Some Finite Double Integral Formulae Involving Generalized Multivariable Gimel-Function

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ABSTRACT
In the present paper we evaluate three finite double integrals involving various products of biorthogonal pair of polynomials, a general class of polynomials, and generalized multivariable Gimel-function with general arguments. Our integrals are quite general in character and a large number of integrals can be deduced as particular cases. Some interesting special cases of our main results have been discussed brefly.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, finite double integral, general class of polynomials, biorthogonal polynomials.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We define a generalized transcendental function of several complex variables.

$$\gimel(z_1, \cdots, z_r) = \gimel^{m_2,n_2;m_3,n_3;\cdots;m_r,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{array} \right|$$

$$[(\mathrm{a}_{2j}; \alpha^{(1)}_{2j}, \alpha^{(2)}_{2j}; A_{2j})]_{1,n_2}, [\tau_{i_2}(a_{2ji_2}; \alpha^{(1)}_{2ji_2}, \alpha^{(2)}_{2ji_2}; A_{2ji_2})]_{n_2+1,p_{i_2}}, [(a_{3j}; \alpha^{(1)}_{3j}, \alpha^{(2)}_{3j}, \alpha^{(3)}_{3j}; A_{3j})]_{1,n_3},$$
$$[(\mathrm{b}_{2j}; \beta^{(1)}_{2j}, \beta^{(2)}_{2j}; B_{2j})]_{1,m_2}, [\tau_{i_2}(b_{2ji_2}; \beta^{(1)}_{2ji_2}, \beta^{(2)}_{2ji_2}; B_{2ji_2})]_{m_2+1,q_{i_2}}, [(b_{3j}; \beta^{(1)}_{3j}, \beta^{(2)}_{3j}, \beta^{(3)}_{3j}; B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(a_{3ji_3}; \alpha^{(1)}_{3ji_3}, \alpha^{(2)}_{3ji_3}, \alpha^{(3)}_{3ji_3}; A_{3ji_3})]_{n_3+1,p_{i_3}}; \cdots; [(\mathrm{a}_{rj}; \alpha^{(1)}_{rj}, \cdots, \alpha^{(r)}_{rj}; A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3}; \beta^{(1)}_{3ji_3}, \beta^{(2)}_{3ji_3}, \beta^{(3)}_{3ji_3}; B_{3ji_3})]_{m_3+1,q_{i_3}}; \cdots; [(\mathrm{b}_{rj}; \beta^{(1)}_{rj}, \cdots, \beta^{(r)}_{rj}; B_{rj})_{1,m_r}],$$

$$[\tau_{i_r}(a_{rji_r}; \alpha^{(1)}_{rji_r}, \cdots, \alpha^{(r)}_{rji_r}; A_{rji_r})_{n_r+1,p_r}] : [(\mathrm{c}^{(1)}_j, \gamma^{(1)}_j; C^{(1)}_j)_{1,n^{(1)}}], [\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}}, \gamma^{(1)}_{ji^{(1)}}; C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r}; \beta^{(1)}_{rji_r}, \cdots, \beta^{(r)}_{rji_r}; B_{rji_r})_{m_r+1,q_r}] : [(\mathrm{d}^{(1)}_j), \delta^{(1)}_j; D^{(1)}_j)_{1,m^{(1)}}], [\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}}, \delta^{(1)}_{ji^{(1)}}; D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left. \begin{array}{l} ;\cdots; [(c^{(r)}_j, \gamma^{(r)}_j; C^{(r)}_j)_{1,m^{(r)}}], [\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}}, \gamma^{(r)}_{ji^{(r)}}; C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}] \\ ;\cdots; [(d^{(r)}_j, \delta^{(r)}_j; D^{(r)}_j)_{1,n^{(r)}}], [\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}}, \delta^{(r)}_{ji^{(r)}}; D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}] \end{array} \right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots, s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \, \mathrm{d}s_1 \cdots \mathrm{d}s_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{m_2}\Gamma^{B_{2j}}(b_{2j}-\sum_{k=1}^{2}\beta_{2j}^{(k)}s_k)\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha_{2ji_2}^{(k)}s_k)\prod_{j=m_2+1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta_{2ji2}^{(k)}s_k)]}$$

$$\frac{\prod_{j=1}^{m_3}\Gamma^{B_{3j}}(b_{3j}-\sum_{k=1}^{3}\beta_{3j}^{(k)}s_k)\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=m_3+1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{m_r}\Gamma^{B_{rj}}(b_{rj}-\sum_{k=1}^{r}\beta_{rj}^{(k)}s_k)\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=m_r+1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \quad (1.2)$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \quad (1.3)$$

