Integration of Multivariable Gimel-Function with Respect to their Parameters

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ABSTRACT
The object of the present paper is to obtain some interesting results by integrating the multivariable Gimel-function with respect to its parameters. Such integrals are useful in the study of certain boundary value problems.

KEYWORDS : Multivariable Gimel-function, multiple integral contours, integration with respect to a parameter.

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1. Introduction and preliminaries.

Throughout this paper, let $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We define a generalized transcendental function of several complex variables.

$$\gimel(z_1, \cdots, z_r) = \gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array} \right|$$

$$[(a_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j};A_{2j})]_{1,n_2}, [\tau_{i_2}(a_{2ji_2};\alpha^{(1)}_{2ji_2},\alpha^{(2)}_{2ji_2};A_{2ji_2})]_{n_2+1,p_{i_2}}; [(a_{3j};\alpha^{(1)}_{3j},\alpha^{(2)}_{3j},\alpha^{(3)}_{3j};A_{3j})]_{1,n_3},$$
$$[\tau_{i_2}(b_{2ji_2};\beta^{(1)}_{2ji_2},\beta^{(2)}_{2ji_2};B_{2ji_2})]_{1,q_{i_2}};$$

$$[\tau_{i_3}(a_{3ji_3};\alpha^{(1)}_{3ji_3},\alpha^{(2)}_{3ji_3},\alpha^{(3)}_{3ji_3};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(a_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj};A_{rj})_{1,n_r}],$$
$$[\tau_{i_3}(b_{3ji_3};\beta^{(1)}_{3ji_3},\beta^{(2)}_{3ji_3},\beta^{(3)}_{3ji_3};B_{3ji_3})]_{1,q_{i_3}};\cdots; \quad \cdots$$

$$[\tau_{i_r}(a_{rji_r};\alpha^{(1)}_{rji_r},\cdots,\alpha^{(r)}_{rji_r};A_{rji_r})_{n_r+1,p_r}]: \; [(c^{(1)}_j,\gamma^{(1)}_j;C^{(1)}_j)_{1,n^{(1)}}], [\tau_{i^{(1)}}(c^{(1)}_{ji^{(1)}},\gamma^{(1)}_{ji^{(1)}};C^{(1)}_{ji^{(1)}})_{n^{(1)}+1,p^{(1)}_i}]$$
$$[\tau_{i_r}(b_{rji_r};\beta^{(1)}_{rji_r},\cdots,\beta^{(r)}_{rji_r};B_{rji_r})_{1,q_r}]: \; [(d^{(1)}_j),\delta^{(1)}_j;D^{(1)}_j)_{1,m^{(1)}}], [\tau_{i^{(1)}}(d^{(1)}_{ji^{(1)}},\delta^{(1)}_{ji^{(1)}};D^{(1)}_{ji^{(1)}})_{m^{(1)}+1,q^{(1)}_i}]$$

$$\left. \begin{array}{l} ;\cdots;[(c^{(r)}_j,\gamma^{(r)}_j;C^{(r)}_j)_{1,m^{(r)}}],[\tau_{i^{(r)}}(c^{(r)}_{ji^{(r)}},\gamma^{(r)}_{ji^{(r)}};C^{(r)}_{ji^{(r)}})_{m^{(r)}+1,p^{(r)}_i}] \\ ;\cdots;[(d^{(r)}_j,\delta^{(r)}_j;D^{(r)}_j)_{1,n^{(r)}}],[\tau_{i^{(r)}}(d^{(r)}_{ji^{(r)}},\delta^{(r)}_{ji^{(r)}};D^{(r)}_{ji^{(r)}})_{n^{(r)}+1,q^{(r)}_i}] \end{array} \right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1,\cdots,s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \, \mathrm{d}s_1 \cdots \mathrm{d}s_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha^{(k)}_{2j}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha^{(k)}_{2ji_2}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji2}+\sum_{k=1}^{2}\beta^{(k)}_{2ji2}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha^{(k)}_{3j}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha^{(k)}_{3ji_3}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji3}+\sum_{k=1}^{3}\beta^{(k)}_{3ji3}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rjir}+\sum_{k=1}^{r}\beta_{rjir}^{(k)}s_k)]} \tag{1.2}$$

and

$$\theta_k(s_k)=\frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)}\in\mathbb{N}$ and verify :

