Some Integrals Involving Generalized Multivariable Gimel-Function

Frédéric Ayant
Teacher in High School , France

ABSTRACT
In this paper some finite and infinite integrals involving products of the hypergeometric function and generalized multivariable Gimel-function. On specializing the parameters of the functions involved in the integrands, new results can be obtained as particular cases, we do not treat the particular cases in this study.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, hypergeometric function, integrals.

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We define a generalized transcendental function of several complex variables.

$$\gimel(z_1, \cdots, z_r) = \gimel^{m_2,n_2;m_3,n_3;\cdots;m_r,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array} \right|$$

$$[(a_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[(b_{2j};\beta^{(1)}_{2j},\beta^{(2)}_{2j};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta^{(1)}_{3j},\beta^{(2)}_{3j},\beta^{(3)}_{3j};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(a_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(b_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj};B_{rj})_{1,m_r}],$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(c^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{m_r+1,q_r}]:[(d^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left. \begin{array}{l} ;\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}] \\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}] \end{array} \right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{m_2}\Gamma^{B_{2j}}(b_{2j}-\sum_{k=1}^{2}\beta^{(k)}_{2j}s_k)\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha^{(k)}_{2ji_2}s_k)\prod_{j=m_2+1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{m_3}\Gamma^{B_{3j}}(b_{3j}-\sum_{k=1}^{3}\beta_{3j}^{(k)}s_k)\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=m_3+1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$\frac{\prod_{j=1}^{m_r}\Gamma^{B_{rj}}(b_{rj}-\sum_{k=1}^{r}\beta_{rj}^{(k)}s_k)\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=m_r+1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)})]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $m_2,n_2,\cdots,m_r,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2\leqslant q_{i_2},0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant m_r\leqslant q_{i_r},0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kj}^{(l)},B_{kj}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n_k);(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$a_{kji_k} \in \mathbb{C}; (j = n_k + 1, \cdots, p_{i_k}); (k = 2, \cdots, r)$.

$b_{kji_k} \in \mathbb{C}; (j = m_k + 1, \cdots, q_{i_k}); (k = 2, \cdots, r)$.

$d^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r)$.

$\gamma^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r)$.

The contour L_k is in the $s_k(k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha^{(k)}_{2j} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha^{(k)}_{3j} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha^{(i)}_{rj}\right) (j = 1, \cdots, n_r), \Gamma^{C^{(k)}_j}\left(1 - c^{(k)}_j + \gamma^{(k)}_j s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$to the right of the contour L_k and the poles of $\Gamma^{B_{2j}}\left(b_{2j} - \sum_{k=1}^{2} \beta^{(k)}_{2j} s_k\right) (j = 1, \cdots, m_2), \Gamma^{B_{3j}}\left(b_{3j} - \sum_{k=1}^{3} \beta^{(k)}_{3j} s_k\right) (j = 1, \cdots, m_3)$ $, \cdots, \Gamma^{B_{rj}}\left(b_{rj} - \sum_{i=1}^{r} \beta^{(i)}_{rj}\right) (j = 1, \cdots, m_r), \Gamma^{D^{(k)}_j}\left(d^{(k)}_j - \delta^{(k)}_j s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A^{(k)}_i \pi$ where

$$A^{(k)}_i = \sum_{j=1}^{m^{(k)}} D^{(k)}_j \delta^{(k)}_j + \sum_{j=1}^{n^{(k)}} C^{(k)}_j \gamma^{(k)}_j - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q^{(k)}_i} D^{(k)}_{ji^{(k)}} \delta^{(k)}_{ji^{(k)}} + \sum_{j=n^{(k)}+1}^{p^{(k)}_i} C^{(k)}_{ji^{(k)}} \gamma^{(k)}_{ji^{(k)}} \right) +$$

$$\sum_{j=1}^{n_2} A_{2j} \alpha^{(k)}_{2j} + \sum_{j=1}^{m_2} B_{2j} \beta^{(k)}_{2j} - \tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha^{(k)}_{2ji_2} + \sum_{j=m_2+1}^{q_{i_2}} B_{2ji_2} \beta^{(k)}_{2ji_2} \right) + \cdots +$$

$$\sum_{j=1}^{n_r} A_{rj} \alpha^{(k)}_{rj} + \sum_{j=1}^{m_r} B_{rj} \beta^{(k)}_{rj} - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha^{(k)}_{rji_r} + \sum_{j=m_r+1}^{q_{i_r}} B_{rji_r} \beta^{(k)}_{rji_r} \right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$ where $i = 1, \cdots, r$:

