Some Finite Series for Generalized Multivariable Gimel-Function

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ABSTRACT
In this paper some finite series of generalized multivariable Gimel-function have been established. Since the finite series of special functions can be employed to obtain identities, recurrence relations and transformations of the special functions, therefore such series occupy a prominent place in the literature of special functions.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, finite series.

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1.Introduction and preliminaries.

Throughout this paper, let $\mathbb{C},\mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

We define a generalized transcendental function of several complex variables.

$$\gimel(z_1,\cdots,z_r)=\gimel^{m_2,n_2;m_3,n_3;\cdots;m_r,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|}\mathrm{z}_1\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\end{array}\right.$$

$$[(\mathrm{a}_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[(\mathrm{b}_{2j};\beta^{(1)}_{2j},\beta^{(2)}_{2j};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta^{(1)}_{3j},\beta^{(2)}_{3j},\beta^{(3)}_{3j};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj};B_{rj})_{1,m_r}],$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]:\ [(\mathrm{c}^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{m_r+1,q_r}]:[(\mathrm{d}^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left.\begin{array}{l};\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}]\\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{m_2}\Gamma^{B_{2j}}(b_{2j}-\sum_{k=1}^{2}\beta^{(k)}_{2j}s_k)\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha^{(k)}_{2ji_2}s_k)\prod_{j=m_2+1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{m_3}\Gamma^{B_{3j}}(b_{3j}-\sum_{k=1}^{3}\beta_{3j}^{(k)}s_k)\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=m_3+1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{m_r}\Gamma^{B_{rj}}(b_{rj}-\sum_{k=1}^{r}\beta_{rj}^{(k)}s_k)\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=m_r+1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)})]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $m_2,n_2,\cdots,m_r,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2\leqslant q_{i_2}, 0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant m_r\leqslant q_{i_r}, 0\leqslant n_r\leqslant p_{i_r}, 0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kj}^{(l)},B_{kj}\in\mathbb{R}^+;(j=1,\cdots,m_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n_k);(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m_k);(k=1,\cdots,r)$.

$a_{kji_k} \in \mathbb{C}; (j = n_k + 1, \cdots, p_{i_k}); (k = 2, \cdots, r).$

$b_{kji_k} \in \mathbb{C}; (j = m_k + 1, \cdots, q_{i_k}); (k = 2, \cdots, r).$

$d^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$\gamma^{(k)}_{ji^{(k)}} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour L_k is in the $s_k (k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha^{(k)}_{2j} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha^{(k)}_{3j} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha^{(i)}_{rj}\right) (j = 1, \cdots, n_r), \Gamma^{C^{(k)}_j}\left(1 - c^{(k)}_j + \gamma^{(k)}_j s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$to the right of the contour L_k and the poles of $\Gamma^{B_{2j}}\left(b_{2j} - \sum_{k=1}^{2} \beta^{(k)}_{2j} s_k\right) (j = 1, \cdots, m_2), \Gamma^{B_{3j}}\left(b_{3j} - \sum_{k=1}^{3} \beta^{(k)}_{3j} s_k\right) (j = 1, \cdots, m_3)$ $, \cdots, \Gamma^{B_{rj}}\left(b_{rj} - \sum_{i=1}^{r} \beta^{(i)}_{rj}\right) (j = 1, \cdots, m_r), \Gamma^{D^{(k)}_j}\left(d^{(k)}_j - \delta^{(k)}_j s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A^{(k)}_i \pi$ where

$$A^{(k)}_i = \sum_{j=1}^{m^{(k)}} D^{(k)}_j \delta^{(k)}_j + \sum_{j=1}^{n^{(k)}} C^{(k)}_j \gamma^{(k)}_j - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q^{(k)}_i} D^{(k)}_{ji^{(k)}} \delta^{(k)}_{ji^{(k)}} + \sum_{j=n^{(k)}+1}^{p^{(k)}_i} C^{(k)}_{ji^{(k)}} \gamma^{(k)}_{ji^{(k)}} \right) +$$

$$\sum_{j=1}^{n_2} A_{2j} \alpha^{(k)}_{2j} + \sum_{j=1}^{m_2} B_{2j} \beta^{(k)}_{2j} - \tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha^{(k)}_{2ji_2} + \sum_{j=m_2+1}^{q_{i_2}} B_{2ji_2} \beta^{(k)}_{2ji_2} \right) + \cdots +$$

$$\sum_{j=1}^{n_r} A_{rj} \alpha^{(k)}_{rj} + \sum_{j=1}^{m_r} B_{rj} \beta^{(k)}_{rj} - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha^{(k)}_{rji_r} + \sum_{j=m_r+1}^{q_{i_r}} B_{rji_r} \beta^{(k)}_{rji_r} \right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$ where $i = 1, \cdots, r$:

$\alpha_i = \min_{\substack{1 \leqslant k \leqslant m_i \\ 1 \leqslant j \leqslant m^{(i)}}} Re \left(\sum_{h=2}^{r} \sum_{h'=1}^{h} B_{hj} \frac{b_{hj}}{\beta^{h'}_{hj}} + D^{(i)}_k \frac{d^{(i)}_k}{\delta^{(i)}_k} \right)$ and $\beta_i = \max_{\substack{1 \leqslant k \leqslant n_i \\ 1 \leqslant j \leqslant n^{(i)}}} Re \left(\sum_{h=2}^{r} \sum_{h'=1}^{h} A_{hj} \frac{a_{hj} - 1}{\alpha^{h'}_{hj}} + C^{(i)}_k \frac{c^{(i)}_k - 1}{\gamma^{(i)}_k} \right)$

Remark 1.
If $m_2 = n_2 = \cdots = m_{r-1} = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the generalized multivariable Gimel-function reduces in the generalized multivariable Aleph- function (extension of multivariable Aleph-function defined by Ayant [1]).

Remark 2.
If $m_2 = n_2 = \cdots = m_r = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [5]).

Remark 3.

If$A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [4].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and Panda [7,8].
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In your investigation, we shall use the following notations.

$$\mathbb{A} = [(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A = [(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B} = [(b_{2j};\beta_{2j}^{(1)},\beta_{2j}^{(2)};B_{2j})]_{1,m_2},[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{m_2+1,q_{i_2}},[(b_{3j};\beta_{3j}^{(1)},\beta_{3j}^{(2)},\beta_{3j}^{(3)};B_{3j})]_{1,m_3},$$

$$[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{m_3+1,q_{i_3}};\cdots;[(\mathrm{b}_{(r-1)j};\beta_{(r-1)j}^{(1)},\cdots,\beta_{(r-1)j}^{(r-1)};B_{(r-1)j})_{1,m_{r-1}}],$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{m_{r-1}+1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B} = [(\mathrm{b}_{rj};\beta_{rj}^{(1)},\cdots,\beta_{rj}^{(r)};B_{rj})_{1,m_r}],[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{m_r+1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B} = [(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U = m_2,n_2;m_3,n_3;\cdots;m_{r-1},n_{r-1};V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

2. Series formulae.

In this section, we give six summation formulae.

Theorem 1.

$$\sum_{s=1}^{u}\frac{(-u)_s\left(\beta-\alpha-\frac{1}{2}-u\right)_s}{s!\left(\frac{3}{2}-\beta+\alpha\right)_s}\gimel_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;m_r+1,n_r+1:V}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};\ (1\text{-}\alpha+s-u;a_1,\cdots,a_r;1),\mathbf{A},(1-\alpha-s;a_1,\cdots,a_r;1):A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ (1\text{-}\beta-s+u;a_1,\cdots,a_r;1),\mathbf{B},(1-\beta+s;a_1,\cdots,a_r;1):B \end{array}\right)=\frac{4^u(1+\alpha-\beta)_u}{(2+2\alpha-2\beta)_u}$$

$$\gimel^{U;m_r+2,n_r+2:V}_{X;p_{i_r}+4,q_{i_r}+4,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbb{A};\ \left(1-\alpha-\frac{u}{2};a_1,\cdots,a_r;1\right),\left(\frac{1}{2}-\alpha-\frac{u}{2};a_1,\cdots,a_r:1\right),\mathbf{A}, \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ \left(1-\beta+\frac{u}{2};a_1,\cdots,a_r:1\right),\left(\frac{3}{2}-\beta+\frac{u}{2};a_1,\cdots,a_r;1\right),\mathbf{B}, \end{array}\right.$$

$$\left.\begin{array}{c} \left(1-\alpha;a_1,\cdots,a_r;1\right),\left(\frac{3}{2}-\beta;a_1,\cdots,a_r;1\right):A \\ \cdot \\ \cdot \\ \left(1-\beta;a_1,\cdots,a_r;1\right),\left(\frac{1}{2}-\alpha;a_1,\cdots,a_r;1\right):B \end{array}\right) \tag{2.1}$$

provided

$a_i>0(i=1,\cdots,r),Re(\alpha)>0,Re(\beta)>0.\ \ |arg\,(z_i)|\ <\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4).