1) $[(c_j^{(1)};\gamma_j^{(1)})]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $m_2,n_2,\cdots,m_r,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2\leqslant q_{i_2},0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant m_r\leqslant q_{i_r},0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kj}^{(l)},B_{kj}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)}, B_{kji_k} \in \mathbb{R}^+; (j = m_k+1,\cdots,q_{i_k}); (k=2,\cdots,r); (l=1,\cdots,k).$

$\delta_{ji^{(k)}}^{(k)} \in \mathbb{R}^+; (i=1,\cdots,R^{(k)}); (j=m^{(k)}+1,\cdots,q_{i^{(k)}}); (k=1,\cdots,r).$

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{R}^+; (i=1,\cdots,R^{(k)}); (j=n^{(k)}+1,\cdots,p_{i^{(k)}}); (k=1,\cdots,r).$

5) $c_j^{(k)} \in \mathbb{C}; (j=1,\cdots,n_k); (k=1,\cdots,r); d_j^{(k)} \in \mathbb{C}; (j=1,\cdots,m_k); (k=1,\cdots,r).$

$a_{kji_k} \in \mathbb{C}; (j=n_k+1,\cdots,p_{i_k}); (k=2,\cdots,r).$

$b_{kji_k} \in \mathbb{C}; (j=m_k+1,\cdots,q_{i_k}); (k=2,\cdots,r).$

$d_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i=1,\cdots,R^{(k)}); (j=m^{(k)}+1,\cdots,q_{i^{(k)}}); (k=1,\cdots,r).$

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i=1,\cdots,R^{(k)}); (j=n^{(k)}+1,\cdots,p_{i^{(k)}}); (k=1,\cdots,r).$

The contour L_k is in the $s_k(k=1,\cdots,r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k\right)(j=1,\cdots,n_2), \Gamma^{A_{3j}}\left(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k\right)$ $(j=1,\cdots,n_3),\cdots,\Gamma^{A_{rj}}\left(1-a_{rj}+\sum_{i=1}^{r}\alpha_{rj}^{(i)}\right)(j=1,\cdots,n_r), \Gamma^{C_j^{(k)}}\left(1-c_j^{(k)}+\gamma_j^{(k)}s_k\right)(j=1,\cdots,n^{(k)})(k=1,\cdots,r)$to the right of the contour L_k and the poles of $\Gamma^{B_{2j}}\left(b_{2j}-\sum_{k=1}^{2}\beta_{2j}^{(k)}s_k\right)(j=1,\cdots,m_2), \Gamma^{B_{3j}}\left(b_{3j}-\sum_{k=1}^{3}\beta_{3j}^{(k)}s_k\right)(j=1,\cdots,m_3)$ $,\cdots,\Gamma^{B_{rj}}\left(b_{rj}-\sum_{i=1}^{r}\beta_{rj}^{(i)}\right)(j=1,\cdots,m_r), \Gamma^{D_j^{(k)}}\left(d_j^{(k)}-\delta_j^{(k)}s_k\right)(j=1,\cdots,m^{(k)})(k=1,\cdots,r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2}A_i^{(k)}\pi$ where

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)}\delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)}\gamma_j^{(k)} - \tau_{i^{(k)}}\left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)}\delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)}\gamma_{ji^{(k)}}^{(k)}\right) +$$

$$\sum_{j=1}^{n_2} A_{2j}\alpha_{2j}^{(k)} + \sum_{j=1}^{m_2} B_{2j}\beta_{2j}^{(k)} - \tau_{i_2}\left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2}\alpha_{2ji_2}^{(k)} + \sum_{j=m_2+1}^{q_{i_2}} B_{2ji_2}\beta_{2ji_2}^{(k)}\right) + \cdots +$$

$$\sum_{j=1}^{n_r} A_{rj}\alpha_{rj}^{(k)} + \sum_{j=1}^{m_r} B_{rj}\beta_{rj}^{(k)} - \tau_{i_r}\left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r}\alpha_{rji_r}^{(k)} + \sum_{j=m_r+1}^{q_{i_r}} B_{rji_r}\beta_{rji_r}^{(k)}\right) \quad (1.4)$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,), max(\,|z_1|,\cdots,|z_r|\,) \rightarrow 0$

$\aleph(z_1,\cdots,z_r) = 0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,), min(\,|z_1|,\cdots,|z_r|\,) \rightarrow \infty$ where $i=1,\cdots,r$:

$\alpha_i = \min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)$ and $\beta_i = \max_{\substack{1\leqslant k\leqslant n_i\\1\leqslant j\leqslant n^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} A_{hj}\frac{a_{hj}-1}{\alpha_{hj}^{h'}} + C_k^{(i)}\frac{c_k^{(i)}-1}{\gamma_k^{(i)}}\right)$

Remark 1.