$0\leqslant m_2,\cdots,0\leqslant m_r,0\leqslant n_2\leqslant p_{i_2},\cdots,0\leqslant n_r\leqslant p_{i_r},0\leqslant m^{(1)}\leqslant q_{i^{(1)}},\cdots,0\leqslant m^{(r)}\leqslant q_{i^{(r)}}$

$0\leqslant n^{(1)}\leqslant p_{i^{(1)}},\cdots,0\leqslant n^{(r)}\leqslant p_{i^{(r)}}$.

3) $\tau_{i_2}(i_2=1,\cdots,R_2)\in\mathbb{R}^+;\tau_{i_r}\in\mathbb{R}^+(i_r=1,\cdots,R_r);\tau_{i^{(k)}}\in\mathbb{R}^+(i=1,\cdots,R^{(k)}),(k=1,\cdots,r)$.

4) $\gamma_j^{(k)},C_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,n^{(k)});(k=1,\cdots,r);\delta_j^{(k)},D_j^{(k)}\in\mathbb{R}^+;(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=m^{(k)}+1,\cdots,p^{(k)});(k=1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)}\in\mathbb{R}^+,(j=n^{(k)}+1,\cdots,q^{(k)});(k=1,\cdots,r)$.

$\alpha_{kj}^{(l)},A_{kj}\in\mathbb{R}^+;(j=1,\cdots,n_k);(k=2,\cdots,r);(l=1,\cdots,k)$.

$\alpha_{kji_k}^{(l)},A_{kji_k}\in\mathbb{R}^+;(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\beta_{kji_k}^{(l)},B_{kji_k}\in\mathbb{R}^+;(j=m_k+1,\cdots,q_{i_k});(k=2,\cdots,r);(l=1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)}\in\mathbb{R}^+;(i=1,\cdots,R^{(k)});(j=n^{(k)}+1,\cdots,p_{i^{(k)}});(k=1,\cdots,r)$.

5) $c_j^{(k)}\in\mathbb{C};(j=1,\cdots,n^{(k)});(k=1,\cdots,r);d_j^{(k)}\in\mathbb{C};(j=1,\cdots,m^{(k)});(k=1,\cdots,r)$.

$a_{kji_k}\in\mathbb{C};(j=n_k+1,\cdots,p_{i_k});(k=2,\cdots,r)$.

$b_{kji_k}\in\mathbb{C};(j=1,\cdots,q_{i_k});(k=2,\cdots,r)$.

$d_{ji^{(k)}}^{(k)}\in\mathbb{C};(i=1,\cdots,R^{(k)});(j=m^{(k)}+1,\cdots,q_{i^{(k)}});(k=1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r)$.

The contour L_k is in the $s_k(k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha_{2j}^{(k)} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha_{3j}^{(k)} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha_{rj}^{(i)}\right) (j = 1, \cdots, n_r), \quad \Gamma^{C_j^{(k)}}\left(1 - c_j^{(k)} + \gamma_j^{(k)} s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$to the right of the contour L_k and the poles of $\Gamma^{D_j^{(k)}}\left(d_j^{(k)} - \delta_j^{(k)} s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as :

$|arg(z_k)| < \frac{1}{2} A_i^{(k)} \pi$ where

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)} \gamma_j^{(k)} - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)} \delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)} \gamma_{ji^{(k)}}^{(k)} \right)$$

$$-\tau_{i_2} \left(\sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha_{2ji_2}^{(k)} + \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta_{2ji_2}^{(k)} \right) - \cdots - \tau_{i_r} \left(\sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha_{rji_r}^{(k)} + \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta_{rji_r}^{(k)} \right) \tag{1.4}$$