Proof

To prove the theorem 1, expressing the generalized multivariable Gimel-function with the help of (1.1) , we obtain (say I)

$$I=\sum_{s=0}^{u}\frac{(-u)_s\left(\beta-\alpha-\frac{1}{2}-u\right)_s}{s!(\frac{3}{2}-\beta+\alpha)_s}\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}$$

$$\frac{\Gamma(\alpha-s+u+\sum_{i=1}^r h_is_i)\Gamma(1-\beta-s+u-\sum_{i=1}^r h_is_i)}{\Gamma(1-\alpha-s-\sum_{i=1}^r h_is_i)\Gamma(\beta-s+\sum_{i=1}^r h_is_i)}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.2}$$

interchanging the order of summation and integrations and using [6], the serie becomes,

$$\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\frac{\Gamma(\alpha+u+\sum_{i=1}^r h_is_i)\Gamma(1-\beta+u-\sum_{i=1}^r h_is_i)}{\Gamma(1-\alpha-\sum_{i=1}^r h_is_i)\Gamma(\beta+\sum_{i=1}^r h_is_i)}$$

$${}_4F_3\left[\begin{array}{c|c} \text{-u},\alpha+\sum_{i=1}^r a_is_i,1-\beta-\sum_{i=1}^r a_is_i,\beta-\alpha-\frac{1}{2}-u & \\ \cdot & 1 \\ 1\text{-}\alpha-u-\sum_{i=1}^r a_is_i,\beta-u+\sum_{i=1}^r a_is_i,\frac{3}{2}-\beta+\alpha & \end{array}\right]\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.3}$$

Now using the relation [3]

$${}_4F_3\left[\begin{array}{c|c} \text{-m,a,b,}\tfrac{1}{2}\text{-a-b-m} & \\ \cdot & 1 \\ \text{1-a-m,1-b-m,}\tfrac{1}{2}\text{+a+b} & \end{array}\right]=\frac{(2a)_m(2b)_m(a+b)_m}{(a)_m(b)_m(2a+2b)_m} \tag{2.4}$$

Substituting (2.4) in (2.3) and interpreting the resulting Mellin-Barnes multiple integrals contour as the generalized multivariable Gimel-function, we obtain the desired theorem 1.

Theorem 2.

$$\sum_{s=1}^{u}\frac{(-u)_s\left(\beta-\alpha-\frac{1}{2}-u\right)_s}{s!\left(\frac{1}{2}-\beta+\alpha\right)_s}\gimel^{U;m_r+1,n_r+1:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};\ (1\text{-}\alpha-s;a_1,\cdots,a_r;1),\mathbf{A},(1-\alpha+s-u;a_1,\cdots,a_r;1):A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ (1\text{-}\beta-s+u;a_1,\cdots,a_r;1),\mathbf{B},(1-\beta+s;a_1,\cdots,a_r;1):B \end{array}\right)=\frac{4^u(1+\alpha-\beta)_u}{(1+2\alpha-2\beta)_u}$$

$$\daleth^{U;m_r+2,n_r+3:V}_{X;p_{i_r}+5,q_{i_r}+5,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbb{A};(1\text{-}\alpha;a_1,\cdots,a_r;1),\left(1-\alpha-\frac{u}{2};a_1,\cdots,a_r;1\right),\left(\frac{1}{2}-\alpha-\frac{u}{2};a_1,\cdots,a_r;1\right),\mathbf{A}, \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ \left(1-\beta+\frac{u}{2};a_1,\cdots,a_r;1\right),\left(\frac{3}{2}-\beta+\frac{u}{2};a_1,\cdots,a_r;1\right),\mathbf{B}, \end{array}\right.$$

$$\left.\begin{array}{c} (1-\alpha-u;a_1,\cdots,a_r;1),\left(\frac{3}{2}-\beta;a_1,\cdots,a_r;1\right):A \\ \cdot \\ \cdot \\ (1\text{-}\beta;a_1,\cdots,a_r;1),\left(\frac{1}{2}-\alpha;a_1,\cdots,a_r;1\right),(1-\alpha-u;a_1,\cdots,a_r;1):B \end{array}\right) \tag{2.5}$$

provided

$a_i>0(i=1,\cdots,r),Re(\alpha)>0,Re(\beta)>0.\quad |arg\,(z_i)|\ <\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4).