$\alpha_i = \min_{\substack{1 \leqslant k \leqslant m_i \\ 1 \leqslant j \leqslant m^{(i)}}} Re \left(\sum_{h=2}^{r} \sum_{h'=1}^{h} B_{hj} \frac{b_{hj}}{\beta^{h'}_{hj}} + D^{(i)}_k \frac{d^{(i)}_k}{\delta^{(i)}_k} \right)$ and $\beta_i = \max_{\substack{1 \leqslant k \leqslant n_i \\ 1 \leqslant j \leqslant n^{(i)}}} Re \left(\sum_{h=2}^{r} \sum_{h'=1}^{h} A_{hj} \frac{a_{hj} - 1}{\alpha^{h'}_{hj}} + C^{(i)}_k \frac{c^{(i)}_k - 1}{\gamma^{(i)}_k} \right)$

Remark 1.
If $m_2 = n_2 = \cdots = m_{r-1} = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the generalized multivariable Gimel-function reduces in the generalized multivariable Aleph- function (extension of multivariable Aleph-function defined by Ayant [1]).

Remark 2.
If $m_2 = n_2 = \cdots = m_r = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [4]).

Remark 3.

If$A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [3].

Remark 4.

If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and Panda [6,7]

In your investigation, we shall use the following notations.

$$\mathbb{A} = [(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \quad (1.5)$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \quad (1.6)$$

$$A = [(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \quad (1.7)$$

$$\mathbb{B} = [(b_{2j};\beta_{2j}^{(1)},\beta_{2j}^{(2)};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta_{3j}^{(1)},\beta_{3j}^{(2)},\beta_{3j}^{(3)};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{(r-1)j};\beta_{(r-1)j}^{(1)},\cdots,\beta_{(r-1)j}^{(r-1)};B_{(r-1)j})_{1,m_{r-1}}],$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{m_{r-1}+1,q_{i_{r-1}}}] \quad (1.8)$$

$$\mathbf{B} = [(\mathrm{b}_{rj};\beta_{rj}^{(1)},\cdots,\beta_{rj}^{(r)};B_{rj})_{1,m_r}],[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{m_r+1,q_{i_r}}] \quad (1.9)$$

$$\mathrm{B} = [(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \quad (1.10)$$

$$U = m_2,n_2;m_3,n_3;\cdots;m_{r-1},n_{r-1};V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \quad (1.11)$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \quad (1.12)$$

2. Required results.

In this section we state the results given by B.L. Sharma [5].

Lemma 1 ([5], p. 3(2.3))

$$\int_0^\infty e^{2\mu\theta}(\sin h\theta)^{2\lambda-1}{}_2F_1\left(\lambda-\mu+\frac{1}{2},\beta;\delta;2e^{-\theta}\sin h\theta\right)\mathrm{d}\theta =$$

$$\frac{\Gamma(\delta)\Gamma(\lambda)\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(\frac{1}{4}-\frac{\lambda}{2}-\frac{\mu}{2}\right)\Gamma\left(\frac{3}{4}-\frac{\lambda}{2}-\frac{\mu}{2}\right)\Gamma\left(\frac{\delta}{2}-\frac{\beta}{2}-\lambda\right)\Gamma\left(\frac{1}{2}+\frac{\delta}{2}-\frac{\beta}{2}-\lambda\right)}{\pi 2^{\lambda+\beta+\frac{\delta}{2}}\Gamma(\delta-\beta)\Gamma\left(\frac{1}{2}+\lambda-\mu\right)\Gamma\left(\frac{\delta}{2}-\lambda\right)\Gamma\left(\frac{1}{2}+\frac{\delta}{2}-\lambda\right)} \tag{2.1}$$

provided $Re(\lambda)>0, Re\left(\frac{1}{2}-\lambda-\mu\right)>0, Re\left(\delta-\beta-\lambda+\mu-\frac{1}{2}\right)>0.$

