A Finite Double Integral Involving a General Multivariable Polynomial and Multivariable Gimel-Function

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Abstract
In this paper we evaluate a general finite double integral involving the product of algebric and exponenetial functions, a general multivariable polynomial and the multivariable Gimel-function. Some new and interesting special cases og our main integral formula have been considered briefly.

Keywords :Multivariable Gimel-function, multiple integrals contour, double integrals, multivariable polynomial.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We define a generalized transcendental function of several complex variables.

$$\gimel(z_1, \cdots, z_r) = \gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2}:R_2;p_{i_3},q_{i_3},\tau_{i_3}:R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array} \right|$$

$$[(a_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(a_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots; \quad \cdots$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]: \ [(c^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}],[\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{1,q_r}]: \ [(d^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}],[\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left. \begin{array}{l} ;\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}] \\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}] \end{array} \right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1,\cdots,s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \, \mathrm{d}s_1 \cdots \mathrm{d}s_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha_{2ji_2}^{(k)}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta_{2ji2}^{(k)}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta_{3ji3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,\cdots,0\leqslant m_r,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)} \in \mathbb{C}; (j = 1, \cdots, n^{(k)}); (k = 1, \cdots, r); d_j^{(k)} \in \mathbb{C}; (j = 1, \cdots, m^{(k)}); (k = 1, \cdots, r)$.

$a_{kji_k} \in \mathbb{C}; (j = n_k + 1, \cdots, p_{i_k}); (k = 2, \cdots, r)$.

$b_{kji_k} \in \mathbb{C}; (j = 1, \cdots, q_{i_k}); (k = 2, \cdots, r)$.

$d_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r)$.

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r)$.

The contour L_k is in the $s_k(k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha_{2j}^{(k)} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha_{3j}^{(k)} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha_{rj}^{(i)}\right) (j = 1, \cdots, n_r), \quad \Gamma^{C_j^{(k)}}\left(1 - c_j^{(k)} + \gamma_j^{(k)} s_k\right) (j = 1, \cdots, n^{(k)}) (k = 1, \cdots, r)$ to the right of the contour L_k and the poles of $\Gamma^{D_j^{(k)}}\left(d_j^{(k)} - \delta_j^{(k)} s_k\right) (j = 1, \cdots, m^{(k)}) (k = 1, \cdots, r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A_i^{(k)} \pi$ where

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)} \gamma_j^{(k)} - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)} \delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)} \gamma_{ji^{(k)}}^{(k)} \right)$$

$$-\tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha_{2ji_2}^{(k)} + \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta_{2ji_2}^{(k)} \right) - \cdots - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha_{rji_r}^{(k)} + \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta_{rji_r}^{(k)} \right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(\,|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}\,), max(\,|z_1|, \cdots, |z_r|\,) \to 0$

$\aleph(z_1, \cdots, z_r) = 0(\,|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}\,), min(\,|z_1|, \cdots, |z_r|\,) \to \infty$ where $i = 1, \cdots, r$:

$\alpha_i = \min_{1 \leqslant j \leqslant m^{(i)}} Re\left[D_j^{(i)} \left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]$ and $\beta_i = \max_{1 \leqslant j \leqslant n^{(i)}} Re\left[C_j^{(i)} \left(\frac{c_j^{(i)} - 1}{\gamma_j^{(i)}}\right)\right]$

Remark 1.
If $n_2 = \cdots = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ $A_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [1].

Remark 2.
If $n_2 = \cdots = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [5].

Remark 3.
If $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [4].

Remark 4.

If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and Panda [9,10].

In your investigation, we shall use the following notations.

$$\mathbb{A} = [(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A = [(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B} = [\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B} = [\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{1,q_{i_r}}] \tag{1.9}$$

$$B = [(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U = 0,n_2;0,n_3;\cdots;0,n_{r-1};V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

The generalized polynomials defined by Srivastava ([8],p. 251, Eq. (C.1)), is given in the following manner :

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s] = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}A[N_1,K_1;\cdots;N_s,K_s]y_1^{K_1}\cdots y_s^{K_s} \tag{1.13}$$

Where $M_1,\cdots,M_s$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_s,K_s]$ are arbitrary constants, real or complex. I we take $s=1$ in the (1.13) and denote $A[N,K]$ thus obtained by $A_{N,K}$, we arrive at general class of polynomials $S_N^M(x)$ study by Srivastava ([7],p. 1, Eq. 1).