If $m_2=n_2=\cdots=m_{r-1}=n_{r-1}=p_{i_2}=q_{i_2}=\cdots=p_{i_{r-1}}=q_{i_{r-1}}=0$ and $A_{2j}=B_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=B_{rj}=A_{rji_r}=B_{rji_r}=1$, then the generalized multivariable Gimel-function reduces in the generalized multivariable Aleph- function (extension of multivariable Aleph-function defined by Ayant [1]).

Remark 2.

If $m_2=n_2=\cdots=m_r=n_r=p_{i_2}=q_{i_2}=\cdots=p_{i_r}=q_{i_r}=0$ and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2==\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the generalized multivariable Gimel-function reduces in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [7]).

Remark 3.

If$A_{2j}=B_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=B_{rj}=A_{rji_r}=B_{rji_r}=1$and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the generalized multivariable Gimel-function reduces in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [6].

Remark 4.

If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and Panda [9,10]

In your investigation, we shall use the following notations.

$$\mathbb{A}=[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A}=[(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A=[(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B}=[(b_{2j};\beta_{2j}^{(1)},\beta_{2j}^{(2)};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta_{3j}^{(1)},\beta_{3j}^{(2)},\beta_{3j}^{(3)};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{(r-1)j};\beta_{(r-1)j}^{(1)},\cdots,\beta_{(r-1)j}^{(r-1)};B_{(r-1)j})_{1,m_{r-1}}],$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{m_{r-1}+1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B}=[(\mathrm{b}_{rj};\beta_{rj}^{(1)},\cdots,\beta_{rj}^{(r)};B_{rj})_{1,m_r}],[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{m_r+1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B}=[(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U = m_2, n_2; m_3, n_3; \cdots ; m_{r-1}, n_{r-1}; V = m^{(1)}, n^{(1)}; m^{(2)}, n^{(2)}; \cdots ; m^{(r)}, n^{(r)} \tag{1.11}$$

$$X = p_{i_2}, q_{i_2}, \tau_{i_2}; R_2; \cdots ; p_{i_{r-1}}, q_{i_{r-1}}, \tau_{i_{r-1}} : R_{r-1}; Y = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}; \cdots ; p_{i^{(r)}}, q_{i^{(r)}}; \tau_{i^{(r)}}; R^{(r)} \tag{1.12}$$

The biorthogonal pair of polynomial sets occurring in this paper are defined in the following manner [3] :

$$J_n^{(\alpha,\beta)}(x;k) = \frac{(\alpha+1)_{kn}}{n!} \sum_{j=0}^{n} (-)^j \binom{n}{j} \frac{(\alpha+\beta+n+1)_{kj}}{(\alpha+1)_{kj}} \left(\frac{1-x}{2}\right)^{kj} \tag{1.13}$$

and

$$K_n^{(\alpha,\beta)}(x;k) = \frac{1}{n!} \sum_{j=1}^{n} (-)^j \binom{\beta+n}{j} \left(\frac{x-1}{2}\right)^j \left(\frac{x+1}{2}\right)^{n-j} \sum_{w=0}^{j} (-)^w \binom{j}{w} \left(\frac{\alpha+w+1}{k}\right)_n \tag{1.14}$$

provided $Re(\alpha), Re(\beta) > -1$.

When $k = 1$ the above polynomials set reduce to the Jacobi polynomials.

Srivastava ([8], p. 1, Eq. 1). have introduced the general class of polynomials :

$$S_N^M(x) = \sum_{K=0}^{[N/M]} \frac{(-N)_{MK}}{K!} A_{N,K} x^K \tag{1.15}$$

where M is an arbitrary positive integer and the coeficients $A_{N,K}$ are arbitrary constants real or complex. On specializing these coefficients $A_{N,K}$, $S_N^M[.]$ yields a number of known polynomials as special cases. These include, among others, Hermite polynomials, Jacobi polynomials, Laguerre polynomials, bessel polynomials and several others, see ([11], p. 158-161).