Following the lines of Braaksma ([2] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$ where $i = 1, \cdots, r$:

$\alpha_i = \min_{1 \leqslant j \leqslant m^{(i)}} Re \left[D_j^{(i)} \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right]$ and $\beta_i = \max_{1 \leqslant j \leqslant n^{(i)}} Re \left[C_j^{(i)} \left(\frac{c_j^{(i)} - 1}{\gamma_j^{(i)}} \right) \right]$

Remark 1.
If $n_2 = \cdots = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ $A_{rj} = A_{rji_r} = B_{rji_r} = 1$, then the multivariable Gimel-function reduces in the multivariable Aleph- function defined by Ayant [1].

Remark 2.
If $n_2 = \cdots = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Gimel-function reduces in a multivariable I-function defined by Prathima et al. [5].

Remark 3.
If$A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1and\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [4].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the multivariable H-function defined by Srivastava and Panda [6,7].

In your investigation, we shall use the following notations.

$\mathbb{A} = [(\mathrm{a}_{2j}; \alpha_{2j}^{(1)}, \alpha_{2j}^{(2)}; A_{2j})]_{1,n_2}, [\tau_{i_2}(a_{2ji_2}; \alpha_{2ji_2}^{(1)}, \alpha_{2ji_2}^{(2)}; A_{2ji_2})]_{n_2+1,p_{i_2}}, [(a_{3j}; \alpha_{3j}^{(1)}, \alpha_{3j}^{(2)}, \alpha_{3j}^{(3)}; A_{3j})]_{1,n_3},$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})_{1,n_{r-1}}],$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})_{n_{r-1}+1,p_{i_{r-1}}}] \tag{1.5}$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})_{1,n_r}],[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})_{\mathfrak{n}+1,p_{i_r}}] \tag{1.6}$$

$$A=[(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n^{(1)}}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})_{n^{(1)}+1,p_i^{(1)}}];\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})_{m^{(r)}+1,p_i^{(r)}}] \tag{1.7}$$

$$\mathbb{B}=[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})_{1,q_{i_{r-1}}}] \tag{1.8}$$

$$\mathbf{B} = [\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})_{1,q_{i_r}}] \tag{1.9}$$

$$\mathrm{B}=[(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,m^{(1)}}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})_{m^{(1)}+1,q_i^{(1)}}];\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,m^{(r)}}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})_{m^{(r)}+1,q_i^{(r)}}] \tag{1.10}$$

$$U=0,n_2;0,n_3;\cdots;0,n_{r-1};V=m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.11}$$

$$X=p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.12}$$

2. Required result.

We have the following result ([3], Erdelyi et al. p .300).

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\Gamma(a+x)\Gamma(b+x)\Gamma(c-x)\Gamma(d-x)}=\frac{\Gamma(a+b+c+d-3)}{\Gamma(a+c-1)\Gamma(a+d-1)\Gamma(b+c-1)\Gamma(b+d-1)} \tag{2.1}$$

where $Re(a+b+c+d)>3$.

3. Main integrals.

Theorem 1.

$$\int_{-\infty}^{\infty}\gimel_{X;p_{i_r},q_{i_r}+4,\tau_{i_r}:R_r:Y}^{U;0,n_r:V}\left(\begin{array}{c|c}\mathrm{z}_1 & \mathbb{A};\ \mathbf{A}:\ A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r & \mathbb{B};\mathbf{B},B_1:B\end{array}\right)dx=\gimel_{X;p_{i_r}+1,q_{i_r}+4,\tau_{i_r}:R_r:Y}^{U;0,n_r+1:V}\left(\begin{array}{c|c}\mathrm{z}_1 & \mathbb{A};A_1,\mathbf{A}:A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r & \mathbb{B};\mathbf{B},B_1':B\end{array}\right) \tag{3.1}$$