Theorem 3.

$$\sum_{s=1}^{u}\frac{(-u)_s\left(\beta-\alpha+\frac{1}{2}-u\right)_s}{s!\left(\frac{1}{2}-\beta+\alpha\right)_s}\daleth^{U;m_r+1,n_r+1:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};\ (2\text{-}\alpha+s-u;a_1,\cdots,a_r;1),\mathbf{A},(1-\alpha-s;a_1,\cdots,a_r;1):A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ (1\text{-}\beta+s;a_1,\cdots,a_r;1),\mathbf{B},(1-\beta-s-u;a_1,\cdots,a_r;1):B \end{array}\right)=\frac{4^u(\alpha-\beta)_u}{(2\alpha-2\beta)_u}$$

$$\daleth^{U;m_r+3,n_r+3:V}_{X;p_{i_r}+6,q_{i_r}+6,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbb{A};(2\text{-}\alpha;a_1,\cdots,a_r;1),\left(\frac{3}{2}-\alpha-\frac{u}{2};a_1,\cdots,a_r;1\right),\left(1-\alpha-\frac{u}{2};a_1,\cdots,a_r;1\right),\mathbf{A}, \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ (1\text{-}\beta;a_1,\cdots,a_r;1),\left(1-\beta+\frac{u}{2};a_1,\cdots,a_r;1\right),\left(\frac{3}{2}-\beta+\frac{u}{2};a_1,\cdots,a_r;1\right),\mathbf{B}, \end{array}\right.$$

$$\left.\begin{array}{c} (1-\alpha;a_1,\cdots,a_r;1),\left(\frac{3}{2}-\beta;a_1,\cdots,a_r;1\right),(1-\beta+u;a_1,\cdots,a_r;1):A \\ \cdot \\ \cdot \\ (1\text{-}\alpha;a_1,\cdots,a_r;1),\left(\frac{3}{2}-\alpha;a_1,\cdots,a_r;1\right),(1-\beta+u;a_1,\cdots,a_r;1):B \end{array}\right) \tag{2.6}$$

provided

$a_i>0(i=1,\cdots,r),Re(\alpha)>0,Re(\beta)>0.\quad |arg\,(z_i)|\ <\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4).

Theorem 4.

$$\sum_{s=1}^{u}\frac{(-u)_s\left(\beta-\alpha+\frac{1}{2}-u\right)_s}{s!\left(\frac{3}{2}-\beta+\alpha\right)_s}\daleth^{U;m_r+2,n_r:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};\mathbf{A},\ (1\text{-}\alpha+s-u;a_1,\cdots,a_r;1),(1-\alpha-s;a_1,\cdots,a_r;1):A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ (1\text{-}\beta+s;a_1,\cdots,a_r;1),(1-\beta-s+u;a_1,\cdots,a_r;1),\mathbf{B}:B \end{array}\right)=\frac{4^u(1+\alpha-\beta)_u\left(\frac{1}{2}+\alpha-\beta\right)_u}{\left(\frac{3}{2}+\alpha-\beta\right)_u(1+2\alpha-2\beta)_u}$$

$$\daleth^{U;m_r+3,n_r+2:V}_{X;p_{i_r}+5,q_{i_r}+5,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbb{A};\left(\frac{1}{2}-\alpha-\frac{u}{2};a_1,\cdots,a_r;1\right),\left(1-\alpha-\frac{u}{2};a_1,\cdots,a_r;1\right),\mathbf{A}, \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ (1\text{-}\beta;a_1,\cdots,a_r;1),\left(1-\beta+\frac{u}{2};a_1,\cdots,a_r;1\right),\left(\frac{3}{2}-\beta+\frac{u}{2};a_1,\cdots,a_r;1\right),\mathbf{B}, \end{array}\right.$$

$$\left.\begin{array}{c} (1-\alpha;a_1,\cdots,a_r;1),\left(\frac{3}{2}-\beta;a_1,\cdots,a_r;1\right),(1-\alpha-u;a_1,\cdots,a_r;1):A \\ \cdot \\ \cdot \\ \left(\frac{1}{2}-\alpha;a_1,\cdots,a_r;1\right),(1-\alpha-u;a_1,\cdots,a_r;1):B \end{array}\right) \tag{2.7}$$

provided

$a_i>0(i=1,\cdots,r),Re(\alpha)>0,Re(\beta)>0.\quad |arg\,(z_i)|\ <\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4).