2. Main integrals.

In this section, we shall evaluate the following three general double integrals :

Theorem 1.

$$\int_0^1 \int_0^1 x^{\lambda-1}(1-x)^{\mu-1} y^{\rho-1}(1-y)^{a-2\rho}(1+ty)^{\rho-a-1} {}_2F_1 \left[\begin{array}{c} \text{a,b} \\ \cdot \\ \text{1+a-b} \end{array} \middle| \frac{(1+t)y}{1+ty} \right] J_s^{(\alpha,\beta)}(1-2x;k)$$

$$S_N^M\left[cy^\gamma(1+ty)^\gamma(1-y)^{-2\gamma}\right] \daleth \left(z_1 x^{a_1}(1-x)^{b_1} \left\{\frac{y(1+ty)}{(1-y)^2}\right\}^{c_1}, \cdots, z_r x^{a_r}(1-x)^{b_r} \left\{\frac{y(1+ty)}{(1-y)^2}\right\}^{c_r} \right) dxdy =$$

$$\frac{2^a[4(1+t)]^{-\rho}\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)(a+1)_{ks}}{\sqrt{\pi}\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)s!} \sum_{K=0}^{[N/M]} \sum_{j=0}^{s} \frac{(-N)_{MK}}{K!} \frac{(\alpha+\beta+s+1)_{kj}(-s)_j}{(\alpha+1)_{kj} j! [4(1+j)]^{\gamma K}} c^K$$

$$\daleth^{U;m_r+2,n_r+3:V}_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y} \left(\begin{array}{c} \frac{z_1}{[4(1+t)]^{c_1}} \\ \cdot \\ \cdot \\ \frac{z_r}{[4(1+t)]^{c_r}} \end{array} \middle| \begin{array}{c} \mathbb{A}; (1-\mu; b_1, \cdots, b_r; 1), (1-\lambda-kj; a_1, \cdots, a_r; 1), \\ \cdot \\ \cdot \\ \mathbb{B}; \left(\frac{1+a}{2}-\rho-\gamma K; c_1, \cdots, c_r : 1\right), \left(\frac{2+a}{2}-\rho-\gamma K; c_1, \cdots, c_r : 1\right), \mathbf{B}, \end{array} \right.$$

$$\left.\begin{matrix} (1\text{-}\rho - K\gamma; c_1,\cdots,c_r;1), \mathbf{A}, (1+a-b-\rho-K\gamma;c_1,\cdots,c_r;1) : A \\ \cdot \\ \cdot \\ (1\text{-}\lambda-\mu-kj;a_1+b_1,\cdots,a_r+b_r;1):B \end{matrix}\right) \tag{2.1}$$

Provided

$a_i, b_i, c_i > 0 (i = 1,\cdots,r), Re(1+a-b) > 0, Re(\alpha) > -1, Re(\beta) > -1, t > -1 Re(1-2b) > 0; 0 \leqslant \gamma$

$$Re(\lambda) + \sum_{i=1}^{r} a_i \min_{\substack{1\leqslant k\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right) > 0$$

$$Re(\mu) + \sum_{i=1}^{r} b_i \min_{\substack{1\leqslant k\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right) > 0$$

$$Re(\rho+\gamma K) + \sum_{i=1}^{r} c_i \min_{\substack{1\leqslant k\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right) > 0$$

$$Re\left(1+a-2\rho-2\gamma K\right) - 2\sum_{i=1}^{r} c_i \min_{\substack{1\leqslant k\leqslant m_i \\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h} B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}} + D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right) > 0$$

$\left|\left(z_i x^{a_i}(1-x)^{b_i}\left\{\frac{y(1+ty)}{(1-y)^2}\right\}^{c_i}\right)\right| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4).

Proof

To prove (3.1), on the left hand side of (2.1), using the series representation of $S_N^M[.]$ with the help of (1.15) and expressing the generalized multivariable Gimel-function as Mellin-Barnes multiple integrals contour with the the help of (1.1), interchanging the order of summation and integration which is justified under the conditions mentioned above, we get (say I)

$$I = \sum_{K=0}^{[N/M]} \frac{(N)_{MK}}{K!} A_{N,K} c^K \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}$$

$$\left[\int_0^1 x^{\lambda+\sum_{i=1}^r a_i s_i -1}(1-x)^{\mu+\sum_{i=1}^r b_i s_i -1} J_s^{(\alpha,\beta)}(1-2x;k)\mathrm{d}x\right]$$

$$\left[\int_0^1 y^{\rho+\gamma K+\sum_{i=1}^r c_i s_i -1}(1-y)^{a-2\rho-2\gamma K-2\sum_{i=1}^r c_i s_i}(1+ty)^{\rho+\gamma K-a-\sum_{i=1}^r c_i s_i -1}{}_2F_1\left(a,b;1+a-b;\frac{(1+t)y}{1+ty}\right)\right]$$

$$\mathrm{d}y\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.2}$$

Now using the results ([4], p. 118, Eq. (3.3.1)) and ([5], p. 254, Eq. (2.1)), and finally interpreting the Mellin-Barnes multiple integrals contour in terms of the generalized multivariable Gimel-function, we get the desired result (2.1).

