A Study on Quasi Class Q Operator

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ABSTRACT: In this paper quasi class Q operator is introduced with example and compared with normaloid operator. Necessary and sufficient condition for a composition operator to be quasi class Q is given with an example.

Also weighted composition operator W and generalized Aluthge transform C_r are characterized to be of quasi class Q.

Keywords: Class Q operators, composition Operators

1. INTRODUCTION

CLASS Q OPERATOR : In [2], B.P.Duggal, C.S.Kubrusly and N.Levan introduced a new class 'Class Q' operator and showed that this class of operators properly induces the paranormal operators and studied various properties of Class Q operators.

In [5], S.Panayappan and D.Senthilkumar have given necessary and sufficient condition for a composition operator to be class Q and studied various properties of class Q composition operators.

An operator $T \in L(H)$ is of class Q if and only if $T^{*2}T^2 - 2T^*T + I \geq 0$. [2]

2. COMPOSITION OPERATOR

The Radon-Nikodym derivative of $\lambda \circ T^{-1}$ with respect to λ is denoted by h and that of $\lambda \circ T^{-k}$ with respect to λ is denoted by h_k where $k \in \mathrm{N}$, the set of natural numbers. Here T^k is obtained by composing T - k times. From [4], w_k denotes $w(w \circ T)(w \circ T^2)....(w \circ T^{k-1})$ so that

$$W^k f = w_k(f \circ T^k),$$

$$W^{*k} f = h_k E(w_k f) \circ T^{-k},$$

$$W^{*k}W^k f = h_k E(w_k^2) \circ T^{-k} f \quad \text{and}$$

$$W^k W^{*k} f = w_k(h_k \circ T^k)E(w_k f).$$

For any operator T, let $T = U|T|$ be the polar decomposition of T. The Aluthge transform of T is the operator $\tilde{T}$ given by $\tilde{T} = |T|^{1/2} U |T|^{1/2}$.

More generally in [1], Charles Burnap, Il Bong Jung and Alan Lambert have defined the family of operators $\{T_r : 0 < r \le 1\}$ where $T_r = |T|^r U |T|^{1-r}$.

In [1], for a composition operator C the polar decomposition is given by $C = U|C|$ where $|C| f = \sqrt{h} f$, and $Uf = \frac{1}{\sqrt{h \circ T}} f \circ T$.

This is valid for all composition operators. In [1], Charles Burnap, Il Bong Jung and Alan Lambert have given more generally Aluthge transformation for composition operators as

$$C_r = |C|^r U |C|^{1-r} \quad \text{for } 0 < r \le 1 \text{ and}$$

$$C_r f = \left(\frac{h}{h \circ T}\right)^{r/2} f \circ T.$$

Here we see that C_r is a weighted composition operator with weight $\pi = \left(\frac{h}{h \circ T}\right)^{r/2}$ where $0 < r \le 1$.

Since C_r is a weighted composition operator it is easy to show that

$$|C_r| f = \sqrt{h[E(\pi^2)] \circ T^{-1}} f$$

$$\Rightarrow C_r^* C_r f = h[E(\pi^2)] \circ T^{-1} f.$$

And $|C_r^*| f = vE(vf)$ where $v = \frac{\pi\sqrt{h \circ T}}{\left[E\left(\pi\sqrt{h \circ T}\right)^2\right]^{\frac{1}{4}}}$ (ie) $\sqrt{C_r C_r^*} f = vE(vf)$.

If $T^{-1}\Sigma = \Sigma$, then E becomes identity operator and hence $C_r C_r^* f = v^4 f$.

Also we have that $C_r^k f = \pi_k (f \circ T^k)$, where $\pi_k = \pi(\pi \circ T)(\pi \circ T^2)....(\pi \circ T^{k-1})$.

$$C_r^{*k} f = h_k E(\pi_k f) \circ T^{-k},$$

$$(C_r^* C_r)^k f = h^k (E(\pi^2) \circ T^{-1})^k f,$$

$$C_r^{*k} C_r^k f = h_k\, E({\pi_k}^2) \circ T^{-k} f \quad \text{and}$$

$$C_r^k C_r^{*k} f = \pi_k (h_k \circ T^k) E(\pi_k f).$$

3. QUASI CLASS Q OPERATOR

In this paper quasi class Q operator is introduced with example and compared with normaloid operator. Necessary and sufficient condition for a composition operator to be quasi class Q is given with example.

