Strong Efficient Edge Domination Number of Some Cycle Related Graphs

M.Annapoopathi[1], *Reg.No. 17231072092002, P.G. & Research Department of Mathematics,*
The M.D.T. Hindu College, Tirunelveli, Tamil Nadu, India
(Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli – 627 012, Tamil Nadu, India)
N.Meena[2], *Assistant Professor, P.G. & Research Department of Mathematics,*
The M.D.T. Hindu College, Tirunelveli, Tamilnadu, India
(Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli – 627 012, Tamil Nadu, India)

Abstract

Let $G = (V, E)$ be a simple graph. A subset S of E(G) is a strong (weak) efficient edge dominating set of G if $|N_s[e] \cap S| = 1$ for all $e \in E(G)$ [$|N_w[e] \cap S| = 1$ for all $e \in E(G)$] where $N_s(e) = \{f / f \in E(G)$ & $deg\, f \geq deg\, e\}$ [$N_w(e) = \{f / f \in E(G)$ & $deg\, f \leq deg\, e\}$] and $N_s[e] = N_s(e) \cup \{e\}$ [$N_w[e] = N_w(e) \cup \{e\}$]. The minimum cardinality of a strong efficient edge dominating set of G (weak efficient edge dominating set of G) is called a strong efficient edge domination number of G and is denoted by $\gamma'_{se}(G)$ [$\gamma'_{we}(G)$]. In this paper, the strong efficient edge domination number of some cycle related graphs are studied.

Keywords - *Domination, edge domination, strong edge domination, efficient edge domination, strong efficient edge domination.*

AMS subject classification (2010):05C69

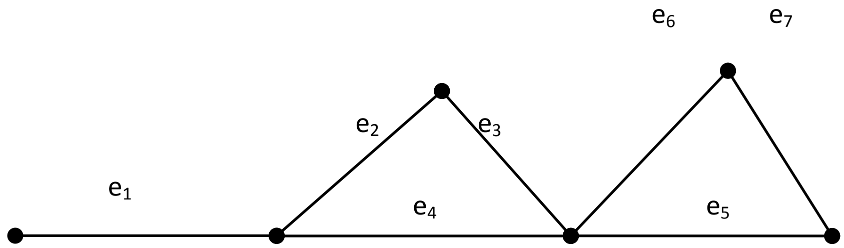
I. INTRODUCTION

Throughout this paper, only finite, undirected and simple graphs are considered. The concept of domination in graphs was introduced by Ore. Two volumes on domination have been published by T.W.Haynes, S.T.Hedetniemi and P.J.Slater [5, 6]. Let $G = (V, E)$ be a graph with p vertices and q edges. The degree of an edge is defined to be deg u + deg v – 2 [4]. An edge uv is called an isolated edge if deg uv = 0. Also $\delta'(G)$ denotes the minimum degree and $\Delta'(G)$ denotes the maximum degree among the edges of G. A subset D of V(G) is called an efficient dominating set of G if for every vertex u ∈ V(G), $|N[u] \cap D| = 1$ [1,2]. Edge dominating sets were studied by S.L.Mitchell and S.T.Hedetniemi [8]. A set F of edges in a graph G is called an edge dominating set of G if every edge in E – F is adjacent to at least one edge in F. Equivalently, a set F of edges in G is called an edge dominating set of G if every edge e ∈ E-F, there exists an edge e_1 ∈ F such that e and e_1 have a vertex in common. The edge domination number $\gamma'(G)$ of a graph G is the minimum cardinality of an edge dominating set of G. The strong (weak) domination number $\gamma_s(G)$ ($\gamma_w(G)$) of G is the minimum cardinality of a strong (weak) dominating set of G and $\Gamma_s(G)$ is the maximum cardinality of a minimal strong dominating set of G[9]. A subset D of E(G) is called an efficient edge dominating set if every edge in E(G) is dominated by exactly one edge in D[3,7,10]. The cardinality of the minimum efficient edge dominating set is called the efficient edge domination number of G. Motivated by these definitions, the authors define strong efficient edge domination in graphs. In this paper, the strong efficient edge domination numbers of some cycle related graphs are studied.

II. MAIN RESULTS

Definition 2.1: Let $G = (V, E)$ be a simple graph. A subset S of E(G) is a strong (weak) efficient edge dominating set of G if $|N_s[e] \cap S| = 1$ for all $e \in E(G)$ [$|N_w[e] \cap S| = 1$ for all $e \in E(G)$] where $N_s(e) = \{ f/f \in E(G)\ \&\ deg\ f \geq \deg e\}$ ($N_w(e) = \{f/f \in E(G)\ \&\ deg\ f \leq \deg e\}$) and $N_s[e] = N_s(e) \cup \{e\}$ ($N_w[e] = N_w(e) \cup \{e\}$). The minimum cardinality of a strong efficient edge dominating set of G (weak efficient edge dominating set of G) is called a strong(weak) efficient edge domination number of G and is denoted by $\gamma'_{se}(G)$ ($\gamma'_{we}(G)$).