The definite in tegral ([5] p. 4 (2.5)) and ([5] p. 4 (2.6))

Lemma 2.

$$\int_0^{\frac{\pi}{2}} e^{\omega(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}{}_2F_1(a,b;\beta;e^{\omega\theta}\cos\theta)\mathrm{d}\theta = \frac{e^{\omega\frac{\pi}{2}\alpha}\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta-a-b)}{\Gamma(\alpha+\beta-a)\Gamma(\alpha+\beta-b)} \tag{2.2}$$

provided $Re(\alpha), Re(\beta), Re(\beta-a-b)>0.$

Lemma 3.

$$\int_0^{\frac{\pi}{2}} e^{\omega(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}{}_2F_1(a,b;\alpha;e^{\omega\left(\theta-\frac{\pi}{2}\right)}\sin\theta)\mathrm{d}\theta = \frac{e^{\omega\frac{\pi}{2}\alpha}\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta-a-b)}{\Gamma(\alpha+\beta-a)\Gamma(\alpha+\beta-b)} \tag{2.3}$$

provided $Re(\alpha), Re(\beta), Re(\alpha-a-b)>0.$

The infinite integrals ([5] p. 4 (2.8)) and ([5] p. 5 (2.9))

lemma 4.

$$\int_0^{\infty} x^{\lambda-1}(1+x)^{-a-1}{}_4F_3\left[\begin{array}{c|c} \text{a},\frac{a}{2}\text{+1,c,d} & \\ . & \frac{x}{1+x} \\ \frac{c}{2}\text{,1+a-c,1+a-d} & \end{array}\right]\mathrm{d}x =$$

$$\frac{2^{a-2\lambda}\Gamma(1+a-c)\Gamma(1+a-d)\Gamma\left(\frac{1+a}{2}-\lambda\right)\Gamma\left(\frac{2+a}{2}-\lambda\right)\Gamma(1+a-c-d-\lambda)\Gamma(\lambda)}{\sqrt{\pi}\Gamma(1+a)\Gamma(1+a-c-d)\Gamma(1+a-c-\lambda)\Gamma(1+a-d-\lambda)} \tag{2.4}$$

provided $Re(\lambda)>0, Re(a-2\lambda+1)>0, Re(a-c-d-\lambda+1)>0$

lemma 5.

$$\int_0^{\infty} x^{\lambda-1}(1+x)^{-a-1}{}_3F_2\left[\begin{array}{c|c} \text{a},\frac{a}{2}\text{+1,b} & \\ . & -\frac{x}{1+x} \\ \frac{a}{2}\text{,1+a-b} & \end{array}\right]\mathrm{d}x = \frac{2^{a-2\lambda}\Gamma(\lambda)\Gamma(1+a-b)\Gamma\left(\frac{1+a}{2}-\lambda\right)\Gamma\left(\frac{2+a}{2}-\lambda\right)}{\sqrt{\pi}\Gamma(1+a)\Gamma(1+a-b-\lambda)} \tag{2.5}$$

provided $Re(\lambda)>0, Re(a-2\lambda+1)>0, Re(a-2b-2\lambda+2)>0$

3. Main integrals.

In this section, we establish several unified integrals involving the hypergeometric function and generalized multivariable Gimel-function.

