Some Transformations and Identities for Multivariable Gimel Function

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ABSTRACT
In this paper, some transformations, summation formulae and identities for multivariable Gimel-function have been evaluated. Many new relations may be derived as particular cases, which are known identities.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, integrals, hypergeometric relations.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We define a generalized transcendental function of several complex variables.

$$\gimel(z_1, \cdots, z_r) = \gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2}:R_2;p_{i_3},q_{i_3},\tau_{i_3}:R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array} \right|$$

$$[(\mathrm{a}_{2j}; \alpha_{2j}^{(1)}, \alpha_{2j}^{(2)}; A_{2j})]_{1,n_2}, [\tau_{i_2}(a_{2ji_2}; \alpha_{2ji_2}^{(1)}, \alpha_{2ji_2}^{(2)}; A_{2ji_2})]_{n_2+1,p_{i_2}}; [(a_{3j}; \alpha_{3j}^{(1)}, \alpha_{3j}^{(2)}, \alpha_{3j}^{(3)}; A_{3j})]_{1,n_3},$$
$$[\tau_{i_2}(b_{2ji_2}; \beta_{2ji_2}^{(1)}, \beta_{2ji_2}^{(2)}; B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3}; \alpha_{3ji_3}^{(1)}, \alpha_{3ji_3}^{(2)}, \alpha_{3ji_3}^{(3)}; A_{3ji_3})]_{n_3+1,p_{i_3}}; \cdots; [(\mathrm{a}_{rj}; \alpha_{rj}^{(1)}, \cdots, \alpha_{rj}^{(r)}; A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3}; \beta_{3ji_3}^{(1)}, \beta_{3ji_3}^{(2)}, \beta_{3ji_3}^{(3)}; B_{3ji_3})]_{1,q_{i_3}}; \cdots; \quad \cdots$$

$$[\tau_{i_r}(a_{rji_r}; \alpha_{rji_r}^{(1)}, \cdots, \alpha_{rji_r}^{(r)}; A_{rji_r})_{n_r+1,p_r}] : \ [(\mathrm{c}_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,n^{(1)}}], [\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)}; C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}]$$
$$[\tau_{i_r}(b_{rji_r}; \beta_{rji_r}^{(1)}, \cdots, \beta_{rji_r}^{(r)}; B_{rji_r})_{1,q_r}] : \ [(\mathrm{d}_j^{(1)}), \delta_j^{(1)}; D_j^{(1)})_{1,m^{(1)}}], [\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)}; D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}]$$

$$\left. \begin{array}{l} ;\cdots; [(c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1,m^{(r)}}], [\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)}; C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \\ ;\cdots; [(d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{1,n^{(r)}}], [\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)}; D_{ji^{(r)}}^{(r)})_{n^{(r)}+1,q_i^{(r)}}] \end{array} \right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots, s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \, \mathrm{d}s_1 \cdots \mathrm{d}s_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1, \cdots, s_r) = \frac{\prod_{j=1}^{n_2} \Gamma^{A_{2j}}(1 - a_{2j} + \sum_{k=1}^{2} \alpha_{2j}^{(k)} s_k)}{\sum_{i_2=1}^{R_2} [\tau_{i_2} \prod_{j=n_2+1}^{p_{i_2}} \Gamma^{A_{2ji_2}}(a_{2ji_2} - \sum_{k=1}^{2} \alpha_{2ji_2}^{(k)} s_k) \prod_{j=1}^{q_{i_2}} \Gamma^{B_{2ji_2}}(1 - b_{2ji2} + \sum_{k=1}^{2} \beta_{2ji2}^{(k)} s_k)]}$$

$$\frac{\prod_{j=1}^{n_3} \Gamma^{A_{3j}}(1 - a_{3j} + \sum_{k=1}^{3} \alpha_{3j}^{(k)} s_k)}{\sum_{i_3=1}^{R_3} [\tau_{i_3} \prod_{j=n_3+1}^{p_{i_3}} \Gamma^{A_{3ji_3}}(a_{3ji_3} - \sum_{k=1}^{3} \alpha_{3ji_3}^{(k)} s_k) \prod_{j=1}^{q_{i_3}} \Gamma^{B_{3ji_3}}(1 - b_{3ji3} + \sum_{k=1}^{3} \beta_{3ji3}^{(k)} s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant m_r,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n_k);(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r)$.