Example 2.2: Consider the following graph



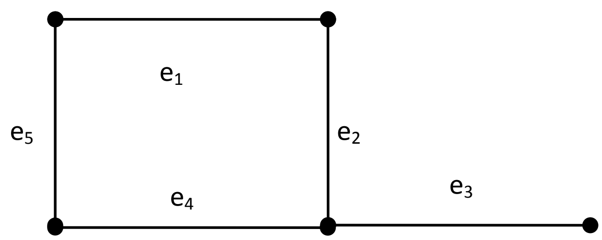
Let S = $\{e_4, e_7\}$. Now $|N_s[e_1] \cap S| = |\{e_1, e_2, e_4\} \cap S| = |\{e_4\}| = 1$, $|N_s[e_2] \cap S| = |\{e_2, e_3, e_4\} \cap S| = |\{e_4\}| = 1$, $|N_s[e_3] \cap S| = |\{e_3, e_4, e_5, e_6\} \cap S| = |\{e_4\}| = 1$, $|N_s[e_4] \cap S| = |\{e_4\} \cap S| = |\{e_4\}| = 1$, $|N_s[e_5] \cap S| = |\{e_3, e_4, e_5, e_6\} \cap S| = |\{e_4\}| = 1$, $|N_s[e_6] \cap S| = |\{e_3, e_4, e_5, e_6\} \cap S| = |\{e_4\}| = 1$, $|N_s[e_7] \cap S| = |\{e_7, e_6, e_5\} \cap S| = |\{e_7\}| = 1$. In G, S is the unique strong efficient edge dominating set of G. Therefore $\gamma'_{se}(G) = 2$.

Theorem 2.3: Every strong efficient edge dominating set is independent.

Proof: Let S be a strong efficient edge dominating set of G. Let $e_1, e_2 \in S$. Suppose the edges e_1 and e_2 are adjacent. Without loss of generality, deg $e_1 \geq$ deg e_2. Then $|N_S[e_2] \cap S| \geq 2$, a contradiction. Therefore S is independent.

Remark 2.4: Not all graphs have strong efficient edge dominating set.

Proof: Consider the following graph G



$S_1 = \{e_2, e_5\}$, $S_2 = \{e_1, e_4\}$, $|N_s[e_1] \cap S_1| = |\{e_1, e_2, e_5\} \cap S_1| = |\{e_2, e_5\}| = 2 > 1$. Therefore S_1 is not a strong efficient edge dominating set of G and $|N_s[e_5] \cap S_2| = |\{e_5, e_1, e_4\} \cap S_2| = |\{e_1, e_4\}| = 2 > 1$. Therefore S_2 is not strong efficient edge dominating set. Since the edges e_2 and e_4 have maximum degree and they are adjacent, any strong efficient edge dominating set must have either e_2 or e_4. Hence the graph G does not have a have strong efficient edge dominating set.

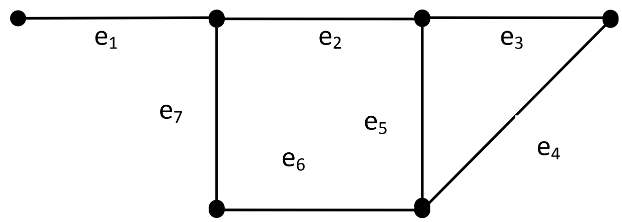
Definition 2.5: Let G = (V, E) be a simple graph. An edge dominating set S is a strong edge dominating set if for every edge $e \in E(G) - S$, there is an edge $f \in S$ with deg f $\geq$ deg e, and e is adjacent to f. The strong edge domination number is denoted by $\gamma'_s(G)$. An edge dominating set S is a weak edge dominating set if for every edge $e \in E(G) - S$, there is a n edge $f \in S$ with deg f $\leq$ deg e, and e is adjacent to f. The weak edge domination number is denoted by $\gamma'_w(G)$.

Remark 2.6: $\gamma'_s(G) \leq \gamma'_{se}(G)$

Proof: Let S be a strong efficient edge dominating set of G. Let $e_1 \in E(G) - S$. Then $|N_S[e] \cap S| = 1$. Then there exists $e_2 \in S$ such that the edges e_1 and e_2 are adjacent and deg $e_2 \geq$ deg e_1. Hence S is a strong efficient edge dominating set of G. Therefore $\gamma'_s(G) \leq \gamma'_{se}(G)$.

Remark 2.7: The strong edge dominating set of a graph G need not be strong efficient edge dominating set.

Proof: Consider the following graph G.



S_1 = {e_2, e_6}, S_2 = {e_5, e_7} are the strong edge dominating sets of G. But $|N_s[e_7] \cap S_1| = |\{e_7, e_2, e_6\} \cap S_1| = |\{e_2, e_6\}|$=2 >1 and $|N_s[e_6] \cap S_2| = |\{e_7, e_2, e_6\} \cap S_1| = |\{e_2, e_6\}|$ =2 > 1. Hence S_1 & S_2 are not strong efficient edge dominating sets of G. Since the edges e_2 and e_5 have maximum degree and they are adjacent, no strong efficient edge dominating set without e_2 or e_5 exists in G.

Theorem 2.8: A graph G does not admit a strong efficient edge dominating set if any two non adjacent maximum degree edges are joined by a single edge.

Proof: Let G = (V, E) be a simple graph and any two non adjacent maximum degree edges are joined by a single edge. Suppose G admits a strong efficient edge dominating set S. Let e_1 and e_2 be the edges of degree $\Delta'(G)$. Let $e_3 \in E(G)$ such that the edges e_1 and e_2 are joined by the edge e_3. Since e_1 and e_2 are not adjacent, they belong to S. Therefore$|N_s[e_3] \cap S| = 2 > 1$, a contradiction. Hence G does not admit a strong efficient edge dominating set of G.