Theorem 1.

$$\int_0^{\infty} e^{2\mu\theta}(\sin h\theta)^{2\lambda-1}{}_2F_1\left(\lambda-\mu+\frac{1}{2},\beta;\delta;2e^{-\theta}\sin h\theta\right)\gimel\left(z_1 e^{2a_1\theta}(\sin\theta)^{2a_1},\cdots,z_r e^{2a_r\theta}(\sin\theta)^{2a_r}\right)\mathrm{d}\theta =$$

$$\frac{2^{-\lambda-\mu-\beta-\frac{3}{2}}\Gamma(\delta)}{\pi\Gamma(\delta-\beta)\Gamma\left(\frac{1}{2}+\lambda-\mu\right)}\gimel^{U;m_r+4,n_r+2:V}_{X;p_{i_r}+4,q_{i_r}+4,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c}\frac{z_1}{4^{a_1}} & \mathbb{A};\ (1-\lambda;a_1,\cdots,a_r;1),\left(\frac{1}{2}-\lambda;a_1,\cdots,a_r;1\right)\\ \cdot & \cdot\\ \cdot & \cdot\\ \frac{z_r}{4^{a_r}} & \mathbb{B};\ \left(-\frac{\lambda+\mu}{2}+\frac{1}{4};a_1,\cdots,a_r:1\right),\end{array}\right.$$

$$\left.\begin{array}{c}A;\ \left(-\lambda+\frac{\delta}{2};a_1,\cdots,a_r;1\right),\left(-\lambda+\frac{\delta}{2}+\frac{1}{2};a_1,\cdots,a_r;1\right):A\\ \cdot\\ \cdot\\ \left(-\frac{\lambda+\mu}{2}+\frac{3}{4};a_1,\cdots,a_r;1\right),\left(-\lambda-\frac{\beta}{2}+\frac{\delta}{2};a_1,\cdots,a_r;1\right),\left(-\lambda-\frac{\beta}{2}+\frac{\delta+1}{2};a_1,\cdots,a_r;1\right),\mathbf{B}:B\end{array}\right) \tag{3.1}$$

provided $a_i>0(i=1,\cdots,r), Re\left(\delta-\beta-\lambda+u-\frac{1}{2}\right)>0$

$$Re(\lambda)+\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\left(\lambda+\mu-\frac{1}{2}\right)+2\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant n_i\\1\leqslant j\leqslant n^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+C_k^{(i)}\frac{c_k^{(i)}-1}{\gamma_k^{(i)}}\right)<0$$

$\left|arg\left(z_ie^{2a_i\theta}(\sin h\theta)^{2a_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4).

Proof

To prove (2.1), on the left hand side of (2.1), expressing the generalized multivariable Gimel-function as Mellin-Barnes multiple integrals contour with the the help of (1.1), interchanging the order of summation and integration which is justified under the conditions mentioned above, we get (say I)

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\int_0^\infty e^{(2\mu+\sum_{i=1}^r a_is_i)\theta}(\sin h\theta)^{2(\lambda+\sum_{i=1}^r a_is_i)-1}$$

$${}_2F_1\left(\lambda-\mu+\frac{1}{2},\beta;\delta;2e^{-\theta}\sin h\theta\right)\mathrm{d}\theta\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.2}$$

Evaluating the inner integral with the help of the lemma 1, it reduces to

$$\frac{2^{-\lambda-\mu-\beta-\frac{3}{2}}\Gamma(\delta)}{\pi\Gamma(\delta-\beta)\Gamma\left(\frac{1}{2}+\lambda-\mu\right)}\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\Gamma(\lambda+\sum_{i=1}^r a_is_i)\Gamma\left(\lambda+\frac{1}{2}+\sum_{i=1}^r a_is_i\right)$$

$$\frac{\Gamma\left(\frac{1}{4}-\frac{\lambda+\mu}{2}-\sum_{i=1}^r a_is_i\right)\Gamma\left(\frac{3}{4}-\frac{\lambda+\mu}{2}-\sum_{i=1}^r a_is_i\right)\Gamma\left(\frac{\delta-\beta}{2}-\lambda-\sum_{i=1}^r a_is_i\right)\Gamma\left(\frac{1+\delta-\beta}{2}-\lambda-\sum_{i=1}^r a_is_i\right)}{\Gamma\left(\frac{1+\delta}{2}-\lambda-\sum_{i=1}^r a_is_i\right)\Gamma\left(\frac{\delta}{2}-\lambda-\sum_{i=1}^r a_is_i\right)}$$

$$4^{-\sum_{i=1}^r a_is_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.3}$$

and finally interpreting the Mellin-Barnes multiple integrals contour in terms of the generalized multivariable Gimel-function, we get the desired result (3.1).

