On the Positive Pellian Equation $y^2 = 48x^2 + 16$

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Abstract:

The hyperbola represented by the binary quadratic equation $y^2 = 48x^2 + 16$ is analyzed for finding its non-zero distinct integer solutions. A few interesting relations among its solutions are presented. Also, knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented. Employing the solutions, a special Pythagorean triangle is constructed.

Keywords: *Binary quadratic, Hyperbola, Parabola, Pell equation, Integral solutions.*

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I. INTRODUCTION

The binary quadratic Diophantine equations of the form $ax^2 - by^2 = N, (a, b, N \neq 0)$ are rich in variety and have been analyzed by many mathematicians for their respective integer solutions for particular values of a, b and N. In this context, one may refer [1-16]. This communication concerns with the problem of obtaining non-zero distinct integer solutions to the binary quadratic equation given by $y^2 = 48x^2 + 16$ representing hyperbola. A few interesting relations among its solutions are presented. Also, knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented. Employing the solutions, a special Pythagorean triangle is constructed.

II. NOTATIONS

- Polygonal number of rank n with size m

$$t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$$

- Pyramidal number of rank n with size m

$$P_n^m = \frac{1}{6}[n(n+1)][(m-2)n + (5-m)]$$

III. METHOD OF ANALYSIS

The binary quadratic equation to be solved for its non-zero distinct integral solution is

$$y^2 = 48x^2 + 16 \tag{1}$$

whose smallest positive integer solution is

$$x_0 = 1 \text{ and } y_0 = 8$$

To obtain the other solutions of (1), consider the Pell equation

$$y^2 = 48x^2 + 1 \tag{2}$$

whose solution is given by

$$\tilde{y}_n = \frac{1}{2} f_n \,,\; \tilde{x}_n = \frac{1}{2\sqrt{48}} g_n$$

where

$$f_n = (7 + \sqrt{48})^{n+1} + (7 - \sqrt{48})^{n+1} \,,\; g_n = (7 + \sqrt{48})^{n+1} - (7 - \sqrt{48})^{n+1}$$

Applying Brahmagupta lemma between the solutions (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{1}{\sqrt{3}} g_n$$

$$y_{n+1} = 4 f_n + \frac{6}{\sqrt{3}} g_n$$

The recurrence relations satisfied by the values of x_{n+1} and y_{n+1} are respectively.

$$x_{n+3} - 14x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 14y_{n+2} + y_{n+1} = 0$$

A few numerical examples are given in the Table: 1 below:

Table: 1 Numerical Examples

n	x_{n+1}	y_{n+1}
-1	1	8
0	15	104
1	209	1448
2	2911	20168
3	40545	280904
4	564719	3912488
5	7865521	54493928

➢ A few interesting relations among the solutions are given below:

- $x_{n+2} - y_{n+1} - 7x_{n+1} = 0$
- $x_{n+1} + x_{n+3} - 14x_{n+2} = 0$
- $4y_{n+1} - 28x_{n+3} + 388x_{n+2} = 0$
- $y_{n+2} - x_{n+3} + 7x_{n+2} = 0$
- $y_{n+3} - 7x_{n+3} + x_{n+2} = 0$
- $y_{n+3} - 7y_{n+2} - 48x_{n+2} = 0$
- $97x_{n+2} - 7x_{n+3} + y_{n+1} = 0$
- $x_{n+1} - x_{n+3} + 2y_{n+2} = 0$
- $28y_{n+1} - 388y_{n+2} + 192x_{n+3} = 0$
- $x_{n+1} - 97x_{n+3} + 14y_{n+3} = 0$
- $x_{n+2} - 7x_{n+3} + y_{n+3} = 0$
- $y_{n+1} - 97y_{n+3} + 672x_{n+3} = 0$
- $48x_{n+1} - y_{n+2} + 7y_{n+1} = 0$
- $48x_{n+2} - 7y_{n+2} + y_{n+1} = 0$

- $672x_{n+1} - y_{n+3} + 97y_{n+1} = 0$
- $48x_{n+1} + 97y_{n+2} - 7y_{n+3} = 0$
- $48x_{n+2} + 7y_{n+2} - y_{n+3} = 0$
- $192x_{n+3} + 4y_{n+2} - 28y_{n+3} = 0$
- $y_{n+3} - 14y_{n+2} + y_{n+1} = 0$
- $7x_{n+1} - x_{n+2} + y_{n+1} = 0$

➢ Each of the following expression represents a cubical integer.