We shall note

$$B_1 = \frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}A[N_1,K_1;\cdots;N_s,K_s] \tag{1.14}$$

2. Required results.

In this section, we give two finite integrals. These results will utilized in the following section.

Lemma 1. ([3], MacRobert p. 450)

$$\int_0^{\frac{\pi}{2}} e^{\omega(a+b)\theta}(\sin\theta)^{a-1}(\cos\theta)^{b-1}\mathrm{d}\theta = e^{\frac{1}{2}\omega\pi a}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{2.1}$$

provided $\min\{Re(a), Re(b)\} > 0$

Lemma 2. (Soni and Rathie, [6], p. 255).

$$\int_0^1 x^{c-1}(1-x)^{d-1}[ax+b(1-x)]^{-c-d}e^{-\frac{zax}{ax+b(1-x)}}{}_2F_1\left[\alpha,\beta;c;\frac{ax}{ax+b(1-x)}\right]\mathrm{d}x =$$

$$e^{-z}\frac{\Gamma(c)\Gamma(d)\Gamma(c+d-\alpha-\beta)}{a^c b^d\Gamma(c+d-\alpha)\Gamma(c+d-\beta)}{}_2F_2\left[d,c+d-\alpha-\beta;c+d-\alpha,c+d-\beta;z\right] \tag{2.1}$$

provided $Re(c), Re(d), Re(c+d-\alpha-\beta) > 0$, a, b are different at zero, $ax+b(1-x)\neq 0$. $0\leqslant x\leqslant 1$

3. Main integral.

Theorem.

$$\int_0^1\int_0^1 x^{c-1}y^{\rho-1}(1-x)^{d-1}(1-y)^{\sigma-1}\left\{\sqrt{1-y^2}+\omega y\right\}^{\rho+2\sigma}[ax+b(1-x)]^{-c-d}exp\left\{-\frac{zax}{ax+b(1-x)}\right\}$$

$${}_2F_1\left(\alpha,\beta;c;\frac{ax}{ax+b(1-x)}\right)S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[t_1y^{u_1}(1-y^2)^{v_1}\left\{\sqrt{1-y^2}+\omega y\right\}^{u_1+2v_1}\left\{\frac{1-x}{ax+b(1-x)}\right\}^{\delta_1},\cdots,\right.$$

$$\left.t_sy^{u_s}(1-y^2)^{v_s}\left\{\sqrt{1-y^2}+\omega y\right\}^{u_s+2v_s}\left\{\frac{1-x}{ax+b(1-x)}\right\}^{\delta_s}\right]\daleth\left(z_1y^{\rho_1}(1-y^2)^{\sigma_1}\left\{\sqrt{1-y^2}+wu\right\}^{\rho_1+2\sigma_1}\right.$$

$$\left.\left\{-\frac{1-x}{ax+b(1-x)}\right\}^{\theta_1},\cdots,z_ry^{\rho_r}(1-y^2)^{\sigma_r}\left\{\sqrt{1-y^2}+wu\right\}^{\rho_r+2\sigma_r}\left\{-\frac{1-x}{ax+b(1-x)}\right\}^{\theta_r}\right)\mathrm{d}x\mathrm{d}y =$$

$$\Gamma(c)\exp\left(-z+\frac{\pi\omega\rho}{2}\right)a^{-c}b^{-d}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}B_1\prod_{j=1}^{s}\left[t_j^{k_j}b^{-\delta_jK_j}exp\frac{(\omega u_jK_j\frac{\pi}{2})}{K_j!}\right]$$

$$\daleth_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;0,n_r+4:V}\left(\begin{array}{c} z_1b^{-\theta_1}exp\left(\frac{\omega\rho_1}{2}\right)\\ \cdot\\ \cdot\\ z_rb^{-\theta_r}exp\left(\frac{\omega\rho_1}{2}\right)\end{array}\middle|\begin{array}{c}\mathbb{A};(1-p-d-\sum_{j=1}^{s}K_j\delta_j;\theta_1,\cdots,\theta_r;1),\\ \cdot\\ \cdot\\ \mathbb{B};\mathbf{B},(1-c-d-p+\alpha-\sum_{j=1}^{s}K_j\delta_j;\theta_1,\cdots,\theta_r;1)\end{array}\right.$$