Theorem 2.

$$\int_0^{\frac{\pi}{2}} e^{\omega(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}{}_2F_1(a,b;\alpha;e^{\omega\theta}\cos\theta)\beth\left(e^{\omega a_1(\theta-\frac{\pi}{2})}(\sin\theta)^{a_1}z_1,\cdots,e^{\omega a_r(\theta-\frac{\pi}{2})}(\sin\theta)^{a_r}z_r\right)\mathrm{d}\theta$$

$$= e^{\omega\frac{\pi}{2}\alpha}\Gamma(\beta)\beth^{U;m_r,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\ (1\text{-}\alpha;a_1,\cdots,a_r;1),(1-\alpha-\beta+a+b;a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\mathbf{B}\ (1\text{-}\alpha-\beta+a;a_1,\cdots,a_r;1),(1-\alpha-\beta+b;a_1,\cdots,a_r;1):B \end{array}\right) \quad (2.4)$$

provided
$a_i > 0(i=1,\cdots,r); Re(\beta) > 0, Re(\beta-a-b) > 0,$

$$Re(\alpha) + \sum_{i=1}^{r} a_i \min_{\substack{1\leqslant k\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right) > 0$$

$\left|arg\left(z_i e^{a_i(\theta-\frac{\pi}{2})}(\sin h\theta)^{2a_i}\right)\right| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4).

Theorem 3.

$$\int_0^{\frac{\pi}{2}} e^{\omega(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}{}_2F_1(a,b;\alpha;e^{\omega(\theta-\frac{\pi}{2})}\sin\theta)\beth\left(e^{\omega a_1\theta}(\cos\theta)^{a_1}z_1,\cdots,(e^{\omega a_r\theta}(\cos\theta)^{a_r}z_r\right)\mathrm{d}\theta$$

$$= e^{\omega\frac{\pi}{2}}\Gamma(\alpha)\beth^{U;m_r,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\ (1\text{-}\beta;a_1,\cdots,a_r;1),(1-\alpha-\beta+a+b;a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\mathbf{B}\ (1\text{-}\alpha-\beta+a;a_1,\cdots,a_r;1),(1-\alpha-\beta+b;a_1,\cdots,a_r;1):B \end{array}\right) \quad (2.5)$$

provided
$a_i > 0(i=1,\cdots,r); Re(\alpha) > 0, Re(\alpha-a-b) > 0,$

$$Re(\beta) + \sum_{i=1}^{r} a_i \min_{\substack{1\leqslant k\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right) > 0$$

$\left|arg\left(z_i e^{a_i\theta}(\cos h\theta)^{2a_i}\right)\right| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4).

Theorem 4.

$$\int_0^{\infty} x^{\lambda-1}(1+x)^{-a-1}{}_4F_3\left[\begin{array}{c|c} \mathrm{a},\frac{a}{2}\text{+1,c,d} & \\ \cdot & \frac{x}{1+x} \\ \frac{c}{2}\text{,1+a-c,1+a-d} & \end{array}\right]\beth\left(z_1(4x^2+4x)^{a_1},\cdots,z_r(4x^2+4x)^{a_r}\right)\mathrm{d}x =$$

$$\frac{2^{\alpha-2\lambda}\Gamma(1+a-c)\Gamma(1+a-d)}{\sqrt{\pi}\Gamma(1+a)\Gamma(1+a-c-d)}\beth^{U;m_r+3,n_r+1:V}_{X;p_{i_r}+3,q_{i_r}+3,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\ (1\text{-}\lambda;a_1,\cdots,a_r;1),\mathbf{A},(1+a-d-\lambda;a_1,\cdots,a_r;1),(1+a-c-\lambda;a_1,\cdots,a_r;1):A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\ (\frac{1+a}{2}-\lambda;a_1,\cdots,a_r;1),(\frac{2+a}{2}-\lambda;a_1,\cdots,a_r;1),(1+a-c-d-\lambda;a_1,\cdots,a_r;1),\mathbf{B}:B \end{array}\right) \quad (2.6)$$

provided $a_i > 0(i=1,\cdots,r), Re(a-c-d-\lambda+1) > 0$

$$Re(\lambda)+2\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\,(2\lambda-a-1)+2\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant n_i\\1\leqslant j\leqslant n^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+C_k^{(i)}\frac{c_k^{(i)}-1}{\gamma_k^{(i)}}\right)<0$$