- $\frac{1}{8}[(8x_{3n+4} - 104x_{3n+3}) + 3(8x_{n+2} - 104x_{n+1})]$
- $\frac{1}{14}[(x_{3n+5} - 181x_{3n+3}) + 3(x_{n+3} - 181x_{n+1})]$
- $\frac{1}{8}[(8y_{3n+3} - 48x_{3n+3}) + 3(8y_{n+1} - 48x_{n+1})]$
- $\frac{1}{56}[(8y_{3n+4} - 720x_{3n+3}) + 3(8y_{n+2} - 720x_{n+1})]$
- $\frac{1}{776}[(8y_{3n+5} - 10032x_{3n+3}) + 3(8y_{n+3} - 10032x_{n+1})]$
- $\frac{1}{8}[(104x_{3n+5} - 1448x_{3n+4}) + 3(104x_{n+3} - 1448x_{n+2})]$
- $\frac{1}{56}[(104y_{3n+3} - 48x_{3n+4}) + 3(104y_{n+1} - 48x_{n+2})]$
- $\frac{1}{8}[(104y_{3n+4} - 720x_{3n+4}) + 3(104y_{n+2} - 720x_{n+2})]$
- $\frac{1}{56}[(104y_{3n+5} - 10032x_{3n+4}) + 3(104y_{n+3} - 10032x_{n+2})]$
- $\frac{1}{194}[(362y_{3n+3} - 12x_{3n+5}) + 3(362y_{n+1} - 12x_{n+3})]$
- $\frac{1}{56}[(1448y_{3n+4} - 720x_{3n+5}) + 3(1448y_{n+2} - 720x_{n+3})]$
- $\frac{1}{8}[(1448y_{3n+5} - 10032x_{3n+5}) + 3(1448y_{n+3} - 10032x_{n+3})]$
- $\frac{1}{8}[(15y_{3n+3} - y_{3n+4}) + 3(15y_{n+1} - y_{n+2})]$
- $\frac{1}{112}[(209y_{3n+3} - y_{3n+5}) + 3(209y_{n+1} - y_{n+3})]$
- $\frac{1}{8}[(209y_{3n+4} - 15y_{3n+5}) + 3(209y_{n+2} - 15y_{n+3})]$

➢ Each of the following expressions represent bi-quadratic integer

- $\frac{1}{8^2}\left[(64x_{4n+5} - 832x_{4n+4}) + 4(8x_{n+2} - 104x_{n+1})^2 - 128\right]$
- $\frac{1}{14^2}\left[(14x_{4n+6} - 2534x_{4n+4}) + 4(x_{n+3} - 181x_{n+1})^2 - 392\right]$

- $\frac{1}{8^2}\left[\left(64y_{4n+4}-384x_{4n+4}\right)+4\left(8y_{n+1}-48x_{n+1}\right)^2-128\right]$
- $\frac{1}{56^2}\left[\left(448y_{4n+5}-40320x_{4n+4}\right)+4\left(8y_{n+2}-720x_{n+1}\right)^2-6272\right]$
- $\frac{1}{776^2}\left[\left(6208y_{4n+6}-7784832x_{4n+4}\right)+4\left(8y_{n+3}-10032x_{n+1}\right)^2-1552\right]$
- $\frac{1}{8^2}\left[\left(832x_{4n+6}-11584x_{4n+5}\right)+4\left(104x_{n+3}-1448x_{n+2}\right)^2-128\right]$
- $\frac{1}{56^2}\left[\left(5824y_{4n+4}-2688x_{4n+5}\right)+4\left(104y_{n+1}-48x_{n+2}\right)^2-6272\right]$
- $\frac{1}{8^2}\left[\left(832y_{4n+5}-5760x_{4n+5}\right)+4\left(104y_{n+2}-720x_{n+2}\right)^2-128\right]$
- $\frac{1}{56^2}\left[\left(5824y_{4n+6}-561792x_{4n+5}\right)+4\left(104y_{n+3}-10032x_{n+2}\right)^2-6272\right]$
- $\frac{1}{194^2}\left[\left(70228y_{4n+4}-2328x_{4n+6}\right)+4\left(362y_{n+1}-12x_{n+3}\right)^2-75272\right]$
- $\frac{1}{56^2}\left[\left(81088y_{4n+5}-40320x_{4n+6}\right)+4\left(1448y_{n+2}-720x_{n+3}\right)^2-6272\right]$
- $\frac{1}{8^2}\left[\left(11584y_{4n+6}-80256x_{4n+6}\right)+4\left(1448y_{n+3}-10032x_{n+3}\right)^2-128\right]$
- $\frac{1}{8^2}\left[\left(120y_{4n+4}-8y_{4n+5}\right)+4\left(15y_{n+1}-y_{n+2}\right)^2-128\right]$
- $\frac{1}{224^2}\left[\left(93632y_{4n+4}-448y_{4n+6}\right)+4\left(418y_{n+1}-2y_{n+3}\right)^2-100352\right]$
- $\frac{1}{8^2}\left[\left(1672y_{4n+5}-120y_{4n+6}\right)+4\left(209y_{n+2}-15y_{n+3}\right)^2-128\right]$