$$\left.\begin{array}{c}\mathbb{A};(1-p-d-c+\alpha+\beta-\sum_{j=1}^{s}K_j\delta_j;\theta_1,\cdots,\theta_r;1),(1-\rho-\sum_{j=1}^{s}u_iK_j;\rho_1,\cdots,\rho_r;1),(1-2\sigma-2\sum_{j=1}^{s}v_iK_j;2\sigma_1,\cdots,2\sigma_r;1),\mathbf{A}:A\\ \cdot\\ \cdot\\ \text{(1-c-d-p+}\beta-\sum_{j=1}^{s}K_j\delta_j;\theta_1,\cdots,\theta_r;1),(1-\rho-2\sigma-\sum_{j=1}^{s}(u_j+2v_j)K_j;\rho_1+2\sigma_1,\cdots,\rho_r+2\sigma_r;1):B\end{array}\right) \tag{3.1}$$

provided

a, b are no zero integers $ax+b(1-x)\neq 0, 0\leqslant x\leqslant 1, u_jv_j,\delta_j>0(j=1,\cdots,s);\rho_i,\sigma_i,\theta_i>0(i=1,\cdots,r)$,

$$Re(d)+\sum_{i=1}^{r}\theta_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0;Re(\rho)+\sum_{i=1}^{r}\rho_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0;$$

$$Re(\sigma)+\sum_{i=1}^{r}\sigma_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0, Re(c)>0$$

$\left|arg\left(z_r y^{\rho_r}(1-y^2)^{\sigma_r}\left\{\sqrt{1-y^2}+wu\right\}^{\rho_r+2\sigma_r}\left\{-\frac{1-x}{ax+b(1-x)}\right\}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

The series on the right-hand side of (3.1) is absolutely convergent.

Proof

We first express the general class of mutivariable polynomials occurring on the left-hand side of (3.1) in the series forms with the help of the equation (1.13) and the multivariable Gimel-function by this multiple integrals contour with the help of (1.1), then interchanging the order of integrations and summations (which is justified due to absolute convergence of the integrals involved in the process under the conditions stated with result and evaluating the x and y-integrals with the help of lemmae 1 and 2, respectively, we find that (say) I

$$I=\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}B_1\prod_{j=1}^{s}\left[\frac{t_j^{k_j}}{K_j!}\right]\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}$$

$$\frac{e^{-z}\Gamma(c)\Gamma(d)\Gamma\left(d+\sum_{j=1}^{s}\delta_jK_j+\sum_{i=1}^{r}\theta_is_i\right)\Gamma\left(c+d-\alpha-\beta+\sum_{j=1}^{s}\delta_jK_j+\sum_{i=1}^{r}\theta_is_i\right)}{a^cb^{d+\sum_{j=1}^{s}\delta_jK_j+\sum_{i=1}^{r}\theta_is_i}\Gamma\left(c+d-\alpha+\sum_{j=1}^{s}\delta_jK_j+\sum_{i=1}^{r}\theta_is_i\right)\Gamma\left(c+d-\beta+\sum_{j=1}^{s}\delta_jK_j+\sum_{i=1}^{r}\theta_is_i\right)}$$

$${}_2F_2\left[d+\sum_{j=1}^{s}K_j\delta_j+\sum_{i=1}^{r}\theta_is_i, d+\sum_{j=1}^{s}K_j\delta_j+\sum_{i=1}^{r}\theta_is_i, c+d-\alpha-\beta+\sum_{j=1}^{s}K_j\delta_j+\sum_{i=1}^{r}\theta_is_i, d+\sum_{j=1}^{s}K_j\delta_j+\sum_{i=1}^{r}\theta_is_i;\right.$$