Definition 2.9: Let $G = (V, E)$ be a simple graph. Let $E(G) = \{e_1, e_2, e_3, e_4, \dots \dots \dots \dots . e_n\}$. An edge e_i is said to be full degree edge if and only if deg e_i = n-1.

Observation 2.10: $\gamma'_{se}(G) = 1$ if and only if G has a full degree edge.

Theorem 2.11: $\gamma'_{se}(C_{3n}) = n, \forall\, n \in N.$

Proof: Let $G = C_{3n}, n \in N$ Let $E(G) = \{e_1, e_2, e_3, e_4, \dots \dots \dots \dots . e_{3n-1}, e_{3n}\}$. Then S_1=$\{e_1, e_4, \dots \dots \dots \dots, e_{3n-2}\}$,S_2=$\{e_2, e_5, \dots \dots \dots \dots, e_{3n-1}\}$, S_3=$\{e_3, e_6, \dots \dots \dots \dots, e_{3n}\}$ are the three strong efficient edge dominating sets of G and $|S_1| = |S_2| = |S_3|$ = n. Therefore $\gamma'_{se}(C_{3n}) \le n$. Since n $= \gamma'_s(C_{3n}) \le \gamma'_{se}(C_{3n})$. Therefore $\gamma'_{se}(C_{3n}) = n, \forall\, n \in N$.

Remark 2.12: C_{3n+1}, C_{3n+2} do not have efficient edge dominating sets , they do not have strong efficient edge dominating sets.

Theorem 2.13: For any path P_m , $\gamma'_{se}(P_m) = \begin{cases} n, & if\ m = 3n+1, n \ge 1 \\ n+1, & if\ m = 3n, n \ge 2 \\ n+1, & if\ m = 3n+2, n \ge 2 \end{cases}$

Proof:

Case1: Let$G = P_{3n}, n \ge 2$Let$E(G) = \{e_1, e_2, e_3, e_4, \dots \dots \dots \dots . e_{3n-1}\}$.Then S=$\{e_1, e_3, \dots \dots \dots \dots e_{3n-3}, e_{3n-1}\}$ is the unique stong efficient edge dominating set of G and $|S| = n+1, n \ge 2$. Therefore $\gamma'_{se}(P_{3n}) = n+1, n \ge 2$.

Case 2: Let $G = P_{3n+1}, n \ge 1$. Let $E(G) = \{e_1, e_2, e_3, e_4, \dots \dots \dots \dots . e_{3n-1}, e_{3n}\}$.Then S=$\{e_2, e_5, e_8, e_{11}, \dots \dots \dots \dots, e_{3n-1}\}$ is the unique stong efficient edge dominating set of G and $|S| = n, n \ge 1$. Therefore $\gamma'_{se}(P_{3n+1}) = n, n \ge 1$.

Case3:Let $G = P_{3n+2}, n \ge 1$. Let $E(G) = \{e_1, e_2, e_3, e_4, \dots \dots \dots \dots . e_{3n-1}, e_{3n}, e_{3n+1}\}$ .Then S_1=$\{e_1, e_3, e_6, e_9, \dots \dots \dots \dots, e_{3n}\}$,S_2=$\{e_2, e_5, e_8, e_{11}, \dots \dots \dots \dots, e_{3n+1}\}$ are the two strong efficient edge dominating sets of G and $|S_1|$ =n+1, n≥1, $|S_2|$ =n+1, n≥1. Therefore $\gamma'_{se}(P_{3n+2}) \le n+1, n \ge 1$.Since n+1=$\gamma'_s(P_{3n+2}) \le \gamma'_{se}(P_{3n+2}), n \ge 1$. Therefore $\gamma'_{se}(P_{3n+2}) = n+1, n \ge 1$.

Definition 2.14: Wheel W_n is defined as the join of C_{n-1}+K_1. The vertex corresponding to K_1 is said to be apex vertex, the vertices corresponding to the cycle are called the rim vertices. The edges corresponding to cycle are called the rim edges and edges joining apex and vertices of the cycle are called spoke edges.

Theorem 2.15: Let $W_{m,}$ $m \geq 2$ be a wheel graph. Then W_m has a strong efficient edge dominating set if and only if m = 3n, m ≥ 2 and $\gamma'_{se}(W_{3n}) = n,$ n ≥ 2.

Proof: Let m = 3n, n ≥ 2. Let u be the central vertex. Let v_1, v_2, …….v_{3n-1} be the vertices of the cycle C_{3n-1}. Let E(G) = { $e_i = uv_i$ / 1 ≤ i ≤ 3n-1}∪ { $f_i = v_i\ v_{i+1}$ / 1 ≤ i ≤ 3n-1}∪ { $f_{3n-1} = v_{3n-1}\ v_1$ } and deg u = 3n-1, deg v_i=3, 1 ≤ i ≤ 3n-1,deg e_i = deg u +deg v_i-2 = 3n, ded f_i =4, 1 ≤ i ≤ 3n. Each e_i strongly dominates all the spoke edges and two rim edges adjacent with e_i. The sub graph induced by remaining rim edges is P_{3n-2}. By theorem 2.13, P_{3n-2} has unique strong efficient edge dominating set containing n-1 elements. These n-1 elements together with e_i form a strong efficient edge dominating set S_i for W_{3n}. Therefore $|S_i|$ =n, 1 ≤ i ≤ 3n-1. Hence $\gamma'_{se}(W_{3n}) = n,$ n ≥ 2.