➢ Each of the following expressions represent Nasty number

- $\frac{1}{8}\left[48x_{2n+3}-624x_{2n+2}+96\right]$
- $\frac{1}{14}\left[6x_{2n+4}-1086x_{2n+2}+168\right]$
- $\frac{1}{8}\left[48y_{2n+2}-288x_{2n+2}+96\right]$
- $\frac{1}{56}\left[48y_{2n+3}-4320x_{2n+2}+672\right]$
- $\frac{1}{8}\left[624y_{2n+3}-4320x_{2n+3}+96\right]$
- $\frac{1}{776}\left[48y_{2n+4}-60192x_{2n+2}+9312\right]$
- $\frac{1}{8}\left[624x_{2n+4}-8688x_{2n+3}+96\right]$
- $\frac{1}{56}\left[624y_{2n+2}-288x_{2n+3}+672\right]$

- $\frac{1}{194}\left[2172y_{2n+2}-72x_{2n+4}+2328\right]$
- $\frac{1}{56}\left[8688y_{2n+3}-4320x_{2n+4}+672\right]$
- $\frac{1}{8}\left[8688y_{2n+4}-60192x_{2n+4}+96\right]$
- $\frac{1}{8}\left[90y_{2n+2}-6y_{2n+3}+96\right]$
- $\frac{1}{112}\left[1254y_{2n+2}-6y_{2n+4}+1344\right]$
- $\frac{1}{8}\left[1254y_{2n+3}-90y_{2n+4}+96\right]$

➢ Each of the following expressions represent quintic integer:

- $\frac{1}{(8)^3}\left[(512x_{5n+6}-6656x_{5n+5})+5(8x_{n+2}-104x_{n+1})^3-5(512x_{n+2}-6656x_{n+1})\right]$
- $\frac{1}{(14)^3}\left[(196x_{5n+7}-35476x_{5n+5})+5(x_{n+3}-181x_{n+1})^3-5(196x_{n+3}-35476x_{n+1})\right]$
- $\frac{1}{(8)^3}\left[(512y_{5n+5}-3072x_{5n+5})+5(8y_{n+1}-48x_{n+1})^3-5(512y_{n+1}-3072x_{n+1})\right]$
- $\frac{1}{(56)^3}\left[\begin{array}{l}(25088y_{5n+6}-2257920x_{5n+5})+5(8y_{n+2}-720x_{n+1})^3\\-5(25088y_{n+2}-2257920x_{n+1})\end{array}\right]$
- $\frac{1}{(776)^3}\left[\begin{array}{l}(4817408y_{5n+7}-6041029632x_{5n+5})+5(8y_{n+3}-10032x_{n+1})^3\\-5(4817408y_{n+3}-6041029632x_{n+1})\end{array}\right]$
- $\frac{1}{(8)^3}\left[\begin{array}{l}(6656x_{5n+7}-92672x_{5n+6})+5(104x_{n+3}-1448x_{n+2})^3\\-5(6656x_{n+3}-92672x_{n+2})\end{array}\right]$
- $\frac{1}{(56)^3}\left[\begin{array}{l}(326144y_{5n+5}-150528x_{5n+6})+5(104y_{n+1}-48x_{n+2})^3\\-5(326144y_{n+1}-150528x_{n+2})\end{array}\right]$
- $\frac{1}{(8)^3}\left[\begin{array}{l}(6656\,y_{5n+6}-46080\,x_{5n+6})+5(104\,y_{n+2}-720\,x_{n+2})^3\\-5(6656\,y_{n+2}-46080\,x_{n+2})\end{array}\right]$
- $\frac{1}{(56)^3}\left[\begin{array}{l}(326144y_{5n+7}-31460352x_{5n+6})+5(104y_{n+3}-10032x_{n+2})^3\\-5(326144y_{n+3}-31460352x_{n+2})\end{array}\right]$
- $\frac{1}{(194)^3}\left[\begin{array}{l}(13624232\,y_{5n+5}-451632\,x_{5n+7})+5(362\,y_{n+1}-12x_{n+3})^3\\-5(13624232\,y_{n+1}-451632\,x_{n+3})\end{array}\right]$
- $\frac{1}{(56)^3}\left[\begin{array}{l}(4540928y_{5n+6}-2257920x_{5n+7})+5(1448y_{n+2}-720x_{n+3})^3\\-5(4540928y_{n+2}-2257920x_{n+3})\end{array}\right]$
- $\frac{1}{(8)^3}\left[\begin{array}{l}(92672y_{5n+7}-642048x_{5n+7})+5(1448y_{n+3}-10032x_{n+3})^3\\-5(92672y_{n+3}-642048x_{n+3})\end{array}\right]$
- $\frac{1}{(8)^3}\left[\begin{array}{l}(960y_{5n+5}-64y_{5n+6})+5(15y_{n+1}-y_{n+2})^3\\-5(960y_{n+1}-64y_{n+2})\end{array}\right]$