Theorem 2.

$$\int_0^1\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}y^{\rho-1}(1-y)^{a-2\rho}(1+ty)^{\rho-a-1}\ {}_3F_2\left[\begin{matrix}\mathrm{a},\frac{a}{2}+1,\mathrm{b} \\ \cdot \\ \frac{c}{2},1+\mathrm{a}\text{-}\mathrm{b}\end{matrix}\ \middle|\ \frac{(1+t)y}{1+ty}\right] J_s^{(\alpha,\beta)}(1-2x;k)$$

$$S_N^M\left[cy^\gamma(1+ty)^\gamma(1-y)^{-2\gamma}\right]\daleth\left(z_1x^{a_1}(1-x)^{b_1}\left\{\frac{y(1+ty}{(1-y)^2}\right\}^{c_1},\cdots,z_rx^{a_r}(1-x)^{b_r}\left\{\frac{y(1+ty}{(1-y)^2}\right\}^{c_r}\right)dxdy=$$

$$\frac{2^a[4(1+t)]^{-\rho}\Gamma(1+a-b)(a+1)_{ks}}{\sqrt{\pi}\Gamma(1+a)s!}\sum_{K=0}^{[N/M]}\sum_{j=0}^{s}\frac{(1-N)_{MK}}{K!}\frac{(\alpha+\beta+s+1)_{kj}(-s)_j}{(\alpha+1)_{kj}j![4(1+j)]^{\gamma K}}c^K$$

$$\daleth_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;m_r+2,n_r+3:V}\left(\begin{array}{c}\frac{z_1}{[4(1+t)]^{c_1}}\\ \cdot\\ \cdot\\ \frac{z_r}{[4(1+t)]^{c_r}}\end{array}\middle|\begin{array}{c}\mathbb{A};\ (1-\mu;b_1,\cdots,b_r;1),(1-\lambda-kj;a_1,\cdots,a_r;1),\mathbf{A},\\ \cdot\\ \cdot\\ \mathbb{B};\ \left(\frac{1+a}{2}-\rho-\gamma K;c_1,\cdots,c_r:1\right),\left(\frac{2+a}{2}-\rho-\gamma K;c_1,\cdots,c_r:1\right),\mathbf{B},\end{array}\right.$$

$$\left.\begin{array}{c}(1\text{-}\rho-K\gamma;c_1,\cdots,c_r;1),(1+a-b-\rho-K\gamma;c_1,\cdots,c_r;1):A\\ \cdot\\ \cdot\\ (1\text{-}\lambda-\mu-kj;a_1+b_1,\cdots,a_r+b_r;1):B\end{array}\right) \quad (2.2)$$

under the same existence conditions of the above theorem except that here $0\leqslant b<1$ instead of $Re(1-2b)>0$.

Theorem 3.

$$\int_0^1\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}y^{\rho-1}(1-y)^{a-2\rho}(1+ty)^{\rho-a-1}\,{}_4F_3\left[\begin{array}{c}\text{a},\frac{a}{2}+1,\text{c,d}\\ \cdot\\ \frac{c}{2},1+\text{a-c},\ 1+\text{a-d}\end{array}\middle|\ \frac{(1+t)y}{1+ty}\right]J_s^{(\alpha,\beta)}(1-2x;k)$$

$$S_N^M\left[cy^\gamma(1+ty)^\gamma(1-y)^{-2\gamma}\right]\daleth\left(z_1x^{a_1}(1-x)^{b_1}\left\{\frac{y(1+ty}{(1-y)^2}\right\}^{c_1},\cdots,z_rx^{a_r}(1-x)^{b_r}\left\{\frac{y(1+ty}{(1-y)^2}\right\}^{c_r}\right)dxdy=$$

$$\frac{2^a[4(1+t)]^{-\rho}\Gamma(1+a-c)\Gamma(1+a-d)(a+1)_{ks}}{\sqrt{\pi}\Gamma(1+a)\Gamma(1+a-c-d)s!}\sum_{K=0}^{[N/M]}\sum_{j=0}^{s}\frac{(-N)_{MK}}{K!}\frac{(\alpha+\beta+s+1)_{kj}(-s)_j}{(\alpha+1)_{kj}j![4(1+j)]^{\gamma K}}c^K$$