Also weighted composition operator W and generalized Aluthge transform C_r are characterized to be of quasi class Q.

Definition: An operator $T \in L(H)$ is said to be of quasi class Q if $\|T^3(x)\|^2 \geq 2\|T^2(x)\|^2 - \|T(x)\|^2, \forall x \in H$ with $\|x\| = 1$.

Theorem 1: An operator $T \in L(H)$ is of quasi class Q iff $T^{*^3}T^3 - 2T^{*^2}T^2 + T^*T \geq 0.$

Proof:

For all $x \in H$, consider

$$T^{*^3}T^3 - 2T^{*^2}T^2 + T^*T \geq 0$$

$$\Leftrightarrow \left\langle (T^{*^3}T^3 - 2T^{*^2}T^2 + T^*T)x, x \right\rangle \geq 0$$

$$\Leftrightarrow \left\langle (T^{*^3}T^3)x, x \right\rangle - 2\left\langle (T^{*^2}T^2)x, x \right\rangle + \left\langle (T^*T)x, x \right\rangle \geq 0 \Leftrightarrow \left\langle T^3x, T^3x \right\rangle - 2\left\langle T^2x, T^2x \right\rangle + \left\langle Tx, Tx \right\rangle \geq 0$$

$$.\Leftrightarrow \|T^3(x)\|^2 - 2\|T^2(x)\|^2 + \|T(x)\|^2 \geq 0$$

$$.\Leftrightarrow \|T^3(x)\|^2 \geq 2\|T^2(x)\|^2 - \|T(x)\|^2.$$

Corollary: A weighted shift T with decreasing weighted sequence (α_n) is quasi class Q if $\alpha_{n+1}{}^2\alpha_{n+2}{}^2 - 2\alpha_{n+1}{}^2 + 1 \geq 0.$

Proof:

Since T is a weighted shift, its adjoint T^* is also a weighted shift and defined by $T(e_n) = \alpha_n e_{n+1}$,

$T^*(e_n) = \alpha_{n-1}e_{n-1}$,

$(T^*T)e_n = \alpha_n{}^2 e_n$,

$(T^{*2}T^2)e_n = \alpha_n{}^2\alpha_{n+1}{}^2 e_n$ and

$(T^{*3}T^3)e_n = \alpha_n{}^2\alpha_{n+1}{}^2\alpha_{n+2}{}^2 e_n$.

Now since T is quasi class Q then by Theorem 1

$T^{*3}T^3-2T^{*2}T^2+T^*T\geq 0.\Rightarrow \alpha_n{}^2\alpha_{n+1}{}^2\alpha_{n+2}{}^2-2\alpha_n{}^2\alpha_{n+1}{}^2+\alpha_n{}^2\geq 0.\Rightarrow \alpha_{n+1}{}^2\alpha_{n+2}{}^2-2\alpha_{n+1}{}^2+1\geq 0.$

Example: Consider the operator $T:l^2\to l^2$ defined by $T(x)=(0,\alpha_1x_1,\alpha_2x_2,............)$ where $\alpha_n=\frac{1}{4},\ \alpha_{n+1}=\frac{1}{2},\ \alpha_{n+2}=\frac{1}{4}$ for $n\geq 1$. Then T is of quasi class Q.

Proof:

Given $T(x)=(0,\alpha_1x_1,\alpha_2x_2,............)$.

