r+2,n_r+1:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbb{A};\left(p-\frac{m-n}{2}+\frac{3}{2};a_1,\cdots,a_r;1\right),\left(p+\frac{m+n}{2}+\frac{3}{2};a_1,\cdots,a_r;1\right),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\left(k+p+\frac{3}{2};a_1,\cdots,a_r:1\right),\mathbf{B},\left(p-k+\frac{1}{2};a_1,\cdots,a_r:1\right):B \end{array}\right)\mathrm{d}n=$$

$$2^{p+\frac{m-n+3}{2}}(x-1)^{-p-\frac{n}{2}-\frac{3}{2}}(x+1)^{\frac{n}{2}}\daleth\left(z_1 2^{-a_1}(x-1)^{a_1},\cdots,z_r 2^{-a_r}(x-1)^{a_r}\right) \tag{4.4}$$

provided

$a_i > 0(i=1,\cdots,r)$

$$Re(n-2p)-2\sum_{i=1}^{r}a_i \min_{\substack{1\leqslant j\leqslant m_i\\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{a_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)<-\frac{3}{2}$$

$$Max\left[Re\left(-\frac{3}{2}-p\right)+\sum_{i=1}^{r}a_i \min_{\substack{1\leqslant j\leqslant n_i\\ 1\leqslant j\leqslant n^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+C_j^{(i)}\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right),\frac{1}{2}Re(m-n)-1\right]<k_1$$

$\left|arg\left(z_1 2^{-a_i}\left(x-1\right)\right)^{a_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

In (4.1) and (4.2) using the following relation due to Fox ([4], p. 230 (1.1))

$P_k^{m,n}(x) = 2^{k+\frac{n}{2}+1}2^{n+1}(1-x)^{-\frac{1}{2}}P^{m,-2k-1}_{-\frac{n+1}{2}}\left(\frac{x-3}{-x-1}\right); -1<x<1$, substituting $\left(\frac{x-3}{-x-1}\right)$ fox $x$and replacing k by $-\frac{n+1}{2}$, n by $-2k-1$ and n_1 by $-2k_1-1$, respectively, we get the above theorems 4 and 5.

Theorem 6.

$$\frac{1}{2\pi\omega}\int_{n_1-\omega\infty}^{n_1+\omega\infty}(2k+1)\Gamma(k+a+1)[\Gamma(k-a+1)]^{-1}{}_2F_1(-k+a,k+a+1;m;-x)$$

$$\daleth^{U;m_r+2,n_r+1:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\left(-p-k-\frac{1}{2};a_1,\cdots,a_r;1\right),\mathbf{A},\left(k-p+\frac{1}{2};a_1,\cdots,a_r;1\right):A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\ \left(-p-a-\frac{1}{2};a_1,\cdots,a_r:1\right),\left(m-p-a-\frac{3}{2};a_1,\cdots,a_r:1\right),\mathbf{A}:B \end{array}\right)\mathrm{d}k=$$

$$x^{-p-\frac{3}{2}}\Gamma(m)\daleth(z_1x^{-a_1},\cdots,z_rx^{-a_r}) \tag{4.5}$$

provided

$a_i>0(i=1,\cdots,r)$

$$\max\left\{Re\left(-\frac{3}{2}-p\right)-\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{a_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right),-Re(a-1)\right\}<k_1$$

$$Re(2p+2a-m)+2\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant j\leqslant n_i\\1\leqslant j\leqslant n^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+c_j^{(i)}\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)<-\frac{5}{2}$$

$\left|arg\left(z_12^{-a_i}(x-1))^{a_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

By applying $P_k^{m,n}(z)=\frac{(z-1)^{-\frac{m}{2}}(z+1)^{\frac{n}{2}}}{\Gamma(1-m)}{}_2F_1\left(1+k-\frac{m-n}{2},-k-\frac{m-n}{2};1-m;\frac{1-z}{2}\right)$, see ([3], p; 285 (0.2)) we obtain the following relations.

Theorem 7.

$$\frac{1}{2\pi\omega}\int_{n_1-\omega\infty}^{n_1+\omega\infty}(2k+1)\Gamma(k+a+1)[\Gamma(k-a+1)]^{1}{}_2F_1(-k,k+a+1;m;-x)$$

$$\daleth^{U;m_r+2,n_r+1:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\left(a+p+\frac{3}{2};a_1,\cdots,a_r;1\right),\left(a+p-m+\frac{5}{2};a_1,\cdots,a_r;1\right),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\ \left(p+k+\frac{1}{2};a_1,\cdots,a_r:1\right),\mathbf{B},\left(p-k+\frac{3}{2};a_1,\cdots,a_r:1\right):B \end{array}\right)\mathrm{d}k=$$

$$x^{-p-a-\frac{3}{2}}\Gamma(m)\daleth(z_1x^{a_1},\cdots,z_rx^{a_r}) \tag{4.6}$$

provided

$a_i>0(i=1,\cdots,r)$

$$Re(2p+2a-m)-2\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant j\leqslant n_i\\1\leqslant j\leqslant n^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+c_j^{(i)}\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)<-\frac{5}{2}$$

$$\max\left\{Re\left(-\frac{3}{2}-p\right)-\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant j\leqslant n_i\\1\leqslant j\leqslant n^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}A_{hj}\frac{a_{hj}-1}{\beta_{hj}^{h'}}+C_j^{(i)}\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right),-Re(a-1)\right\}<k_1$$