Theorem 5.

$$\sum_{s=1}^{u}\frac{(-u)_s\left(\beta-\alpha-\frac{1}{2}-u\right)_s}{s!\left(\frac{3}{2}-\beta+\alpha\right)_s}\daleth^{U;m_r,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};\ (1\text{-}\alpha+s-u;a_1,\cdots,a_r;1),(1-\alpha-s;a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\mathbf{B},\ (1\text{-}\beta+s;a_1,\cdots,a_r;1),(2-\beta-s+u;a_1,\cdots,a_r;1):B \end{array}\right)=\frac{4^u(1+\alpha-\beta)_u}{(2+2\alpha-2\beta)_u}$$

$$\daleth^{U;m_r+2,n_r+3:V}_{X;p_{i_r}+5,q_{i_r}+5,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbb{A};(1\text{-}\alpha;a_1,\cdots,a_r),\left(\frac{1}{2}-\alpha-\frac{u}{2};a_1,\cdots,a_r;1\right),\left(1-\alpha-\frac{u}{2};a_1,\cdots,a_r;1\right),\mathbf{A}, \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\ \left(\frac{3}{2}-\beta+\frac{u}{2};a_1,\cdots,a_r;1\right),\left(2-\beta+\frac{u}{2};a_1,\cdots,a_r;1\right),\mathbf{B}, \end{array}\right.$$

$$\left.\begin{array}{c} (2-\beta+u;a_1,\cdots,a_r;1),\left(\frac{3}{2}-\beta;a_1,\cdots,a_r;1\right):A \\ \cdot \\ \cdot \\ (1-\beta;a_1,\cdots,a_r;1),(2-\beta+u;a_1,\cdots,a_r;1),\left(\frac{1}{2}-\alpha;a_1,\cdots,a_r;1\right):B \end{array}\right) \tag{2.8}$$

provided

$a_i>0(i=1,\cdots,r),Re(\alpha)>0,Re(\beta)>0.\quad |arg\,(z_i)|\ <\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4).

Theorem 5.

$$\sum_{s=1}^{u}\frac{(-u)_s}{s!}\daleth^{U;m_r,n_r+3:V}_{X;p_{i_r}+3,q_{i_r}+3,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} z_1 & \mathbb{A};\ \left(-\frac{\alpha}{2}-s;\frac{a_1}{2},\cdots,\frac{a_r}{2};1\right),\left(\frac{1-\alpha}{2}-s;\frac{a_1}{2},\cdots,\frac{a_r}{2};1\right),(1-\beta-u-s;a_1,\cdots,a_r;1),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & \mathbb{B};\mathbf{B},\ \left(-\frac{\beta}{2}-s;\frac{a_1}{2},\cdots,\frac{a_r}{2};1\right),\left(\frac{1-\beta}{2}-s;\frac{a_1}{2},\cdots,\frac{a_r}{2};1\right),(-\alpha;h_1,\cdots,h_r;1),(-\beta-2u;h_1,\cdots,a_r;1):B \end{array}\right)$$

$$=(\beta-\alpha)_u\,\daleth^{U;m_r,n_r+4:V}_{X;p_{i_r}+4,q_{i_r}+4,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\ \left(-\frac{\alpha}{2};\frac{a_1}{2},\cdots,\frac{a_r}{2};1\right),\left(\frac{1-\alpha}{2};\frac{a_1}{2},\cdots,\frac{a_r}{2};1\right),(-\beta;a_1,\cdots,a_r;1),(1-\beta-2,h_1,\cdots,h_r),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\mathbf{B},\ \left(-\frac{\beta}{2};\frac{a_1}{2},\cdots,\frac{a_r}{2};1\right),\left(\frac{1-\beta}{2};\frac{a_1}{2},\cdots,\frac{a_r}{2};1\right),(-\alpha;h_1,\cdots,h_r;1),(-\beta-2u;h_1,\cdots,a_r;1):B \end{array} \right) \quad (2.9)$$

provided

$a_i > 0 (i = 1,\cdots,r), Re(\alpha) > 0, Re(\beta) > 0.$ $|arg\,(z_i)| < \dfrac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4).

To prove the theorems 2 to 6, we use the similar method that theorem 1.

Remark 6.
If $m_2 = n_2 = \cdots = m_{r-1} = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$, then we can obtain the same finite series relations in the generalized multivariable Aleph- function (extension of multivariable Aleph-function defined by Ayant [1])

Remark 7.
If $m_2 = n_2 = \cdots = m_r = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then we can obtain the same finite series relations in a generalized multivariable I-function (extension of multivariable I-function defined by Prathima et al. [5]).

Remark 8.
If$A_{2j} = B_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = B_{rj} = A_{rji_r} = B_{rji_r} = 1$and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then we can obtain the same finite series relations in generalized of multivariable I-function (extension of multivariable I-function defined by Prasad [4]).

Remark 9.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the generalized multivariable H-function (extension of multivariable H-function defined by Srivastava and Panda [7,8] and then we can obtain the same finite series relations .

3. Conclusion.

The importance of our finite series formulae lies in their manifold generality. By specializing the various parameters and variables involved in the generalized multivariable Gimel-function, we get a several fractional integral formulae involving in remarkably wide variety of useful function (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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