$b_{kji_k}\in\mathbb{C};(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r)$.

$d_{ji^{(k)}}^{(k)}\in\mathbb{C};(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{C};(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

The contour L_k is in the $s_k(k=1,\cdots,r)$- plane and run from $\sigma-i\infty$ to $\sigma+i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k\right)(j=1,\cdots,n_2),\Gamma^{A_{3j}}\left(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k\right)$ $(j=1,\cdots,n_3),\cdots,\Gamma^{A_{rj}}\left(1-a_{rj}+\sum_{i=1}^{r}\alpha_{rj}^{(i)}\right)(j=1,\cdots,n_r),\Gamma^{C_j^{(k)}}\left(1-c_j^{(k)}+\gamma_j^{(k)}s_k\right)(j=1,\cdots,n^{(k)})(k=1,\cdots,r)$to the right of the contour L_k and the poles of $\Gamma^{D_j^{(k)}}\left(d_j^{(k)}-\delta_j^{(k)}s_k\right)(j=1,\cdots,m^{(k)})(k=1,\cdots,r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)|<\frac{1}{2}A_i^{(k)}\pi$ where

$$A_i^{(k)}=\sum_{j=1}^{m^{(k)}}D_j^{(k)}\delta_j^{(k)}+\sum_{j=1}^{n^{(k)}}C_j^{(k)}\gamma_j^{(k)}-\tau_{i^{(k)}}\left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}}D_{ji^{(k)}}^{(k)}\delta_{ji^{(k)}}^{(k)}+\sum_{j=n^{(k)}+1}^{p_i^{(k)}}C_{ji^{(k)}}^{(k)}\gamma_{ji^{(k)}}^{(k)}\right)+$$

$$-\tau_{i_2}\left(\sum_{j=n_2+1}^{p_{i_2}}A_{2ji_2}\alpha_{2ji_2}^{(k)}+\sum_{j=1}^{q_{i_2}}B_{2ji_2}\beta_{2ji_2}^{(k)}\right)-\cdots-\tau_{i_r}\left(\sum_{j=n_r+1}^{p_{i_r}}A_{rji_r}\alpha_{rji_r}^{(k)}+\sum_{j=1}^{q_{i_r}}B_{rji_r}\beta_{rji_r}^{(k)}\right)\tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,),max(\,|z_1|,\cdots,|z_r|\,)\rightarrow 0$

$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,),min(\,|z_1|,\cdots,|z_r|\,)\rightarrow\infty$ where $i=1,\cdots,r:$

$$\alpha_i=\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] \text{ and } \beta_i=\max_{1\leqslant j\leqslant n^{(i)}}Re\left[C_j^{(i)}\left(\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)\right]$$

Remark 1.
If $n_2=\cdots=n_{r-1}=p_{i_2}=q_{i_2}=\cdots=p_{i_{r-1}}=q_{i_{r-1}}=0$ and $A_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=A_{rji_r}=B_{rji_r}=1$ $A_{rj}=A_{rji_r}=B_{rji_r}=1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [1].

Remark 2.
If $n_2=\cdots=n_r=p_{i_2}=q_{i_2}=\cdots=p_{i_r}=q_{i_r}=0$ and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1,$ then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [6].

Remark 3.
If$A_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=A_{rji_r}=B_{rji_r}=1$and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [5].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and Panda [8,9].

In your investigation, we shall use the following notations.

$$\mathbb{A}=[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}]\tag{1.5}$$

$$\mathbf{A} = [(a_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A = [(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B} = [\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B} = [\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{1,q_{i_r}}] \tag{1.9}$$

$$B = [(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U = 0,n_2;0,n_3;\cdots;0,n_{r-1};V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

2. Required results.

The following result will be utilized in this paper from Luke [4] we have

Lemma 1.

$${}_3F_2\left[\begin{matrix}a,\ b,c+1\\ .\\ d,e\end{matrix}\ \middle|\ z\right] = {}_2F_1\left[\begin{matrix}a,\ b\\ .\\ d\end{matrix}\ \middle|\ z\right] + \frac{abz}{cd}\,{}_2F_1\left[\begin{matrix}a+1,\ b+1\\ .\\ d+1\end{matrix}\ \middle|\ z\right] \tag{2.1}$$

We have the following relation, Rainville [7].