$\left|arg\left(z_1x^{-a_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

We have the following relation , see ([3], p. 246(1.39))

$e^{\omega m\pi}\left\{Q_k^{-m,-n}(t)-Q_{k-1}^{-m,-n}(t)\right\}=2^{m-n-1}\Gamma(\beta+1)\Gamma(\delta+1)\Gamma(-\alpha)(-\gamma)\frac{\sin(2k\pi}{\pi}P_k^{m,n}(x)$ and we get

Theorem 8.

$$\frac{1}{2\pi\omega}\int_{n_1-\omega\infty}^{n_1+\omega\infty}\frac{(2k+1)}{\sin(2k\pi)}[\Gamma(\delta+1)\Gamma(-\alpha)\Gamma(-\gamma)\Gamma(\gamma+1)]^{-1}\{Q_k^{-m,-n}(t)-Q_{k-1}^{-m,-n}(t)\}$$

$$\daleth_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;m_r+2,n_r+1:V}\left(\begin{array}{c|c}z_1 & \mathbb{A};\left(-k-p-\frac{1}{2};a_1,\cdots,a_r;1\right),\mathbf{A}\left(k-p+\frac{1}{2};a_1,\cdots,a_r;1\right):A\\ \cdot & \cdot\\ \cdot & \cdot\\ z_r & \mathbb{B};\left(-p+\frac{m-n}{2}-\frac{1}{2};a_1,\cdots,a_r:1\right),\left(-p-\frac{m+n}{2}-\frac{1}{2};a_1,\cdots,a_r:1\right),\mathbf{B}:B\end{array}\right)\mathrm{d}k=$$

$$2^{p+\frac{m-n+1}{2}}\pi^{-1}e^{-\omega m\pi}x^{-p-\frac{n+3}{2}}(x+1)^{\frac{n}{2}}\daleth\left(\frac{2^{a_1}z_1}{(x-1)^{a_1}},\cdots,\frac{2^{a_r}z_r}{(x-1)^{a_r}}\right)\tag{4.7}$$

provided

$a_i>0(i=1,\cdots,r)$

$$Re\left(k+p+\frac{3}{2}\right)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant j\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{a_{hj}}{\beta_{hj}^{h'}}+D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$$

$$Re(n+2p)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant j\leqslant n_i\\1\leqslant j\leqslant n^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+C_j^{(i)}\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)<-\frac{3}{2}$$

Theorem 9.

$$\frac{1}{2\pi\omega}\int_{n_1-\omega\infty}^{n_1+\omega\infty}\frac{(2k+1)}{\sin(2k\pi)}[\Gamma(\delta+1)\Gamma(-\alpha)\Gamma(-\gamma)\Gamma(\gamma+1)]^{-1}\{Q_k^{-m,-n}(t)-Q_{k-1}^{-m,-n}(t)\}$$

$$\daleth_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;m_r+2,n_r+1:V}\left(\begin{array}{c|c}z_1 & \mathbb{A};\left(\frac{3}{2}-\frac{m-n}{2};a_1,\cdots,a_r;1\right)\left(\frac{3}{2}+\frac{m+n}{2}+p;a_1,\cdots,a_r;1\right),\mathbf{A}:A\\ \cdot & \cdot\\ \cdot & \cdot\\ z_r & \mathbb{B};\left(p+k+\frac{3}{2};a_1,\cdots,a_r:1\right),\mathbf{B},\left(\frac{1}{2}-k+p;a_1,\cdots,a_r:1\right):B\end{array}\right)\mathrm{d}k=$$

$$2^{p+\frac{m-n+1}{2}}\pi^{-1}e^{-\omega m\pi}x^{-p-\frac{n+3}{2}}(x+1)^{\frac{n}{2}}\daleth\left(\frac{z_1(x-1)^{a_1}}{2^{a_1}},\cdots,\frac{z_r(x-1)^{a_r}}{2^{a_r}}\right) \tag{4.8}$$

provided

$a_i > 0 (i = 1, \cdots, r)$

$$Re\,(n+2p) - 2\sum_{i=1}^{r} a_i \min_{\substack{1\leqslant j\leqslant m_i\\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{a_{hj}}{\beta_{hj}^{h'}} + D_j^{(i)}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) < -\frac{3}{2}$$

$$Re\left(k+p-\frac{3}{2}\right) - \sum_{i=1}^{r} a_i \min_{\substack{1\leqslant j\leqslant n_i\\ 1\leqslant j\leqslant n^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}} + C_j^{(i)}\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right) > 0$$

Remark :
We obtain the same integrals with the functions cited in the section 1.

5. Conclusion.

The importance of our all the results lies in their manifold generality. Firstly, in view of general arguments utilized in these contour integrals, we can obtain a large simpler contour integrals. Secondly, by specialising the various parameters as well as variables in the generalized multivariable Gimel-function, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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