$\left|arg\left(z_i(4x^2+4x)^{a_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4).

Theorem 5.

$$\int_0^{\infty}x^{\lambda-1}(1+x)^{-a-1}{}_3F_2\left[\begin{matrix}\text{a},\frac{a}{2}+1,\text{b}\\ \cdot\\ \frac{a}{2},1+\text{a-b}\end{matrix}\middle|-\frac{x}{1+x}\right]\gimel\left(z_1(4x^2+4x)^{a_1},\cdots,z_r(4x^2+4x)^{a_r}\right)\mathrm{d}x=\frac{2^{\alpha-2\lambda}\Gamma(1+a-b)}{\sqrt{\pi}\Gamma(1+a)}$$

$$\gimel^{U;m_r+2,n_r+1:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}\left(\begin{matrix}\text{z}_1\\ \cdot\\ \cdot\\ \text{z}_r\end{matrix}\middle|\begin{matrix}\mathbb{A};\ (1\text{-}\lambda;a_1,\cdots,a_r;1),\mathbf{A},(1+a-b-\lambda;a_1,\cdots,a_r;1):A\\ \cdot\\ \cdot\\ \mathbb{B};\ (\frac{1+a}{2}-\lambda;a_1,\cdots,a_r;1),(\frac{2+a}{2}-\lambda;a_1,\cdots,a_r;1),\mathbf{B}:B\end{matrix}\right) \quad (2.7)$$

provided $a_i>0(i=1,\cdots,r),Re(a-2b-2\lambda+2)>0$

$$Re(\lambda)+2\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\,(2\lambda-a-1)+2\sum_{i=1}^{r}a_i\min_{\substack{1\leqslant k\leqslant n_i\\1\leqslant j\leqslant n^{(i)}}}Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}}+C_k^{(i)}\frac{c_k^{(i)}-1}{\gamma_k^{(i)}}\right)<0$$

$\left|arg\left(z_i(4x^2+4x)^{a_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4).

Proceeding as in (3.1), the above integrals (2.4), (2.5), (2.6) and (2.7) can be established with the help of lemmae 2, 3 , 4 and 5 respectively.

Remark :
We obtain the same integrals with the functions cited in the section 1.

5. Conclusion.

The importance of our all the results lies in their manifold generality. Firstly, in view of general arguments utilized in these finite and infinite integrals, we can obtain a large simpler single finite or infinite integrals. Secondly, by specialising the various parameters as well as variables in the generalized multivariable Gimel-function, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

REFERENCES.

[1] F. Ayant, An integral associated with the Aleph-functions of several variables. International Journal of Mathematics Trends and Technology (IJMTT), 31(3) (2016), 142-154.

[2] B.L.J. Braaksma, Asymptotics expansions and analytic continuations for a class of Barnes-integrals, Compositio Math. 15 (1962-1964), 239-341.

[3] Y.N. Prasad, Multivariable I-function , Vijnana Parishad Anusandhan Patrika 29 (1986) , 231-237.

[4] J. Prathima, V. Nambisan and S.K. Kurumujji, A Study of I-function of Several Complex Variables, International Journal of Engineering Mathematics Vol (2014), 1-12.

[5] B.L. Sharma, Integrals involving G-function, Seminario Mat. De Barcelona, 16(1) (1964), 3-12.

[6] H.M. Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables. Comment. Math. Univ. St. Paul. 24 (1975),119-137.

[7] H.M. Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables II. Comment. Math. Univ. St. Paul. 25 (1976), 167-197.