$$\left.c+d-\alpha+\sum_{j=1}^{s}K_j\delta_j+\sum_{i=1}^{r}\theta_is_i, c+d-\beta+\sum_{j=1}^{s}K_j\delta_j+\sum_{i=1}^{r}\theta_is_i, c+d-\beta+\sum_{j=1}^{s}K_j\delta_j+\sum_{i=1}^{r}\theta_is_i, d+\sum_{j=1}^{s}K_j\delta_j+\sum_{i=1}^{r}\theta_is_i;z\right]$$

$$e^{\rho+\sum_{j=1}^{s}u_jK_j+\sum_{i=1}^{r}\rho_is_i}\Gamma\left(\rho+\sum_{j=1}^{s}K_ju_j+\sum_{i=1}^{r}\rho_is_i\right)\frac{\Gamma\left(2\sigma+2\sum_{j=1}^{s}v_iK_j+2\sum_{i=1}^{r}\sigma_is_i\right)}{\Gamma\left(\rho+2\sigma+\sum_{j=1}^{s}(u_j+2v_i)K_j+\sum_{i=1}^{r}(\rho_i+2\sigma_i)s_i\right)}$$

$$\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.2}$$

On expressing ${}_2F_2(x)$ as a hypergeometric serie and changing the order of integrations and summation, interpreting the resulting expression in multiple integrals contour with the help of (1.1), we obtain the desired theorem 1.

3. Special cases.

In this session, we shall give several particular cases.

Taking $u_j=v_j=0(j=1,\cdots,s),\rho_i=\sigma_i=0(i=1,\cdots,r)$ in (2.1) and evaluating the y-integral, we obtain the following integral.

Corollary 1.

$$\int_0^1 x^{c-1}(1-x)^{d-1}\left\{[ax+b(1-x)]^{-c-d}exp\left\{-\frac{zax}{ax+b(1-x)}\right\}{}_2F_1\left(\alpha,\beta;c;\frac{ax}{ax+b(1-x)}\right)\right.$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[t_1\left\{\frac{1-x}{ax+b(1-x)}\right\}^{\delta_1},\cdots,t_r\left\{\frac{1-x}{ax+b(1-x)}\right\}^{\delta_r}\right]$$

$$\daleth\left(z_1\left\{-\frac{1-x}{ax+b(1-x)}\right\}^{\theta_1},\cdots,z_r\left\{-\frac{1-x}{ax+b(1-x)}\right\}^{\theta_r}\right)\mathrm{d}x=$$

$$\Gamma(c)\,a^{-c}b^{-d}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}B_1\prod_{j=1}^{s}\left[\frac{t_j^{k_j}b^{-\delta_jK_j}}{K_j!}\right]\daleth^{U;0,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c}\mathrm{z}_1b^{-\theta_1} & \mathbb{A};(1-p-d-\sum_{j=1}^{s}K_j\delta_j;\theta_1,\cdots,\theta_r;1),(1-c-d+\alpha+\beta-\sum_{j=1}^{s}K_j\delta_j;\theta_1,\cdots,\theta_r;1),\mathbf{A}:A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_rb^{*\theta_r} & \mathbb{B};\mathbf{B},(1-c-d-p+\alpha-\sum_{j=1}^{s}K_j\delta_j;\theta_1,\cdots,\theta_r;1),(1-c-d-p+\beta-\sum_{j=1}^{s}K_j\delta_j;\theta_1,\cdots,\theta_r;1)\end{array}\right)\quad(3.1)$$

provided

a,b are no zero integers, $ax+b(1-x)\neq 0, 0\leqslant x\leqslant 1, \delta_j>0(j=1,\cdots,s);\theta_i>0(i=1,\cdots,r), Re(c)>0$

$$Re(d)+\sum_{i=1}^{r}\theta_i\min_{1\leqslant j\leqslant m^{(i)}}Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0;$$

$\left|arg\left(z_r\left\{-\frac{1-x}{ax+b(1-x)}\right\}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

The series on the right-hand side of (3.1) is absolutely converge.