- $\frac{1}{(112)^3}\left[\begin{array}{l}(2621696y_{5n+5}-12544y_{5n+7})+5(209y_{n+1}-y_{n+3})^3\\-5(2621696y_{n+1}-12544y_{n+3})\end{array}\right]$
- $\frac{1}{(8)^3}\left[\begin{array}{l}(21736y_{5n+6}-21720y_{5n+7})+5(209y_{n+2}-15y_{n+3})^3\\-5(21736y_{n+2}-21720y_{n+3})\end{array}\right]$
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IV. REMARKABLE OBSERVATIONS

1. Employing the linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in Table: 2 below:

Table 2: Hyperbolas

S.No	Hyperbolas	(X_n, Y_n)
1	$48X_n^2-Y_n^2=12288$	$[(8x_{n+2}-104x_{n+1}),(720x_{n+1}-48x_{n+2})]$
2	$48X_n^2-Y_n^2=37632$	$[(x_{n+3}-181x_{n+1}),(1254x_{n+1}-6x_{n+3})]$
3	$48X_n^2-Y_n^2=12288$	$[(8y_{n+1}-48x_{n+1}),(384x_{n+1}-48y_{n+1})]$
4	$48X_n^2-Y_n^2=602112$	$[(8y_{n+2}-720x_{n+1}),(4992x_{n+1}-48y_{n+2})]$
5	$48X_n^2-Y_n^2=115617792$	$[(8y_{n+3}-10032x_{n+1}),(69504x_{n+1}-48y_{n+3})]$
6	$48X_n^2-Y_n^2=12288$	$[(104x_{n+3}-1448x_{n+2}),(10032x_{n+2}-720x_{n+3})]$
7	$48X_n^2-Y_n^2=602112$	$[(104y_{n+1}-48x_{n+2}),(384x_{n+2}-720y_{n+1})]$
8	$48X_n^2-Y_n^2=12288$	$[(104y_{n+2}-720x_{n+2}),(4992x_{n+2}-720y_{n+2})]$
9	$48X_n^2-Y_n^2=602112$	$[(104y_{n+3}-10032x_{n+2}),(69504x_{n+2}-720y_{n+3})]$
10	$48X_n^2-Y_n^2=7226112$	$[(362y_{n+1}-12x_{n+3}),(96x_{n+3}-2508y_{n+1})]$
11	$48X_n^2-Y_n^2=602112$	$[(1448y_{n+2}-720x_{n+3}),(4992x_{n+3}-103280y_{n+2})]$

12	$48X_n^2 - Y_n^2 = 12288$	$[(1448y_{n+3} - 10032x_{n+3}), (69504x_{n+3} - 10032y_{n+3})]$
13	$48X_n^2 - Y_n^2 = 12288$	$[(15y_{n+1} - y_{n+2}), (8y_{n+2} - 104y_{n+1})]$
14	$48X_n^2 - Y_n^2 = 2408448$	$[(209y_{n+1} - y_{n+3}), (384y_{n+3} - 69504y_{n+1})]$
15	$48X_n^2 - Y_n^2 = 12288$	$[(209y_{n+2} - 15y_{n+3}), (1448y_{n+2} - 104y_{n+3})]$