Then

$T^*(x)=(\alpha_1x_2,\alpha_2x_3,............).\ T^*T(x)=(\alpha_1{}^2x_1,\alpha_2{}^2x_2,\alpha_3{}^2x_3........).$

$T^2(x)=(0,0,\alpha_2\alpha_1x_1,\alpha_3\alpha_2x_2,\alpha_4\alpha_3x_3............).\ T^*T^2(x)=(0,\alpha_2{}^2\alpha_1x_1,\alpha_3{}^2\alpha_2x_2,\alpha_4{}^2\alpha_3x_3............).$

$T^{*2}T^2(x)=(\alpha_2{}^2\alpha_1{}^2x_1,\alpha_3{}^2\alpha_2{}^2x_2,\alpha_4{}^2\alpha_3{}^2x_3........).\ T^3(x)=(0,0,\ 0,\ \alpha_3\alpha_2\alpha_1x_1,\ \alpha_4\alpha_3\alpha_2x_2,............).$

$T^*T^3(x)=(0,0,\ \alpha_3{}^2\alpha_2\alpha_1x_1,\ \alpha_4{}^2\alpha_3\alpha_2x_2,............).\ T^{*2}T^3(x)=(0,\ \alpha_3{}^2\alpha_2{}^2\alpha_1x_1,\ \alpha_4{}^2\alpha_3{}^2\alpha_2x_2,............).$

$T^{*3}T^3(x)=(\alpha_3{}^2\alpha_2{}^2\alpha_1{}^2x_1,\ \alpha_4{}^2\alpha_3{}^2\alpha_2{}^2x_2,\ \alpha_5{}^2\alpha_4{}^2\alpha_3{}^2x_3........).$ Now consider $\left\langle (T^{*3}T^3-2T^{*2}T^2+T^*T)x,x\right\rangle$

$$=\Big\langle(\alpha_3{}^2\alpha_2{}^2\alpha_1{}^2x_1,\alpha_4{}^2\alpha_3{}^2\alpha_2{}^2x_2,....)\ -2(\alpha_2{}^2\alpha_1{}^2x_1,\alpha_3{}^2\alpha_2{}^2x_2,.....)\ +(\alpha_1{}^2x_1,\alpha_2{}^2x_2,........),$$
$$(x_1,x_2,...)\Big\rangle=\Big\langle((\alpha_3{}^2\alpha_2{}^2\alpha_1{}^2-2\alpha_2{}^2\alpha_1{}^2+\alpha_1{}^2)x_1\ +(\alpha_4{}^2\alpha_3{}^2\alpha_2{}^2-2\alpha_3{}^2\alpha_2{}^2+\alpha_2{}^2)x_2+...),(x_1,x_2,...)\Big\rangle$$
$$=\Big\langle(\alpha_3{}^2\alpha_2{}^2\alpha_1{}^2-2\alpha_2{}^2\alpha_1{}^2+\alpha_1{}^2)x_1,x_1\Big\rangle+\Big\langle(\alpha_4{}^2\alpha_3{}^2\alpha_2{}^2-2\alpha_3{}^2\alpha_2{}^2+\alpha_2{}^2)x_2,x_2\Big\rangle+.........$$
$$=(\alpha_3{}^2\alpha_2{}^2\alpha_1{}^2-2\alpha_2{}^2\alpha_1{}^2+\alpha_1{}^2)\|x_1\|^2+(\alpha_4{}^2\alpha_3{}^2\alpha_2{}^2-2\alpha_3{}^2\alpha_2{}^2+\alpha_2{}^2)\|x_2\|^2+.......>0.$$

Example: Let T be a weighted shift with weighted sequence $\alpha_n=\frac{1}{2^n}$. Then T is of quasi class Q but it is not normaloid.

Proof:

$T(e_n)=\frac{1}{2^n}e_{n+1}.$

$T^*(e_n)=\frac{1}{2^{n-1}}e_{n-1}.$

$T^{*3}T^3-2T^{*2}T^2+T^*T\geq 0,\quad \forall n\in N.$

$\Rightarrow T$ is of quasi class Q.

But T is not normoloid. [3] .

4. QUASI CLASS Q COMPOSITION OPERATOR.

Theorem 2: Let $C \in B(L^2(\lambda))$. Then C is of quasi class Q iff $f_0^{(3)} - 2f_0^{(2)} + f_0 \geq 0$ a.e.