Where

$$B_1=\left(1-a-x;\sum_{i=1}^r a_i;1\right),\left(1-b-x;\sum_{i=1}^r b_i;1\right),\left(1-c+x;\sum_{i=1}^r c_i;1\right),\left(1-d+x;\sum_{i=1}^r d_i;1\right) \tag{3.2}$$

$$A_1=(4-a-b-c-d;a_1+b_1+c_1+d_1,\cdots,a_r+b_r+c_r+d_r;1) \tag{3.3}$$

$$B_1' = \left(2-a-c;\sum_{i=1}^{r}(a_i+c_i);1\right),\left(2-a-d;\sum_{i=1}^{r}(a_i+d_i);1\right),\left(2-b-c;\sum_{i=1}^{r}(b_i+c_i);1\right),$$

$$\left(2-b-d;\sum_{i=1}^{r}(b_i+d_i);1\right) \tag{3.4}$$

Provided

$a_i,b_i,c_i,d_i \geqslant 0 (i=1,\cdots,r)$, $Re(a+b+c+d-3)>0$

$|arg\,(z_1)| < \frac{1}{2}\left(A_i^{(k)}-a_i-b_i-c_i-d_i\right)\pi$ where $A_i^{(k)}$ is defined by (1.4).

proof

To establish the theorem 1, expressing the multivariable Gimel-function in the Mellin-Barnes multiple integrals contour with the help of (1.1) and interchanging the order of integrations, which is justified under the conditions mentioned above, we get (say I)

$$I = \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}$$

$$\left[\int_{-\infty}^{\infty}\frac{1}{\Gamma\left(a+\sum_{i=1}^{r}a_is_i+x\right)\Gamma\left(b+\sum_{i=1}^{r}b_is_i+x\right)\Gamma\left(c+\sum_{i=1}^{r}c_is_i-x\right)\Gamma\left(d+\sum_{i=1}^{r}d_is_i-x\right)}\mathrm{d}x\right]$$

$$\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.5}$$

Now we use the lemma, we obtain

$$I = \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\frac{\Gamma(a+b+c+d-3+\sum_{i=1}^{r}(a_i+b_i+c_i+d_i)s_i)}{\Gamma\left(a+c-1+\sum_{i=1}^{r}(a_i+c_i)s_i\right)\Gamma\left(a+d-1+\sum_{i=1}^{r}(a_i+d_i)s_i\right)}$$

$$\frac{1}{\Gamma\left(b+c-1+\sum_{i=1}^{r}(b_i+c_i)s_i\right)\Gamma\left(b+d-1+\sum_{i=1}^{r}(b_i+d_i)s_i\right)}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{3.6}$$

Now interpreting the resulting expression in Mellin-Barnes multiple integrals contour with the help of (1.1), we obtain the desired theorem

Theorem 2.

$$\int_{-\infty}^{\infty}\gimel_{X;p_{i_r},q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;0,n_r+1:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\,\mathbf{A},\,\mathrm{A}_2:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\mathbf{B},B_2:B \end{array}\right)dx = \gimel_{X;p_{i_r}+1,q_{i_r}+4,\tau_{i_r}:R_r:Y}^{U;0,n_r+1:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};A_2,\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\mathbf{B},B_2':B \end{array}\right) \tag{3.7}$$

Where

$$A_2 = (d-x;\sum_{i=1}^{r}d_i;1)$$

$$B_2 = \left(1-a-x;\sum_{i=1}^{r}a_i;1\right),\left(1-b-x;\sum_{i=1}^{r}b_i;1\right),\left(1-c+x;\sum_{i=1}^{r}c_i;1\right) \tag{3.8}$$

$$A_2 = (4-a-b-c-d;a_1+b_1+c_1-d_1,\cdots,a_r+b_r+c_r-d_r;1) \tag{3.9}$$

$$B_2' = \left(2-a-c;\sum_{i=1}^{r}(a_i+c_i);1\right),\left(2-b-c;\sum_{i=1}^{r}(b_i+c_i);1\right),\left(2-a-d;\sum_{i=1}^{r}(a_i+d_i);1\right),$$

$$\left(2-b-d;\sum_{i=1}^{r}(b_i+d_i);1\right) \tag{3.10}$$

Provided

$a_i, b_i, c_i, d_i \geqslant 0 (i = 1, \cdots, r)$, $Re(a+b+c+d-3) > 0$; $a_i, b_i \geqslant d_i$