Similarly, if we take $\delta_j=0(j=1,\cdots,s)$ and $\theta_i=0(i=1,\cdots,r)$ in (2.1) , we get the following resulting

Corollary 2.

$$\int_0^1 y^{\rho-1}(1-y)^{\sigma-1}\left\{\sqrt{1-y^2}+\omega y\right\}^{\rho+2\sigma}$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left[t_1y^{u_1}(1-y^2)^{v_1}\left\{\sqrt{1-y^2}+\omega y\right\}^{u_1+2v_1},\cdots,t_sy^{u_s}(1-y^2)^{v_s}\left\{\sqrt{1-y^2}+\omega y\right\}^{u_s+2v_s}\right]$$

$$\daleth\left(z_1y^{\rho_1}(1-y^2)^{\sigma_1}\left\{\sqrt{1-y^2}+wu\right\}^{\rho_1+2\sigma_1},\cdots,z_ry^{\rho_r}(1-y^2)^{\sigma_r}\left\{\sqrt{1-y^2}+wu\right\}^{\rho_r+2\sigma_r}\right)\mathrm{d}x\mathrm{d}y=$$

$$\exp\left(\frac{\pi\omega\rho}{2}\right)\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}B_1\prod_{j=1}^{s}\left[t_j^{k_j}exp\frac{(\omega u_jK_j\frac{\pi}{2})}{K_j!}\right]\daleth^{U;0,n_r+2:V}_{X;p_{i_r}+2,q_{i_r}+1,\tau_{i_r}:R_r:Y}$$

$$\left(\begin{array}{c|c}\mathrm{z}_1exp\left(\frac{\omega\rho_1}{2}\right) & \mathbb{A};(1-\rho-\sum_{j=1}^{s}K_ju_j;\rho_1,\cdots,\rho_r;1),(1-2\sigma-\sum_{j=1}^{s}K_ju_j;2\sigma_1,\cdots,2\sigma_r;1),\mathbf{A}:A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_rexp\left(\frac{\omega\rho_1}{2}\right) & \mathbb{B};\mathbf{B},(1-\rho-2\sigma-\sum_{j=1}^{s}K_j(u_j+2v_j);rho_1+2\sigma_1,\cdots,\rho_r;+2\sigma_r;1):B\end{array}\right)\quad(3.2)$$

provided

a,b are no zero integers $ax+b(1-x)\neq 0, 0\leqslant x\leqslant 1, u_jv_j>0(j=1,\cdots,s);\rho_i,\sigma_i>0(i=1,\cdots,r),$

$$Re(\rho)+\sum_{i=1}^{r}\rho_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0;Re(\sigma)+\sum_{i=1}^{r}\sigma_i \min_{1\leqslant j\leqslant m^{(i)}} Re\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0,$$

$\left|arg\left(z_r y^{\rho_r}(1-y^2)^{\sigma_r}\left\{\sqrt{1-y^2}+wu\right\}^{\rho_r+2\sigma_r}\right)\right|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4).

The series on the right-hand side of (3.2) is absolutely converge.

We consider a class of polynomial of one variable.

Srivastava ([7],p. 1, Eq. 1). have introduced the general class of polynomials :

$$S_N^M(x)=\sum_{K=0}^{[N/M]}\frac{(-N)_{MK}}{K!}A_{N,K}x^K \tag{3.3}$$

where M is an arbitrary positive integer and the coeficients $A_{N,K}$ are arbitrary constants real or complex. On specializing these coefficients $A_{N,K}$, $S_N^M[.]$ yields a number of known polynomials as special cases. These include, among others, Hermite polynomials, Jacobi polynomials, Laguerre polynomials, bessel polynomials and several others, see ([11], p. 158-161).

Taking $S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s]$ by $S_N^M(y)$

We shall note

$$a_{NK}=\frac{(-N)_{MK}}{K!}A_{N,K} \tag{3.4}$$

and we obtain

Corollary 3.

$$\int_0^1\int_0^1 x^{c-1}y^{\rho-1}(1-x)^{d-1}(1-y)^{\sigma-1}\left\{\sqrt{1-y^2}+\omega y\right\}^{\rho+2\sigma}[ax+b(1-x)]^{-c-d}exp\left\{-\frac{zax}{ax+b(1-x)}\right\}$$

$${}_2F_1\left(\alpha,\beta;c;\frac{ax}{ax+b(1-x)}\right)S_N^M\left[ty^u(1-y^2)^v\left\{\sqrt{1-y^2}+\omega y\right\}^{u+2v}\left\{\frac{1-x}{ax+b(1-x)}\right\}^{\delta}\right]$$