Conversely:

Case 1: Let m = 3n+1. Suppose S is a strong efficient edge dominating set of W_m. Clearly S contains one of e_i's and two rim edges adjacent with e_i. Without loss of generality e_1 belongs to S.e_1 strongly dominates e_2, e_3, e_4………..., e_{3n} , f_1 and f_{3n}. Clearly f_2 and f_{3n-1} do not belong to S. Then the edges f_3, f_6, ……., f_{3n-3} belong to S. f_{3n-3} strongly dominates f_{3n-2}. If the edge f_{3n-1} belongs to S then f_{3n} is strongly dominated by two edges f_{3n-1} and f_2. Hence f_{3n-1} does not belong to S. Therefore there is no edge in S to strongly dominate f_{3n-1} , a contradiction. Hence W_{3n+1} has no strong efficient edge dominating set.

Case 2: Let m=3n+2. Suppose S is a strong efficient edge dominating set of W_m. As in case 1, suppose e_1 belongs to S.e_1 strongly dominates e_2, e_3, e_4……….. e_{3n+1} , f_1 and f_{3n+1}. Clearly f_2 and f_{3n} do not belong to S. Then the edges f_3, f_6, ……., f_{3n-3} belong to S. f_{3n-3} strongly dominates f_{3n-4} & $f_{3n-2.}$ If the edge f_{3n-1} belongs to S then f_{3n+2} is strongly dominated by two edges f_{3n-1} and f_{3n-3}. If f_{3n} belongs to S then f_{3n+1} is dominated by two edges f_{3n+1} & e_1. Hence there is no edge in S to strongly dominate f_{3n} & f_{3n-1} , a contradiction. Hence W_m , m=3n+2 has no strong efficient edge dominating set.

Definition 2.16: Consider two cycles C_1 and C_2. Connect a vertex of C_1 to a vertex of C_2 with a new edge. The new graph obtained called the join sum of C_1 and C_2.

Theorem 2.17: Let G be the joint sum of two cycles C_i and C_j where i, j ∈ N then γ'_{se} (G) = 2n+1, n ≥ 1.

Proof:

Case (i): Let i = 3n, j = 3n, n ≥ 1. Let V (G) = {v_i, u_j / 1 ≤ i, j ≤ 3n}. Let e be an edge joining a vertex v_1 of C_i and a vertex u_1 of C_j & E(G) = {$v_i\ v_{i+1}$ / 1 ≤ i ≤ 3n - 1}∪ { $v_{3n}v_1$}∪ {u_ju_{j+1} / 1 ≤ j ≤ 3n – 1}∪{$u_{3n}u_1$}∪{ e }. Without loss of generalidity, let e = $v_1\ u_1$. Deg u_1 = 3, deg v_1 = 3, deg u_i = 2, deg v_i = 2, 2≤ i ≤ 3n. The edge e is the only maximum degree edge of G. deg v_1u_1 = 4, deg $v_1\ v_2$ = 3, deg v_1v_{3n} = 3. Similarly deg u_1u_2 = 3, deg u_1u_{3n} = 3. The remaining edges have degree two. S = {v_1u_1, v_2v_3, v_5v_6,……, $v_{3n-1}v_{3n}$, u_2u_3, u_5u_6,……,$u_{3n-1}u_{3n}$} is the unique strong efficient edge dominating set of G. Therefore │S│ = 2n+1. Hence γ'_{se} (G) = 2n+1, n ≥ 1.

Case (ii): i = 3n, j = 3n+1, n ≥ 1. Let e = $v_1\ u_1$. Let V(G) = {v_i / 1 ≤ i ≤ 3n}∪ {u_j / 1 ≤ j ≤ 3n+1} , E(G) = {$v_i\ v_{i+1}$ / 1 ≤ i ≤ 3n - 1}∪ { $v_{3n}v_1$}∪ {u_ju_{j+1} / 1 ≤ j ≤ 3n }∪{$u_{3n+1}u_1$}∪{ e }. Then degree of all vertices and all edges are same as in case (i). S_1 = { v_1u_1, v_2v_3, v_5v_6,…… $v_{3n-1}v_{3n}$, u_2u_3, $u_5\ u_6$,…… $u_{3n-1}u_{3n}$ } and S_2 = { v_1u_1, v_2v_3, v_5v_6,…… $v_{3n-1}v_{3n}$, u_3u_4, u_6u_7,…… $u_{3n}u_{3n+1}$ } are the two strong efficient edge dominating set of G. Therefore │S_1│= │S_2│= 2n+1. Hence γ_{se} ' (G) ≤ 2n+1, n ≥ 1. Any strong efficient edge dominating set must contain the edge e. Also it contains the edge $u_{3n}u_{3n+1}$ or $u_{3n-1}u_{3n}$. Therefore no set of 2n edges is a strong efficient edge dominating set. Hence γ'_{se} (G) = 2n+1, n ≥ 1.

Case (iii): i = 3n, j = 3n+2, n ≥ 1. Let e = $v_1\ u_1$. Let V(G) = {v_i / 1 ≤ i ≤ 3n}∪ {u_j / 1 ≤ j ≤ 3n+2} , E(G) = {v_iv_{i+1} / 1 ≤ i ≤ 3n - 1}∪ { $v_{3n}v_1$} ∪ {u_ju_{j+1} / 1 ≤ j ≤ 3n+1} ∪ {$u_{3n+2}u_1$} ∪ { e }. Then degree of all the vertices and all the edges are same as in case (i). S = {v_1u_1, v_2v_3, v_5v_6,……,$v_{3n-1}v_{3n}$, u_3u_4, u_6u_7,…… $u_{3n}u_{3n+1}$} is the unique strong efficient edge dominating set of G. Therefore │S│= 2n+1. Hence γ'_{se} (G) = 2n+1, n ≥ 1.