$|arg\,(z_1)| < \frac{1}{2}\left(A_i^{(k)} - a_i - b_i - c_i + d_i\right)\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 3.

$$\int_{-\infty}^{\infty} \daleth_{X;p_{i_r}+2,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;0,n_r+1:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\ \mathbf{A},\ \mathrm{A}_3 : A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\mathbf{B},B_3;B \end{array}\right)dx = \daleth_{X;p_{i_r}+1,q_{i_r}+4,\tau_{i_r}:R_r:Y}^{U;0,n_r+1:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};A_3',\mathbf{A} : A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\mathbf{B},B_3' : B \end{array}\right) \tag{3.11}$$

where

$$A_3 = \left(c-x;\sum_{i=1}^{r}c_i;1\right),\left(d-x;\sum_{i=1}^{r}d_i;1\right) \tag{3.12}$$

$$B_3 = \left(1-a-x;\sum_{i=1}^{r}a_i;1\right),\left(1-b-x;\sum_{i=1}^{r}b_i;1\right),\left(1-c+x;\sum_{i=1}^{r}c_i;1\right) \tag{3.13}$$

$$A_3' = (4-a-b-c-d; a_1+b_1-c_1-d_1,\cdots,a_r+b_r-c_r-d_r;1) \tag{3.14}$$

$$B_3' = \left(2-a-c;\sum_{i=1}^{r}(a_i+c_i);1\right),\left(2-b-c;\sum_{i=1}^{r}(b_i+c_i);1\right),\left(2-a-d;\sum_{i=1}^{r}(a_i+d_i);1\right),$$

$$\left(2-b-d;\sum_{i=1}^{r}(b_i+d_i);1\right) \tag{3.15}$$

Provided

$a_i, b_i, c_i, d_i \geqslant 0 (i = 1, \cdots, r)$, $Re(a+b+c+d-3) > 0$; $a_i, b_i \geqslant c_i, d_i$

$|arg\,(z_1)| < \frac{1}{2}\left(A_i^{(k)} - a_i - b_i + c_i + d_i\right)\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 4.

$$\int_{-\infty}^{\infty} \daleth_{X;p_{i_r}+3,q_{i_r}+1,\tau_{i_r}:R_r:Y}^{U;0,n_r+1:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};\ \mathbf{A},\ \mathrm{A}_4 : A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\mathbf{B},B_4;B \end{array}\right)dx = \daleth_{X;p_{i_r}+3,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;0,n_r+1:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathbb{A};A_4',\mathbf{A} : A \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r & \mathbb{B};\mathbf{B},B_4' : B \end{array}\right) \tag{3.16}$$

Where

$$A_4 = \left(b+x;\sum_{i=1}^{r}c_i;1\right) \tag{3.17}$$

$$B_4=\left(1-a-x;\sum_{i=1}^{r}a_i;1\right),\left(c-x;\sum_{i=1}^{r}c_i;1\right),\left(d-x;\sum_{i=1}^{r}cd_i;1\right) \tag{3.18}$$

$$A'_4=(4-a-b-c-d;a_1-b_1-c_1-d_1,\cdots,a_r-b_r-c_r-d_r;1) \tag{3.19}$$

$$B'_4=\left(2-a-c;\sum_{i=1}^{r}(a_i+c_i);1\right),\left(2-a-d-\sum_{i=1}^{r}(a_i+d_i);1\right),\left(b+c-1;\sum_{i=1}^{r}(a_i+d_i);1\right),$$
$$\left(b+d-1;\sum_{i=1}^{r}(b_i+d_i);1\right) \tag{3.20}$$

Provided

$a_i,b_i,c_i,d_i\geqslant 0(i=1,\cdots,r)$, $Re(a+b+c+d-3)>0$; $a_i\geqslant b_i+c_i+d_i$

$|arg\,(z_1)|<\frac{1}{2}\left(A_i^{(k)}-a_i-3b_i+c_i+d_i\right)\pi$ where $A_i^{(k)}$ is defined by (1.4).