$$\daleth_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;m_r+2,n_r+3:V}\left(\begin{array}{c}\frac{z_1}{[4(1+t)]^{c_1}}\\ \cdot\\ \cdot\\ \frac{z_r}{[4(1+t)]^{c_r}}\end{array}\middle|\begin{array}{c}\mathbb{A};\ (1-\mu;b_1,\cdots,b_r;1),(1-\lambda-kj;a_1,\cdots,a_r;1),\\ \cdot\\ \cdot\\ \mathbb{B};\ \left(\frac{1+a}{2}-\rho-\gamma K;c_1,\cdots,c_r:1\right),\left(\frac{2+a}{2}-\rho-\gamma K;c_1,\cdots,c_r:1\right),\end{array}\right.$$

$$\left.\begin{array}{c}(1\text{-}\rho-\gamma K:c_1,\cdots,c_r;1),\mathbf{A},(1+a-c-\rho-K\gamma;c_1,\cdots,c_r;1),(1+a-d-\rho-K\gamma;c_1,\cdots,c_r;1):A\\ \cdot\\ \cdot\\ (1\text{+a-c-d-}\rho-K\gamma;c_1,\cdots,c_r),\mathbf{B},(1-\lambda-\mu-kj,a_1+b_1,\cdots,a_r+b_r;1):B\end{array}\right) \quad (2.3)$$

Provided

$a_i,b_i,c_i>0(i=1,\cdots,r),Re(1+a-c)>0,Re(1+a-2c-2d)>0,Re(1+a-d)>0;$

$$Re(\lambda)+\sum_{i=1}^r a_i\min_{\substack{1\leqslant k\leqslant m_i\\1\leqslant j\leqslant m^{(i)}}}Re\left(\sum_{h=2}^r\sum_{h'=1}^h B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re(\mu)+\sum_{i=1}^{r}b_i \min_{\substack{1\leqslant k\leqslant m_i\\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re(\rho+\gamma K)+\sum_{i=1}^{r}c_i \min_{\substack{1\leqslant k\leqslant m_i\\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$Re\,(1+a-2\rho-2\gamma K)-2\sum_{i=1}^{r}c_i \min_{\substack{1\leqslant k\leqslant m_i\\ 1\leqslant j\leqslant m^{(i)}}} Re\left(\sum_{h=2}^{r}\sum_{h'=1}^{h}B_{hj}\frac{b_{hj}}{\beta_{hj}^{h'}}+D_k^{(i)}\frac{d_k^{(i)}}{\delta_k^{(i)}}\right)>0$$

$$\left|\left(z_i x^{a_i}(1-x)^{b_i}\left\{\frac{y(1+ty}{(1-y)^2}\right\}^{c_i}\right)\right|<\frac{1}{2}A_i^{(k)}\pi\ ,\quad \text{where } A_i^{(k)} \text{ is defined by (1.4).}$$

To prove the second and third integrals, we use the similar method to that of the theorem 1 with the only difference that here we make use the integral ([5], p. 254 , Eq. (2.2)) and ([5], p. 255 , Eq. (2.3)) respectively instead of ([5], p. 254 , Eq. (2.1))

3. Special cases.

Taking $m=1$ and $A_{N,K}=\dfrac{(a'+1)_N(\alpha'+\beta'+N+1)_K}{N!(\beta'+1)_K}$ in (2.1) we obtain the following result

Corollary 1.

$$\int_0^1\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}y^{\rho-1}(1-y)^{a-2\rho}(1+ty)^{\rho-a-1}{}_2F_1\left(a,b;1+a-b;\frac{(1+t)y}{1+ty}\right)J_s^{(\alpha,\beta)}(1-2x;k)$$

$$P_N^{(\alpha',\beta')}\left(1-2\left[cy^{\gamma}(1+ty)^{\gamma}(1-y)^{-2\gamma}\right]\right)\gimel\left(z_1x^{a_1}(1-x)^{b_1}\left\{\frac{y(1+ty}{1-y}\right\}^{c_1},\cdots,z_rx^{a_r}(1-x)^{b_r}\left\{\frac{y(1+ty}{1-y}\right\}^{c_r}\right)$$