Case (iv): i = 3n+1, j = 3n+1, n ≥ 1. Let e = $v_1\ u_1$. Let V(G) = {v_i / 1 ≤ i ≤ 3n+1}∪ {u_j / 1 ≤ j ≤ 3n+1} , E(G) = {v_iv_{i+1} / 1 ≤ i ≤ 3n}∪ {$v_{3n+1}v_1$}∪ {u_ju_{j+1} / 1 ≤ j ≤ 3n}∪{$u_{3n+1}u_1$}∪{e}. As in case (i), the degree of all the vertices and all the edges are same. S_1 = {v_1u_1, v_2v_3, v_5v_6,…… $v_{3n-1}v_{3n}$, u_2u_3, $u_5\ u_6$,…… $u_{3n-1}u_{3n}$} and S_2 = {v_1u_1, v_2v_3, v_5v_6,…… $v_{3n-1}v_{3n}$, u_3u_4, u_6u_7,…… $u_{3n}u_{3n+1}$}, S_3 = {v_1u_1, v_3v_4, v_6v_7,…… $v_{3n}v_{3n+1}$, u_2u_3, u_5u_6,…… $u_{3n-1}u_{3n}$} and S_4 = {v_1u_1, v_3v_4, v_6v_7,…… $v_{3n}v_{3n+1}$, u_3u_4, u_6u_7,…… $u_{3n}u_{3n+1}$} are the four strong efficient edge dominating set of G. Therefore │S_1│= │S_2│= │S_3│= │S_4│= 2n+1. Hence γ'_{se} (G) ≤ 2n+1, n ≥ 1. Any strong efficient edge dominating set must contain the edge e. Also it contains the edge $u_{3n-1}u_{3n}$ or $u_{3n}u_{3n+1}$ or $v_{3n-1}v_{3n}$ or $v_{3n}v_{3n+1}$ Therefore no set of 2n edges is a strong efficient edge dominating set. Hence γ'_{se} (G) = 2n+1, n ≥ 1.

Case (v): i = 3n+1, j = 3n+2, n ≥ 1. Let e = v_1 u_1. As in case (iv), we have to define the vertex set and the edge set. S_1 = {v_1u_1, v_2v_3, v_5v_6,…… $v_{3n-1}v_{3n}$, u_3u_4, u_6u_7,……,$u_{3n}u_{3n+1}$} and S_2 = {v_1u_1, v_3v_4, v_6v_7,…… $v_{3n}v_{3n+1}$, u_3u_4, u_6u_7,……,$u_{3n}u_{3n+1}$} are the two strong efficient edge dominating set of G. Therefore │S_1│= │S_2│= 2n+1. Hence γ'_{se} (G) = 2n+1, n ≥ 1.

Case (vi): i = 3n+2, j = 3n+2, n ≥ 1. Let e = v_1 u_1. As in case (iv), we have to define the vertex set and the edge set. S = {v_1u_1, v_3v_4, v_6v_7,……,$v_{3n}v_{3n+1}$, u_3u_4, u_6u_7,……,$u_{3n}u_{3n+1}$} is the unique strong efficient edge dominating set of G. Therefore │S│= 2n+1. Hence γ'_{se} (G) ≤ 2n+1, n ≥ 1. Any strong efficient edge dominating set must contain the edge e. Also it contains the edge $v_{3n-1}v_{3n}$ or $v_{3n}v_{3n+1}$ Therefore no set of 2n edges is a strong efficient edge dominating set. Hence γ'_{se} (G) = 2n+1, n ≥ 1.

Theorem 2.18: Let G = $C_{3n} \circ mK_1$, n ≥ 1. Then γ'_{se} (G) = 2n, n ≥ 1.

Proof: Let G = $C_{3n} \circ mK_1$, m ≥ 1, n ≥ 1. Let V(C_{3n}) = {v_i / 1 ≤ i ≤ 3n}. Let v_{i1}, v_{i2}, ……., v_{im} be the vertices joined with v_i, 1 ≤ i ≤ 3n. Let $e_i = v_i v_{i+1}$, 1 ≤ i ≤ 3n – 1, $e_{3n} = u_{3n}u_1$ and let $e_{ik} = v_i v_{ik}$, 1 ≤ k ≤ m, 1 ≤ i ≤ 3n. deg e_i = 2m+2, 1 ≤ i ≤ 3n, deg e_{ik} = m+1, 1 ≤ k ≤ m, 1 ≤ i ≤ 3n. Clearly S_1 = {e_i/ i = 1, 4, 7, ……, 3n-2}∪ {e_{jk} / j ≠ i, i+1, 1 ≤ j ≤ 3n, 1 ≤ k ≤ m}, S_2 = {e_i/ i = 2, 5, 8, ……, 3n-2}∪ {e_{jk} / j ≠ i, i+1, 1 ≤ j ≤ 3n, 1 ≤ k ≤ m}, S_3 = {e_i/ i = 3, 6, 9, ……, 3n}∪ {e_{jk} / j ≠ i, i+1, 1 ≤ j ≤ 3n, 1 ≤ k ≤ m} are the strong efficient edge dominating sets of G and │S_i│ = 2n, 1 ≤ i ≤ 3. Therefore γ'_{se} (G) ≤ 2n, n ≥ 1. It is verified that no set of 2n – 1 edges are not strong efficient edge dominating set of G. Hence γ_{se} ‘(G) ≥ 2n. Therefore γ'_{se} (G) = 2n, n ≥ 1.