Lemma 2.

$$\frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)} = \frac{(-)^n}{(\alpha)_n} \tag{2.2}$$

3. Main formulae.

Theorem 1.

$$\sum_{s=0}^{\infty}\frac{z^s}{s!}\beth_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;0,n_r+4:V}$$

$$\left(\begin{matrix}z_1\\ .\\ .\\ z_r\end{matrix}\ \middle|\ \begin{matrix}\mathbb{A};\ (a\text{-}s;a_1,\cdots,a_r;1),(b-s;b_1,\cdots,b_r;1),(c;c_1,\cdots,c_r;1),(c-s-1;c_1,\cdots,c_r;1),\mathbf{A},A\\ .\\ .\\ \mathbb{B};\ \mathbf{B},\ (d\text{-}s;d_1,\cdots,d_r;1),(c-s;c_1,\cdots,c_r;1),(c-1;c_1,\cdots,c_r;1):B\end{matrix}\right)$$

$$= \sum_{s=1}^{\infty} \frac{z^s}{s!} \gimel^{U;0,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y} \left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\ (\text{a-s};\mathrm{a}_1,\cdots,a_r;1),(b-s;b_1,\cdots,b_r;1),\mathbf{A},A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\ \mathbf{B},\ (\text{d-s};\mathrm{d}_1,\cdots,d_r;1):B \end{array} \right) + \sum_{s=0}^{\infty} \frac{z^{s+1}}{(s+1)!}$$

$$\gimel^{U;0,n_r+3:V}_{X;p_{i_r}+3,q_{i_r}+2,\tau_{i_r}:R_r:Y} \left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\ (\text{a-s-1};\mathrm{a}_1,\cdots,a_r;1),(b-s-1;b_1,\cdots,b_r;1),(c;c_1,\cdots,c_r;1),\mathbf{A},A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\ \mathbf{B},\ (\text{d-s-1};\mathrm{d}_1,\cdots,d_r;1),(c-1;c_1,\cdots,c_r;1):B \end{array} \right) \quad (3.1)$$

provided

$$a_i,b_i,c_i,d_i>0 (i=1,\cdots,r),\quad |arg\,(z_i)|\ < \frac{1}{2}\left(A_i^{(k)}-\sum_{i=1}^{r}(a_i+b_i-d_i)\right)\pi\ ,$$

Theorem 2.

$$\sum_{s=0}^{\infty} \frac{z^s}{s!} \gimel^{U;0,n_r+2:V}_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\ (\text{c-s-1};\mathrm{c}_1,\cdots,c_r;1),(c;c_1,\cdots,c_r;1),\mathbf{A},(a-s;a_1,\cdots,a_r;1),(b-s;b_1,\cdots,b_r;1),A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\ \mathbf{B},\ (\text{d-s};\mathrm{d}_1,\cdots,d_r;1),(c-s;c_1,\cdots,c_r;1),(c-1;c_1,\cdots,c_r;1):B \end{array} \right)$$

$$= \sum_{s=1}^{\infty} \frac{z^s}{s!} \gimel^{U;0,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y} \left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\ \mathbf{A},(\text{a-s};\mathrm{a}_1,\cdots,a_r;1),(b-s;b_1,\cdots,b_r;1):A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\ \mathbf{B},\ (\text{d-s};\mathrm{d}_1,\cdots,d_r;1):B \end{array} \right) + \sum_{s=0}^{\infty} \frac{z^{s+1}}{(s+1)!}$$

$$\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+3,q_{i_r}+2,\tau_{i_r}:R_r:Y} \left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\ (\text{c};\mathrm{c}_1,\cdots,c_r;1),\mathbf{A},(a-s-1;a_1,\cdots,a_r;1),(b-s-1;b_1,\cdots,b_r;1):A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\ \mathbf{B},\ (\text{d-s-1};\mathrm{d}_1,\cdots,d_r;1),(c-1;c_1,\cdots,c_r;1):B \end{array} \right) \quad (3.2)$$

provided

$$a_i,b_i,c_i,d_i>0 (i=1,\cdots,r),\quad |arg\,(z_i)|\ < \frac{1}{2}\left(A_i^{(k)}-\sum_{i=1}^{r}(a_i+b_i+d_i)\right)\pi$$

Theorem 3.