$$dxdy=\frac{2^a[4(1+t)]^{-\rho}\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)(a+1)_{ks}}{\sqrt{\pi}\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)s!}\frac{(\alpha+1)_N}{N!}$$

$$\sum_{K=0}^{[N/M]}\sum_{j=0}^{s}\frac{(-N)_K}{K!j!}\frac{(\alpha'+\beta'+N+1)_K}{(\beta'+1)_K}\frac{(\alpha+\beta+s+1)_{kj}(-s)_j}{(\alpha+1)_j[4(1+j)]^{\gamma K}}c^K$$

$$\gimel^{U;m_r+2,n_r+3:V}_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c}\frac{z_1}{[4(1+t)]^{c_1}} & \mathbb{A};\,(1-\mu;b_1,\cdots,b_r;1)\,,(1-\lambda-j;a_1,\cdots,a_r;1),\mathbf{A},\\ \cdot & \cdot\\ \cdot & \cdot\\ \frac{z_r}{[4(1+t)]^{c_r}} & \mathbb{B};\,\left(\frac{1+a}{2}-\rho-\gamma K;c_1,\cdots,c_r:1\right),\left(\frac{2+a}{2}-b-\rho-\gamma K;c_1,\cdots,c_r:1\right),\mathbf{B},\end{array}\right.$$

$$\left.\begin{array}{c}(1\text{-}\rho-K\gamma;c_1,\cdots,c_r;1),(1+a-b-\rho-K\gamma;c_1,\cdots,c_r;1):A\\ \cdot\\ \cdot\\ (1\text{-}\lambda-\mu-kj;a_1+b_1,\cdots,a_r+b_r;1):B\end{array}\right)\tag{3.1}$$

under the same conditions that (2.1).

Consider the above corollary, taking $k=1$ and $s=n$, we obtain

Corollary 2.

$$\int_0^1\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}y^{\rho-1}(1-y)^{a-2\rho}(1+ty)^{\rho-a-1}{}_2F_1\left(a,b;1+a-b;\frac{(1+t)y}{1+ty}\right)P_n^{(\alpha,\beta)}(1-2x)$$

$$P_N^{(\alpha',\beta')}\left(1-2\left[cy^{\gamma}(1+ty)^{\gamma}(1-y)^{-2\gamma}\right]\right)\beth\left(z_1x^{a_1}(1-x)^{b_1}\left\{\frac{y(1+ty}{1-y}\right\}^{c_1},\cdots,z_rx^{a_r}(1-x)^{b_r}\left\{\frac{y(1+ty}{1-y}\right\}^{c_r}\right)$$

$$dxdy=\frac{2^a[4(1+t)]^{-\rho}\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)(a+1)_{ks}}{\sqrt{\pi}\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)s!}\frac{(\alpha+1)_N}{N!}$$

$$\sum_{K=0}^{[N/M]}\sum_{j=0}^{s}\frac{(-N)_K}{K!j!}\frac{(\alpha'+\beta'+N+1)_K}{(\beta'+1)_K}(-)^j\frac{(\alpha+\beta+s+1)_j}{(\alpha+1)_j[4(1+j)]^{\gamma K}}c^K$$

$$\beth^{U;m_r+2,n_r+3:V}_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}\left(\begin{array}{c}\frac{z_1}{[4(1+t)]^{c_1}}\\ \cdot\\ \cdot\\ \frac{z_r}{[4(1+t)]^{c_r}}\end{array}\left|\begin{array}{c}\mathbb{A};(1-\mu;b_1,\cdots,b_r;1),(1-\lambda-kj;a_1,\cdots,a_r;1),\mathbf{A},\\ \cdot\\ \cdot\\ \mathbb{B};\left(\frac{1+a}{2}-\rho-\gamma K;c_1,\cdots,c_r:1\right),\left(\frac{2+a}{2}-b-\rho-\gamma K;c_1,\cdots,c_r:1\right),\mathbf{B},\end{array}\right.\right.$$

$$\left.\begin{array}{c}(1+a-b-\rho-K\gamma;c_1,\cdots,c_r;1):A\\ \cdot\\ \cdot\\ (1\text{-}\lambda-\mu-j;a_1+b_1,\cdots,a_r+b_r):B\end{array}\right) \quad (3.2)$$

Taking $N=\gamma=0$ in (2.1) , $S_N^M(x)$ reduces to $A_{0,0}$ and we get the following double integral

Corollary 3.