Definition 2.19: A gear graph G_n is obtained from the wheel graph W_n by adding a vertex between every pair of adjacent vertices in the cycle C_n.

Theorem 2.20: Let G_m be a gear graph. Then G_m has a strong efficient edge dominating set if and only if m = 3n + 2, n ≥ 1 and γ'_{se} (G_{3n+2}) = 2n + 1, n ≥ 1.

Proof: Let m = 3n + 2, n ≥ 1. Let V(G) = { u, v_1, v_2, v_3, ……… v_n, v_{n+1}, v_{n+2}…….. ,v_{3n+1}, u_1, u_2, u_3, ……… u_{3n+1}}. Let E(G) = { $f_i = v_iu_i$ / 1 ≤ i ≤ 3n+1 }∪ { $e_i = u_iv_{i+1}$ / 1 ≤ i ≤ 3n }∪ { $g_i = uv_i$ / 1 ≤ i ≤ 3n+1 }∪ { $e_{3n+1} = u_{3n+1}v_1$ / 1 ≤ i ≤ 3n } and deg u = 3n + 1, deg v_i = 3 , deg u_i = 2, deg f_i = 3, deg e_i = 3, deg g_i = 3n+2, 1 ≤ i ≤ 3n + 1. Each g_i strongly dominates all the spoke edges and two rim edges adjacent with g_i. The subgraph induced by the remaining rim edges is the path $P_{6n} = P_{3(2n)}$. By theorem 2.12, $P_{3(2n)}$ has a unique strong efficient edge dominating set containing 2n + 1 elements. These 2n + 1 elements together with g_i form a strong efficient edge dominating set S_i for G_{3n+2} .Therefore │ S_i │= 2n+1, 1 ≤ i ≤ 3n+1. Hence γ'_{se} (G_{3n+2}) = 2n + 1, n ≥ 1.

Conversely:

Case 1: Let m = 3n. Suppose S is the strong efficient edge dominating set of G_m. Clearly S contains one of g_i's and two rim edges adjacent with g_i. Without loss of generality g_1 belongs to S. g_1 strongly dominates all g_j, 2 ≤ j ≤ 3n-1 , f_1 and e_{3n-1}. Clearly e_1 and f_{3n-1} do not belong to S. Then the edges f_2, e_3, f_5, e_6, f_8, e_9, f_{11},…….,e_{3n-3}, f_{3n-1} belong to S. The edge f_{3n-1} strongly dominates e_{3n-2} and e_{3n-1}. If the edge f_{3n-1} belongs to S then the edge e_{3n-1} is strongly dominated by two edges f_{3n-1} and e_1. Hence f_{3n-1} does not belong to S. Therefore there is no edge in S to strongly dominate $f_{3n-1,}$ a contradiction. Hence G_{3n}, n ≥ 1 has no strong efficient edge dominating set.

Case 2: Let m = 3n+1. Suppose S is a strong efficient edge dominating set of G_m. As in case 1, suppose g_1 belongs to S. g_1 strongly dominates all g_j, 2 ≤ j ≤ 3n , f_1 and e_{3n}. Clearly e_1 and f_{3n} do not belong to S. Then the edges f_2, e_3, f_5, e_6, f_8, e_9, f_{11},…….,e_{3n}, f_{3n-1} belongs to S. The edge e_{3n} strongly dominates f_{3n} and e_1. If the edge e_{3n} belongs to S then the edge f_{3n} is strongly dominated by two edges f_{3n-1} and e_{3n}. Hence f_{3n-1} does not belong to S. Therefore there is no edge in S to strongly dominate $f_{3n-1,}$ a contradiction. Hence G_{3n+1}, n ≥ 1 has no strong efficient edge dominating set.

Definition 2.21: The Helm H_n is the graph obtained from the wheel W_n with n spokes by adding a pendant edge at each vertex on the wheel's rim.

Theorem 2.22: Let $H_n = W_n \circ K_1$. Then γ'_{se} (H_m) = 2n, n ≥ 2, m = 3n.

Proof: Let m = 3n, n ≥ 2. Let V(G) = { u, v_i, u_i, / 1 ≤ i ≤ 3n - 1}, E(G) = { $e_i = uv_i$ / 1 ≤ i ≤ 3n - 1 }∪ { $f_i = v_iv_{i+1}$ / 1 ≤ i ≤ 3n - 1 }∪ { $f_{3n-1} = v_{3n-1}v_1$ }∪ { $g_i = v_iu_i$ / 1 ≤ i ≤ 3n – 1} and deg u = 3n-1, deg v_i = 4 , deg u_i = 1, 1 ≤ i ≤ 3n – 1, deg e_i = 3n + 1, deg f_i = 6, deg g_i = 3, deg f_{3n-1} = 6. All e_i's are adjacent with each other. To dominate them, any one e_i is considered. Without loss of generalidity, let it be e_1 which is adjacent to two rim edges. The subgraph induced by the remaining rim edges is the path P_{3n-2}. By theorem 2.12, P_{3n-2} has the unique strong efficient edge dominating set containing n - 1 elements. The edges g_2, g_5, g_8, …………, g_{3n-1} also belongs to strong efficient edge dominating set. Therefore the total number of elements in the strong efficient edge dominating set is 2n. Hence $\gamma'_{se}(H_{3n}) \leq 2n$, $n \geq 2$. Since $2n = \gamma'_s(H_{3n}) \leq \gamma'_{se}(H_{3n})$. Therefore $\gamma'_{se}(H_{3n}) = 2n$, $n \geq 2$.