$$\sum_{s=0}^{\infty} \frac{\Gamma(a+r)z^s}{s!} \gimel^{U;0,n_r+3:V}_{X;p_{i_r}+3,q_{i_r}+3,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\ (\text{b-s};\mathrm{b}_1,\cdots,b_r;1),(c-s-1;c_1,\cdots,c_r;1),(c;c_1,\cdots,c_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\ \mathbf{B},\ (\text{d-s};\mathrm{d}_1,\cdots,d_r;1),(c-s;c_1,\cdots,c_r;1),(c-1;c_1,\cdots,c_r;1):B \end{array} \right)$$

$$= \sum_{s=0}^{\infty} \frac{\Gamma(a+r)z^s}{s!} \gimel^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+1,\tau_{i_r}:R_r:Y} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_r \end{array} \middle| \begin{array}{c} \mathbb{A}; \text{(b-s;b}_1, \cdots, b_r; 1), \mathbf{A} : A \\ \cdot \\ \cdot \\ \mathbb{B}; \mathbf{B}, \text{(d-s;d}_1, \cdots, d_r; 1) : B \end{array} \right) + \sum_{s=0}^{\infty} \frac{\Gamma(a+s+1)z^{s+1}}{(s+1)!}$$

$$\gimel^{U;0,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_r \end{array} \middle| \begin{array}{c} \mathbb{A}; \text{(b-s-1;b}_1, \cdots, b_r; 1), (c; c_1, \cdots, c_r; 1), (b-s-1; b_1, \cdots, b_r; 1), \mathbf{A} : A \\ \cdot \\ \cdot \\ \mathbb{B}; \mathbf{B}, \text{(d-s-1;d}_1, \cdots, d_r; 1), (c-1; c_1, \cdots, c_r; 1) : B \end{array} \right) \quad (3.3)$$

provided

$$a_i, b_i, c_i, d_i > 0 (i = 1, \cdots, r), \quad |arg\,(z_i)| < \frac{1}{2} \left(A_i^{(k)} - \sum_{i=1}^{r} (b_i - d_i) \right) \pi, |z| < 1$$

Proof

To prove the theorem 1, expressing the multivariable Gimel-function in multiple integrals contour with the help of (1.1) , we say I

$$I = \sum_{s=0}^{\infty} \frac{z^s}{s!} \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots, s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \frac{\Gamma(1-a+s+\sum_{i=1}^r s_i a_i)\Gamma(1-b+s+\sum_{i=1}^r s_i b_i)}{\Gamma(2-c+\sum_{i=1}^r c_i s_i)\Gamma(1-c+s+\sum_{i=1}^r s_i c_i)}$$

$$\frac{\Gamma(2-c+s+\sum_{i=1}^r s_i c_i)\Gamma(1-c+\sum_{i=1}^r s_i c_i)}{\Gamma(1-d+s+\sum_{i=1}^r s_i d_i)} \mathrm{d}s_1 \cdots \mathrm{d}s_r \quad (3.4)$$

Now interchanging the order of summations and integrations which is justified under the above conditions, we have

$$I = \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots, s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \frac{\Gamma(1-c+\sum_{i=1}^r s_i c_i)}{\Gamma(2-c+\sum_{i=1}^r s_i c_i)}$$

$${}_3F_2 \left[\begin{array}{c} \text{1-a+}\sum_{i=1}^r a_i s_i, 1-b+\sum_{i=1}^r b_i s_i, 2-c+\sum_{i=1}^r c_i s_i, \\ \cdot \\ \text{1-d+s+}\sum_{i=1}^r c_i s_i, 1-c+\sum_{i=1}^r c_i s_i, \end{array} \middle| \; z \right] \mathrm{d}s_1 \cdots \mathrm{d}s_r \quad (3.5)$$

Now applying the lemma 1, expressing both the Gauss hypergeometric function as series, changing the order of integration ans summation and interchanging the result thus obtained in view of lemma 1, we get the right hand side of (3.1). The theorems 2 and 3 can be established in the same way.

4. Particular cases.

In this section we derive some infinite summations form of transformations formulae discussed in the above section.
In (3.1) and (3.2) putting $z = 1$ on both sides and right hand side, expressed as multivariable Gimel-function as multiple integrals contour, changing the order of summation and integration and evaluating the summation inside the contour with the help pf the Gauss theorem [3] ,we get the following summations.