$$\int_0^1\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}y^{\rho-1}(1-y)^{a-2\rho}(1+ty)^{\rho-a-1}{}_2F_1\left(a,b;1+a-b;\frac{(1+t)y}{1+ty}\right)J_s^{(\alpha,\beta)}(1-2x;k)$$

$$\beth\left(z_1x^{a_1}(1-x)^{b_1}\left\{\frac{y(1+ty}{1-y}\right\}^{c_1},\cdots,(z_rx^{a_r}(1-x)^{b_r}\left\{\frac{y(1+ty}{1-y}\right\}^{c_r}\right)dxdy=$$

$$\frac{2^a[4(1+t)]^{-\rho}\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)(a+1)_{ks}}{\sqrt{\pi}\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)s!}\sum_{j=0}^{s}\frac{(\alpha+\beta+s+1)_{kj}(-s)_j}{(\alpha+1)_{kj}j!}c^K$$

$$\beth^{U;m_r+2,n_r+3:V}_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}\left(\begin{array}{c}\frac{z_1}{[4(1+t)]^{c_1}}\\ \cdot\\ \cdot\\ \frac{z_r}{[4(1+t)]^{c_r}}\end{array}\left|\begin{array}{c}\mathbb{A};(1-\mu;b_1,\cdots,b_r;1),(1-\lambda-kj;a_1,\cdots,a_r;1),\mathbf{A},\\ \cdot\\ \cdot\\ \mathbb{B};\left(\frac{1+a}{2}-\rho;c_1,\cdots,c_r:1\right),\left(\frac{2+a}{2}-b-\rho;c_1,\cdots,c_r:1\right),\mathbf{B},\end{array}\right.\right.$$

$$\left.\begin{array}{c}(1\text{-}\rho;c_1,\cdots,c_r;1),(1+a-b-\rho;c_1,\cdots,c_r;1):A\\ \cdot\\ \cdot\\ (1\text{-}\lambda-\mu-kj;a_1+b_1,\cdots,a_r+b_r;1):B\end{array}\right) \quad (3.3)$$

Now taking $a_i, b_i \to 0 (i = 1, \cdots, r)$ in theorem 1 and evaluate the x-integral in the resulting expression, we obtain the following single integral.

Corollary 4.

$$\int_0^1 y^{\rho-1}(1-y)^{a-2\rho}(1+ty)^{\rho-a-1}{}_2F_1\left(a,b;1+a-b;\frac{(1+t)y}{1+ty}\right)$$

$$S_N^M\left[cy^{\gamma}(1+ty)^{\gamma}(1-y)^{-2\gamma}\right]\gimel\left(z_1\left\{\frac{y(1+ty}{1-y}\right\}^{c_1},\cdots,z_r\left\{\frac{y(1+ty}{1-y}\right\}^{c_r}\right)dxdy =$$

$$\frac{2^a[4(1+t)]^{-\rho}\Gamma\left(1+\frac{a}{2}\right)\Gamma(1+a-b)\Gamma(\mu)}{\sqrt{\pi}\Gamma(1+a)\Gamma\left(1+\frac{a}{2}-b\right)}\sum_{K=0}^{[N/M]}\frac{(-N)_{MK}}{K!}\frac{c^K}{[4(1-t)]^{K\gamma}}A_{N,K}\,c^K\gimel^{U;m_r+2,n_r+1:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c}\frac{z_1}{[4(1+t)]^{c_1}} & \mathbb{A};\,(1\text{-}\rho-K\gamma;a_1,\cdots,a_r;1),\mathbf{A},(1+a-b-\rho-K\gamma;c_1,\cdots,c_r;1);A\\ \cdot & \cdot\\ \cdot & \cdot\\ \frac{z_r}{[4(1+t)]^{c_r}} & \mathbb{B};\,\left(\frac{1+a}{2}-\rho-\gamma K;c_1,\cdots,c_r:1\right),\left(\frac{2+a}{2}-b-\rho-\gamma K;c_1,\cdots,c_r:1\right),\mathbf{B}:B\end{array}\right) \quad (3.4)$$

The conditions of validity of the integrals (3.1) through (3.4) can be deduced from the existence conditions mentioned with the theorem 1.

Remark :
We obtain the same integrals with the functions cited in the section 1.
We can also evaluate three more integrals by replacing $J_n^{(\alpha,\beta)}(x;k)$ by $K_n^{(\alpha,\beta)}(x;k)$.

5. Conclusion.

The importance of our all the results lies in their manifold generality. Firstly, in view of general arguments utilized in these double finite integrals, we can obtain a large simpler double or single finite integrals. Secondly, by specialising the various parameters as well as variables in the generalized multivariable Gimel-function, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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