Conversely:

Case 1: Let m = 3n+1. Suppose S is a strong efficient edge dominating set of H_m. Clearly S contains atleast one of e_i's and two rim edges adjacent with e_i. Without loss of generality e_1 belongs to S. e_1 strongly dominates all g_j, $2 \leq j \leq 3n-1$, f_1 and e_{3n-1}. Clearly e_1 and f_{3n-1} do not belong to S. Then the edges $e_2, e_3, \ldots, e_{3n}$, f_1, g_1 and f_{3n}. Clearly f_2 and f_{3n-1} do not belong to S. Also the edges $f_3, f_6, \ldots, f_{3n-3}$ belong to S. The edge f_{3n-3} strongly dominates f_{3n-2} and e_{3n-1}. If the edge f_{3n-1} belongs to S then the edge f_{3n} is strongly dominated by two edges f_{3n-1} and f_2. Hence f_{3n-1} does not belong to S. Therefore there is no edge in S to strongly dominate f_{3n-1}, a contradiction. Hence H_{3n+1}, $n \geq 1$ has no strong efficient edge dominating set.

Case 2: Let m = 3n+2. Suppose S is a strong efficient edge dominating set of H_m. As in case 1, suppose e_1 belongs to S. e_1 strongly dominates all e_j, $2 \leq j \leq 3n+2$, f_1, g_1 and f_{3n+1}. Clearly f_2 and f_{3n+1} do not belong to S. Then the edges $f_3, f_6, f_9, \ldots, f_{3n-2}$ belong to S. The edge f_{3n-2} strongly dominates f_{3n-3} and f_{3n-1}. If the edge f_{3n} belongs to S then the edge f_{3n+1} is strongly dominated by two edges f_{3n} and f_1. Hence f_{3n+1} does not belong to S. Therefore there is no edge in S to strongly dominate f_{3n}, a contradiction. Hence H_{3n+2}, has no strong efficient edge dominating set.

Theorem 2.23: Let G be a graph obtained by joining the central vertices of two copies of a wheel W_m, $m \geq 4$. Then G has a strong efficient edge dominating set if and only if m = 3n + 1 , $n \geq 1$.

Proof: Let m = 3n + 1. Let u and v be the central vertex of W_1 and W_2 respectively. Let u_i, $1 \leq i \leq 3n$ and v_i, $1 \leq i \leq 3n$ be the vertices of the wheel W_1 and W_2 respectively. Let e = uv and $E(G) = \{ e_i = uu_i / 1 \leq i \leq 3n \} \cup \{ f_i = u_iu_{i+1} / 1 \leq i \leq 3n \} \cup \{ f_{3n} \}$ where $f_{3n} = u_{3n}u_1 \cup \{ g_j = vv_j / 1 \leq j \leq 3n \} \cup \{ h_j = v_j v_{j+1} / 1 \leq j \leq 3n \} \cup \{ h_{3n} = v_{3n} v_1 \} \cup \{ e = uv \}$. Deg u = 3n + 1, deg v = 3n + 1, deg u_i = 3, deg v_i = 3, $1 \leq i \leq 3n$, deg e_i = 3n + 2, deg g_j = 3n + 2, deg f_i = 4, deg h_j = 4, $1 \leq i, j \leq 3n$ and deg e = 6n. The edge e is the unique maximum degree edge of G. The edge e strongly dominates all the spoke edges of W_{m1} and W_{m2}. The subgraph induced by the remaining rim edges are the cycles C_i and C_j, i = j = 3n, $n \geq 1$. By the theorem 2.11, these cycles have a strong efficient edge dominating sets containing 2n elements. Then 2n elements together with the edge e form a strong efficient edge dominating set S of G. Hence $\gamma'_{se}(G) = 2n+1$, $n \geq 1$.

Conversely: Let $i \neq 3n$, $j \neq 3n$, $n \geq 1$. The edge e strongly dominates all the spoke edges of W_i and W_j, the subgraph induced by the remaining rim edges are the cycles C_i and C_j, $i \neq 3n$, $j \neq 3n$, $n \geq 1$. By remark 2.12, C_{3n+1}, C_{3n+2} do not have efficient edge dominating sets. Hence the graph G has no strong efficient edge dominating set

Definition 2.24: A flower F_n is constructed from a helm H_n by joining each vertex of degree one to the centre.

Theorem 2.25: A flower graph F_m has a strong efficient edge dominating set if and only if m = 3n, $n \geq 1$. Then $\gamma'_{se}(F_{3n}) = 2n$, $n \geq 2$.