Corollary 1.

$$\sum_{s=0}^{\infty} \frac{1}{s!} \gimel^{U;0,n_r+4:V}_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};\ (\text{a-s};a_1,\cdots,a_r;1),(b-s;b_1,\cdots,b_r;1),(c;c_1,\cdots,c_r;1),(c-s-1;c_1,\cdots,c_r;1),\mathbf{A},A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ \mathbf{B},\ (\text{d-s};d_1,\cdots,d_r;1),(c-s;c_1,\cdots,c_r;1),(c-1;c_1,\cdots,c_r;1):B \end{array}\right)$$

$$=\gimel^{U;0,n_r+3:V}_{X;p_{i_r}+4,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};\ (\text{a};a_1,\cdots,a_r;1),(b;b_1,\cdots,b_r;1),(d+2-a-b;d_1-a_1-b_1,\cdots,d_r-a_r-b_r;1),\mathbf{A},A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ \mathbf{B},(\text{d-a+1};d_1-a_1,\cdots,d_r-a_r;1),(d-b+1;d_1-b_1,\cdots,d_r-b_r;1),(c-1;c_1,\cdots,c_r;1):B \end{array}\right)$$

$$+\gimel^{U;0,n_r+4:V}_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};\ (\text{a-1};a_1,\cdots,a_r;1),(b-1;b_1,\cdots,b_r;1),(d+3-a-b;d_1-a_1-b_1,\cdots,d_r-a_r-b_r;1),\mathbf{A},A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ \mathbf{B},(\text{d-a+1};d_1-a_1,\cdots,d_r-a_r;1),(d-b+1;d_1-b_1,\cdots,d_r-b_r;1),(c-1;c_1,\cdots,c_r;1):B \end{array}\right) \qquad (3.1)$$

provided

$$a_i,b_i,c_i,d_i,d_i-a_i-b_i>0(i=1,\cdots,r),\quad |arg\,(z_i)|<\frac{1}{2}\left(A_i^{(k)}-\sum_{i=1}^{r}(a_i+b_i-d_i)\right)\pi\ ,$$

Corollary 2.

$$\sum_{s=0}^{\infty}\frac{1}{s!}\gimel^{U;0,n_r+2:V}_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};\ (\text{c-s-1};c_1,\cdots,c_r;1),(c;c_1,\cdots,c_r;1),\mathbf{A},(a-s;a_1,\cdots,a_r;1),(b-s;b_1,\cdots,b_r;1),A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ \mathbf{B},\ (\text{d-s};d_1,\cdots,d_r;1),(c-s;c_1,\cdots,c_r;1),(c-1;c_1,\cdots,c_r;1):B \end{array}\right)$$

$$=\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+3,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};\ (\text{d-a-b+2};d_1-a_1-b_1\cdots,d_r-a_r-b_r;1),\mathbf{A},(a;a_1,\cdots,a_r;1),(b;b_1,\cdots,b_r;1):A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ \mathbf{B},(\text{d-a+1};d_1-a_1,\cdots,d_r-a_r;1),(d-b+1;d_1-b_1,\cdots,d_r-b_r;1):B \end{array}\right)$$

$$+\gimel^{U;0,n_r+2:V}_{X;p_{i_r}+4,q_{i_r}+4,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};\ (\text{d+3-a-b};d_1-a_1-b_1,\cdots,d_r-a_r-b_r;1),\mathbf{A},(a;a_1,\cdots,a_r;1),(b;b_1,\cdots,b_r;1):A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ \mathbf{B},(\text{d-a-b+3};d_1-a_1-b_1,\cdots,d_r-a_r-b_r;1),(d-b+a;a_1+d_1-b_1,\cdots,a_r+d_r-b_r;1),(c-1;c_1,\cdots,c_r;1):B \end{array}\right)(3.2)$$

provided

$$a_i, b_i, c_i, d_i, d_i - a_i - b_i > 0(i = 1, \cdots, r), \quad |arg\,(z_i)| < \frac{1}{2}\left(A_i^{(k)} - \sum_{i=1}^{r}(b_i - d_i)\right)\pi \ ,$$

We obtain the same double finite series relations with the functions cited in the section I.

3. Conclusion.

The importance of our tranformation series formulae and identities lies in their manifold generality. By specializing the various parameters and variables involved in the generalized multivariable Gimel-function, we get a several series formulae involving in remarkably wide variety of useful function (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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