Theorem 5.

$$\int_{-\infty}^{\infty}\gimel_{X;p_{i_r}+2,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;0,n_r+1:V}\left(\begin{array}{c|c}\mathrm{z}_1 & \mathbb{A};\,\mathbf{A},\,\mathrm{A}_5:A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r & \mathbb{B};\mathbf{B},B_5;B\end{array}\right)dx=\gimel_{X;p_{i_r}+2,q_{i_r}+3,\tau_{i_r}:R_r:Y}^{U;0,n_r+1:V}\left(\begin{array}{c|c}\mathrm{z}_1 & \mathbb{A};A'_5,\mathbf{A}:A\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_r & \mathbb{B};\mathbf{B},B'_5:B\end{array}\right) \tag{3.21}$$

Where

$$A_4=\left(b+x;\sum_{i=1}^{r}c_i;1\right),\left(d-x;\sum_{i=1}^{r}d_i;1\right) \tag{3.22}$$

$$B_5=\left(1-a-x;\sum_{i=1}^{r}a_i;1\right),\left(1-c+x;\sum_{i=1}^{r}c_i;1\right) \tag{3.23}$$

$$A'_5=(4-a-b-c-d;a_1-b_1+c_1-d_1,\cdots,a_r-b_r+c_r-d_r;1) \tag{3.24}$$

$$B'_5=\left(2-a-c;\sum_{i=1}^{r}(a_i+c_i);1\right),\left(2-a-d;\sum_{i=1}^{r}(a_i+d_i);1\right),\left(b+d-1;\sum_{i=1}^{r}(b_i+d_i);1\right),$$
$$\left(2-b+c;\sum_{i=1}^{r}(c_i-b_i);1\right) \tag{3.25}$$

Provided

$a_i,b_i,c_i,d_i\geqslant 0(i=1,\cdots,r)$, $Re(a+b+c+d-3)>0$; $a_i\geqslant d_i;c_i\geqslant b_i$;

$|arg\,(z_1)|<\frac{1}{2}\left(A_i^{(k)}-a_i-b_i-c_i-d_i\right)\pi$ where $A_i^{(k)}$ is defined by (1.4).

The theorems 2 , 3, 4 and 5 can similarly be established by replacing $\{a,b,c,d\}$ respectively by
$$\left\{a+\sum_{i=1}^{r}a_is_i,b+\sum_{i=1}^{r}b_is_i,c+\sum_{i=1}^{r}c_is_i,d+\sum_{i=1}^{r}d_is_i\right\},\left\{a+\sum_{i=1}^{r}a_is_i,b+\sum_{i=1}^{r}b_is_i,c-\sum_{i=1}^{r}c_is_i,d-\sum_{i=1}^{r}d_is_i\right\},$$

$$\left\{a+\sum_{i=1}^{r}a_is_i,b-\sum_{i=1}^{r}b_is_i,c-\sum_{i=1}^{r}c_is_i,d-\sum_{i=1}^{r}d_is_i\right\},\left\{a+\sum_{i=1}^{r}a_is_i,b-\sum_{i=1}^{r}b_is_i,c+\sum_{i=1}^{r}c_is_i,d-\sum_{i=1}^{r}d_is_i\right\}.$$

Remark :

We obtain the same double finite integrals with the functions defined in section I.

4. Conclusion.

The Gimel-function of several variables presented in this paper, are quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various known and (news) integrals with respect to parameters concerning the special functions of one variable and several variables. Secondly, in view of general arguments utilized in these integrals, we can obtain a large number of particular cases of integrals with respect to parameters.

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