Proof:

From Theorem 1, C is of quasi class Q

$$\Leftrightarrow \; C^{*3}C^3 - 2C^{*2}C^2 + C^*C \geq 0$$

$$\Leftrightarrow \left\langle (C^{*3}C^3 - 2C^{*2}C^2 + C^*C)f, f \right\rangle \geq 0, \; \forall \; f \in L^2(\lambda) \Leftrightarrow \int_E (f_0^{(3)} - 2f_0^{(2)} + f_0)|f|^2 d\lambda \geq 0, \quad \forall E \in \Sigma$$

$$\Leftrightarrow \; f_0^{(3)} - 2f_0^{(2)} + f_0 \geq 0 \text{ a.e.}$$

Example 1: Let $X = N$, the set of all natural numbers and λ be the counting measure on it. Define $T : N \to N$, by $T(1) = 1, \quad T(2) = 1, \quad T(3) = 1, \quad T(4) = 1, \quad T(5) = 2, T(6n+m) = n+1, \quad m = 0,1,2,3,4,5$ and $n \in N$.

Since $f_0^{(3)} - 2f_0^{(2)} + f_0 \geq 0, \; \forall n \in N$, C is of quasi class Q.

Example 2: Let $X = N$, the set of all natural numbers and λ be the counting measure on it. Define $T : N \to N$ by $T(1) = 1, \; T(4n+m-2) = n+1, \quad m = 0,1,2,3$ and $n \in N$.

Since $f_0^{(3)} - 2f_0^{(2)} + f_0 \geq 0, \; \forall n \in N$, C is of quasi class Q.

Theorem 3: The weighted composition operator W is of quasi class Q iff $(h_3E(w_3^2) \circ T^{-3}) - 2(h_2E(w_2^2) \circ T^{-2}) + (hE(w^2) \circ T^{-1}) \geq 0$ a.e.

Proof:

From theorem 1, W is of quasi class Q

$$\Leftrightarrow W^{*3}W^3 - 2W^{*2}W^2 + W^*W \geq 0$$

$$\Leftrightarrow \left\langle (W^{*3}W^3 - 2W^{*2}W^2 + W^*W)f, f \right\rangle \geq 0, \forall f \in L^2(\lambda) \; \Leftrightarrow \int_E (W^{*3}W^3 - 2W^{*2}W^2 + W^*W)|f|^2 d\lambda \geq 0, \forall E \in \Sigma$$

$$\Leftrightarrow (h_3E(w_3^2) \circ T^{-3}) - 2(h_2E(w_2^2) \circ T^{-2}) + (hE(w^2) \circ T^{-1}) \geq 0 \text{ a.e.}$$

Corollary: The generalized Aluthge transformation C_r is of quasi class Q $\Leftrightarrow (h_3E(\pi_3^2) \circ T^{-3}) - 2(h_2E(\pi_2^2) \circ T^{-2}) + (hE(\pi^2) \circ T^{-1}) \geq 0$ a.e.

Proof:

From theorem 1, C_r is of quasi class Q

$$\Leftrightarrow \; C_r^{*3}C_r^3 - 2C_r^{*2}C_r^2 + C_r^*C_r \geq 0$$

$$\Leftrightarrow \left\langle (C_r^{*3}C_r^{\,3} - 2C_r^{*2}C_r^{\,2} + C_r^{*}C_r)\, f, f \right\rangle \geq 0, \ \forall\, f \in L^2(\lambda)$$

$$\Leftrightarrow \int_E (C_r^{*3}C_r^{\,3} - 2C_r^{*2}C_r^{\,2} + C_r^{*}C_r) \left| f \right|^2 d\lambda \geq 0, \ \ \forall E \in \Sigma$$

$$\Leftrightarrow (h_3 E(\pi_3^{\,2}) \circ T^{-3}) - 2(h_2 E(\pi_2^{\,2}) \circ T^{-2}) + (hE(\pi^2) \circ T^{-1}) \geq 0.$$

5. REFERENCES

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