2. Employing the linear combination among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in Table: 3 below:

Table 3: Parabolas

S.No	Parabolas	(X_n, Y_n)
1	$384X_n - Y_n^2 = 12288$	$[(16 + 8x_{2n+3} - 104x_{2n+2}), (720x_{n+1} - 48x_{n+2})]$
2	$672X_n - Y_n^2 = 37632$	$[(28 + x_{2n+4} - 181x_{2n+2}), (1254x_{n+1} - 6x_{n+3})]$
3	$384X_n - Y_n^2 = 12288$	$[(16 + 8y_{2n+2} - 48x_{2n+2}), (384x_{n+1} - 48y_{n+1})]$
4	$2688X_n - Y_n^2 = 602112$	$[(112 + 8y_{2n+3} - 720x_{2n+2}), (4992x_{n+1} - 48y_{n+2})]$
5	$37248X_n - Y_n^2 = 115617792$	$[(1552 + 8y_{2n+4} - 10032x_{2n+2}), (69504x_{n+1} - 48y_{n+3})]$
6	$384X_n - Y_n^2 = 12288$	$[(16 + 104x_{2n+4} - 1448x_{2n+3}), (10032x_{n+2} - 720x_{n+3})]$
7	$2240X_n - Y_n^2 = 602112$	$[(112 + 104y_{2n+2} - 48x_{2n+3}), (384x_{n+2} - 720y_{n+1})]$
8	$384X_n - Y_n^2 = 12288$	$[(16 + 104y_{2n+3} - 720x_{2n+3}), (4992x_{n+2} - 720y_{n+2})]$
9	$2688X_n - Y_n^2 = 602112$	$[(112 + 104y_{2n+4} - 10032x_{2n+3}), (69504x_{n+2} - 720y_{n+3})]$
10	$9312X_n - Y_n^2 = 7226112$	$[(388 + 362y_{2n+2} - 12x_{2n+4}), (96x_{n+3} - 2508y_{n+1})]$
11	$2688X_n - Y_n^2 = 602112$	$[(112 + 1448y_{2n+3} - 720x_{2n+4}), (4992x_{n+3} - 10032y_{n+2})]$

12	$384X_n - Y_n^2 = 12288$	$[(16+1448y_{2n+4} - 10032x_{2n+4}),(69504x_{n+3} - 10032y_{n+3})]$
13	$384X_n - Y_n^2 = 12288$	$[(16 - y_{2n+3} + 15y_{2n+2}),(8y_{n+2} - 104y_{n+1})]$
14	$5376X_n - Y_n^2 = 2408448$	$[(224+209y_{2n+2} - y_{2n+4}),(384y_{n+3} - 69504y_{n+1})]$
15	$384X_n - Y_n^2 = 12288$	$[(16+209y_{2n+3} - 15y_{2n+4}),(1448y_{n+2} - 104y_{n+3})]$

V. GENERATORS OF THE PYTHAGOREAN TRIANGLE

Let p, q be the non-zero distinct integers such that $p = x_{n+1} + y_{n+1}$, $q = x_{n+1}$
Note that p > q > 0.Treat p, q as the generators of the Pythagorean triangle $T(X,Y,Z)$

$$X = 2pq \; , \; Y = p^2 - q^2 \; , \; Z = p^2 + q^2, \; p > q > 0$$

Let A, P represent the area and perimeter of T.

Then the following interesting relations are observed.

- $X - 24Y + 23Z = -16$
- $13X - 12Y + 11Z - \frac{48A}{P} = -16$
- $25X - Z - \frac{96A}{P} = -16$
- $\frac{2A}{P} = x_{n+1}y_{n+1}$

VI. RELATIONS BETWEEN SOLUTIONS AND SPECIAL POLYGONAL NUMBERS

- $9(P_y^3 * t_{3,x})^2 = 48(P_x^5 * t_{3,y+1})^2 + 16(t_{3,x} * t_{3,y+1})^2$
- $(P_y^5 * t_{3,x+1})^2 = 432(P_x^3 * t_{3,y})^2 + 16(t_{3,y} * t_{3,x+1})^2$
- Let $\{s_{n+1}\}$ and $\{u_{n+1}\}$ be sequences of positive integers defined by

$$s_{n+1} = \frac{1}{2}(x_{n+1} - 1), \; u_{n+1} = \frac{1}{2}(y_{n+1}), \; n = 0,1,2......$$

Observation:

- $t_{4,u_{n+1}} - 96t_{3,s_{n+1}} = 16$

- Let $\{s_{n+1}\}$ and $\{u_{n+1}\}$ be sequences of positive integers defined by

$$s_{n+1} = \frac{1}{2}(x_{n+1} - 1), \; u_{n+1} = \frac{1}{4}(y_{n+1}), \; n = 0,1,2......$$

Observation:

- $t_{4,u_{n+1}} - 24t_{3,s_{n+1}} = 4$

VII.CONCLUSION

In this paper, we have presented infinitely many integer solutions for the Diophantine equation, represented by hyperbola is given by $y^2 = 48x^2 + 16$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of equations and determine their integer solutions along with suitable properties

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