Proof: Let m = 3n, $n \geq 2$. Let $V(G) = \{u, v_i, u_i / 1 \leq i \leq 3n - 1\}$. Let $E(G) = \{e_i = uv_i / 1 \leq i \leq 3n-1\} \cup \{ f_i = v_iv_{i+1} / 1 \leq i \leq 3n-1\} \cup \{f_{3n-1} = v_{3n-1}v_1\} \cup \{g_i = v_iu_i / 1 \leq i \leq 3n-1\} \cup \{h_i = uu_i / 1 \leq i \leq 3n-1\}$. Deg u = 6n-2, deg u_i = 2, deg v_i = 4 , deg f_i = 6, $1 \leq i \leq 3n-1$, deg e_i = 6n, deg g_i = 4, deg h_i = 6n-2, $1 \leq i \leq 3n-1$. Each e_i strongly dominates all the spoke edges and two rim edges adjacent with e_i and the edges h_i, $1 \leq i \leq 3n-1$.The subgraph induced by the remaining edges is the path $P_{3n-2} \circ K_1$. By theorem 2.12, $P_{3n-2} \circ K_1$ has a unique strong efficient edge dominating set containing 2n-1 edges. These 2n -1 edges together with the edge e form a strong efficient edge dominating set S of F_{3n}. Therefore $|S| = 2n$. Hence $\gamma_{se}'(F_{3n}) \leq 2n$, $n \geq 2$. Since $P_{3n-2} \circ K_1$ has a unique strong efficient edge dominating set, no other set of 2n-1 edges cannot be strong efficient edge dominating set of F_{3n}. Therefore $\gamma_{se}'(F_{3n}) \geq 2n$, $n \geq 2$. Hence $\gamma'_{se}(F_{3n}) = 2n$, $n \geq 2$.

Conversely:

Case 1: Let m = 3n+1. Suppose S is a strong efficient edge dominating set of F_m. Clearly S contains one of e_i's and two rim edges adjacent with e_i. Without loss of generality e_1 belongs to S.e_1 strongly dominates e_j, $2 \leq j \leq 3n$ f_1, f_{3n}, g_1 and all the edges h_i, $1 \leq i \leq 3n$. Also the edges $g_2, g_5, g_8, \ldots g_{3n-2}$ belong to S. Clearly f_2 and f_{3n-1} do not belong to S. Then the edges $f_3, f_6, \ldots, f_{3n-3}$ belong to S. The edge f_{3n-3} strongly dominates f_{3n-2}. If the edge f_{3n-1} belongs to S then the edge f_{3n} is strongly dominated by two edges f_{3n-1} and f_1. Hence f_{3n-1} does not belong to S. Therefore there is no edge in S to strongly dominate f_{3n-1}, a contradiction. Hence F_{3n+1} has no strong efficient edge dominating set.

Case 2: Let m = 3n+2. Suppose S is a strong efficient edge dominating set of F_m. As in case 1, suppose e_1 belongs to S.e_1 strongly dominates e_2, e_3, e_4, ……….., e_{3n}, f_1, f_{3n}, g_1 and all the edges h_i, $1 \leq i \leq 3n+1$. Also the edges g_2, g_5, g_8, ……., g_{3n-1} belong to S. Clearly f_2 and f_{3n} do not belong to S. Then the edges f_3, f_6,……., f_{3n-3} belong to S. The edge f_{3n-3} strongly dominates f_{3n-4} & f_{3n-2}. If the edge f_{3n-1} belongs to S then f_{3n} is strongly dominated by two edges f_{3n-1} and f_{3n+1}. Therefore there is no edge in S to strongly dominate f_{3n+1}, a contradiction. Hence F_m, m = 3n+2 has no strong efficient edge dominating set.

III. CONCLUSION

In this paper, strong efficient edge domination number of some standard graphs and cycle related graphs are determined.

REFERENCES

[1] D.W.Bange, A.E.Barkauskas, L.H.Host, and P.J.Slater. Generalized Domination and Efficient Domination in Graphs. Discrete Math., 159:1 – 11, 1996.

[2] D.W.Bange, A.E.Barkauskas, and P.J.Slater. Efficient Dominating Sets in Graphs. In R.D.Ringeisen and F.S.Roberts, Editors, Applications of Discrete Mathematics, pages 189 – 199. SIAM, Philadelphia, PA, 1988.

[3] Dominngos M.Cardoso, J.Orestes Cerdefra Charles Delorme, Pedro C.Silva , Efficient Edge Domination in Regular Graphs, Discrete Applied Mathematics 156 , 3060 - 3065(2008).

[4] V.R.Kulli, Theory of Domination in Graphs, Vishwa International Publications, Gulbarga,India (2010).

[5] Teresa W.Haynes, Stephen T.Hedetniemi, Peter J.Slater (Eds), Domination in Graphs: Advanced Topics, Marcel Decker, Inc., New York 1998.

[6] Teresa W.Haynes, Stephen T.Hedetniemi, Peter J.Slater, Fundamentals of Domination in Graphs, Marcel Decker, Inc., New York 1998.

[7] C.L.Lu, M-T.Ko, C.Y.Tang, Perfect Edge Domination and Efficient Edge Domination in Graphs, Discrete Appl.Math. 19227-250(2002)

[8] S.L.Mitchell and S.T.Hedetniemi, Edge Domination in Trees. Congr. Number.19489-509 (1977)

[9] E.Sampath Kumar and L.Pushpalatha, Strong Weak Domination and Domination Balance in a Graph, Discrete Math., 161:235 - 242, 1996.

[10] G.Santhosh , A Study on Integral Sum Graphs and Various Graph Theoretic Parameters, Ph.D thesis, 2003.