$$\gimel\left(z_1y^{\rho_1}(1-y^2)^{\sigma_1}\left\{\sqrt{1-y^2}+wu\right\}^{\rho_1+2\sigma_1}\left\{-\frac{1-x}{ax+b(1-x)}\right\}^{\theta_1},\cdots,z_ry^{\rho_r}(1-y^2)^{\sigma_r}\left\{\sqrt{1-y^2}+wu\right\}^{\rho_r+2\sigma_r}\right.$$

$$\left.\left\{-\frac{1-x}{ax+b(1-x)}\right\}^{\theta_r}\right)\mathrm{d}x\mathrm{d}y=\Gamma(c)\exp\left(-z+\frac{\pi\omega\rho}{2}\right)a^{-c}b^{-d}\sum_{K=0}^{[N/M]}a_{NK}\left[t^kb^{-\delta K}exp\frac{(\omega uK\frac{\pi}{2})}{K!}\right]$$

$$\gimel_{X;p_{i_r}+4,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;0,n_r+4:V}\left(\begin{array}{c} z_1b^{-\theta_1}exp\left(\frac{\omega\rho_1}{2}\right)\\ \cdot\\ \cdot\\ z_rb^{-\theta_r}exp\left(\frac{\omega\rho_1}{2}\right)\end{array}\middle|\begin{array}{c}\mathbb{A};(1-p-d-K\delta;\theta_1,\cdots,\theta_r;1),\\ \cdot\\ \cdot\\ \mathbb{B};\mathbf{B},(1-c-d-p+\alpha-K\delta;\theta_1,\cdots,\theta_r;1)\end{array}\right.$$

$$\left.\begin{array}{c}\mathbb{A};(1-p-d-c+\alpha+\beta-K\delta;\theta_1,\cdots,\theta_r;1),(1-\rho-uK;\rho_1,\cdots,\rho_r;1),(1-2\sigma-2vK;2\sigma_1,\cdots,2\sigma_r;1),\mathbf{A}:A\\ \cdot\\ \cdot\\ \text{(1-c-d-p}+\beta-K\delta;\theta_1,\cdots,\theta_r;1),(1-\rho-2\sigma-(u+2v)K;\rho_1+2\sigma_1,\cdots,\rho_r+2\sigma_r;1):B\end{array}\right) \tag{3.5}$$

provided

a, b are no zero integers $ax + b(1-x) \neq 0, 0 \leqslant x \leqslant 1, u, v, \delta > 0; \rho_i, \sigma_i, \theta_i > 0 (i = 1, \cdots, r)$,

$$Re(d) + \sum_{i=1}^{r} \theta_i \min_{1 \leqslant j \leqslant m^{(i)}} Re \left[D_j^{(i)} \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] > 0; Re(\rho) + \sum_{i=1}^{r} \rho_i \min_{1 \leqslant j \leqslant m^{(i)}} Re \left[D_j^{(i)} \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] > 0;$$

$$Re(\sigma) + \sum_{i=1}^{r} \sigma_i \min_{1 \leqslant j \leqslant m^{(i)}} Re \left[D_j^{(i)} \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] > 0, Re(c) > 0$$

$$\left| arg \left(z_r y^{\rho_r} (1-y^2)^{\sigma_r} \left\{ \sqrt{1-y^2} + wu \right\}^{\rho_r + 2\sigma_r} \left\{ -\frac{1-x}{ax + b(1-x)} \right\} \right) \right| < \frac{1}{2} A_i^{(k)} \pi \text{ where } A_i^{(k)} \text{ is defined by (1.4).}$$

The series on the right-hand side of (3.5) is absolutely converge.

Remark :
We obtain the same double finite integrals with the functions defined in section I.

4. Conclusion.

The importance of our all the results lies in their manifold generality. Firstly, in view of general arguments utilized in these double integrals, we can obtain a large simpler double or single finite integrals. Secondly by specialising the various parameters as well as variables in the generalized multivariable polynomials, we obtain a large number of formulae involving simpler special functions (ultraspherical -Gegenbauer, Legendre, Tchebyshev, Bateman's, Hermite, Laguerre polynomials and others). Thirdy by specialising the various parameters as well as variables in the multivariable Gimel-function, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several intersting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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