Existential and Uniqueness Results for Initial Value Problems associated with Nonlinear Singular Interface Problems on Time Scales using Fixed Point Theorems

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The authors dedicate this work to the Founder Chancellor of Sri Sathya Sai Institute of Higher Learning, Bhagwan Sri Sathya Sai Baba.

Abstract. In this paper we present existential and uniqueness results for IVPs associated with general nonlinear singular interface problems on Time Scales. We discuss these results for a fourth order IVP associated with nonlinear singular interface problems using the classical fixed point theorems of Banach and Schauder.

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1. Introduction

In literature we find a class of problems wherein two different differential equations are defined on adjacent intervals with a common point of interface. We term these problems as interface problems.

If the interface problem has a well defined boundary, we call the problem to be a *regular boundary value problem(RBVP).* The interface problem with a boundary that has singularity at the end points is called a *singular boundary value problem(SBVP).* If there is a singularity at the point of interface, we term the problem to be a *singular interface problem(SIP).* Solving these type of boundary value problems with singularities remains a challenge for mathematicians.

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While *regular* boundary value problems, those over finite intervals with well-behaved coefficients pose no difficulties, the problems wherein the domain of the problem is not well defined, or the continuity and/or smoothness of the functions, coefficients involved are not guaranteed in some parts of the domain, sometimes in the boundary or parts of the boundary are difficult to tackle. There are quite a number of different approaches that we come across in the literature to tackle these *singular* problems [1],[4],[6],[8],[9],[10].

In literature, we see that work has been done on initial and boundary value problems associated with a pair of linear differential operators with conditions at the interface for both regular and singular cases. Some publications include [15]-[22],[26]-[31].

The singular interface problem requires a special mention. In this case existing theory based on the conventional analysis may not come handy.

We feel that the new framework of dynamic equations on time scales(an arbitrary closed subset of real numbers)[2] with facilities of the two jump operators with various definitions of continuity and derivatives makes one's job simple to study these singular interface problems. These dynamic equations are nothing but the differential equations when $\mathbb{T} = \mathbb{R}$ and are difference equations when $\mathbb{T} = \mathbb{Z}$.

Our preliminary investigation about the feasibility of this study for linear second order interface problems has resulted in the work [16, 17, 25].

From the above we observe that substantial amount of work has been done for regular and singular boundary value problems involving linear differential operators. It is clear that there is a need for these singular interface problems to be discussed for the case where the problem involves nonlinear differential operators.

A systematic study of Initial Value Problems, Boundary Value Problems and eigen Value problems associated with these nonlinear singular interface problems involving nonlinear second order pair of dynamic equations is done in [34]-[41].

In this paper we study the existence and uniqueness of solution for a fourth order Initial Value Problem associated with these nonlinear singular interface problems. Schauder and Banach's fixed point theorems are used for proving the exsistential and uniqueness results.

2. Mathematical Preliminaries

Definition 2.1. Let $\mathbb{T}$ be a time scale(an arbitrary closed subset of real numbers). For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) > t$, we say that t is *right-scattered,* while $\rho(t) < t$ we say that t is *left-scattered.* Points that are right-scattered and left-scattered at the same time are called *isolated.* Also, if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense,* and if $t >\ \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense.* Points that are right-dense and left-dense at the same time are called *dense.* Finally, the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

Definition 2.2. $\mathbb{T}^\kappa = \left\{ \begin{array}{ll} \mathbb{T} - \{m\} & \text{if sup } \mathbb{T} < \infty \\ \mathbb{T} & \text{if sup } \mathbb{T} = \infty \end{array} \right\}$ where m is the left scattered maximum of $\mathbb{T}$.

Definition 2.3. Let f be a function defined on $\mathbb{T}$. We say that f is *delta differentiable* at $t \in \mathbb{T}^\kappa$ provided there exists an α such that for all $\epsilon > 0$ there is a neighborhood $\mathcal{N}$ around t with

$$|f(\sigma(t) - f(s) - \alpha(\sigma(t) - s)| \leq \epsilon\,|\sigma(t) - s| \text{ for all } s \in \mathcal{N}.$$

Definition 2.4. For a function $f : \mathbb{T} \to \mathbb{R}$ we shall talk about the second derivative $f^{\Delta\Delta}$ provided f^Δ is differentiable on $\mathbb{T}^{\kappa 2} = (\mathbb{T}^\kappa)^\kappa$ with derivative $f^{\Delta\Delta} = \left(f^\Delta\right)^\Delta : \mathbb{T}^{\kappa 2} \to \mathbb{R}$. Similarly we define the higher order derivatives $f^{\Delta^n} : \mathbb{T}^{\kappa n} \to \mathbb{R}$.

Theorem 2.5. *(Banach Contraction Mapping Theorem)*
If $T : X \to X$ is contractive on a complete metric space X then T has a unique fixed point in X.

Theorem 2.6. *(Schauder's Fixed Point Thorem)*
Let L be a convex subset of a normed linear space E. Then each compact map $T : L \to L$ has a fixed point.

Let $I = [c, d]$ with $c < \rho(d)$. We define $I_c = [c, \infty)$ in case $sup\mathbb{T} = +\infty$. By $\mathcal{C}^B_{TS}(I_a)$ we mean the linear space of all continuous functions $f : I_c \to \mathbb{R}$ such that $sup_{t\in I_c}|f(t)| < \infty$.

Now we quote the time scales version of the Arzela-Ascoli theorem [1].

Theorem 2.7. *(Arzela-Ascoli Theorem)*
Let X be a subset of $\mathcal{C}^B_{TS}(I_a)$ having the following properties.
(i) X is bounded.
(ii) On every compact subinterval J of $[c, \infty)$ we have: For any $\epsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in J$, $|t_1 - t_2| < \delta$ implies $|f(t_1) - f(t_2)| < \epsilon$ for all $f \in X$.
(iii) For every $\epsilon > 0$ there exists $b \in I_c$ such that $t_1, t_2 \in [b, \infty)$ implies $|f(t_1) - f(t_2)| < \epsilon$ for all $f \in X$.
Then X is relatively compact.

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales. Let $C(\mathbb{T})$ denote the space of all continuous functions on the time scale $\mathbb{T}$.

Definition 2.8. By $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ we mean that $t_1 \in \mathbb{T}_1$ and $t_2 \in \mathbb{T}_2$ with the product topology on $\mathbb{T}_1 \times \mathbb{T}_2$.

Definition 2.9. Let $Z(\mathbb{T})$ be a function space on the time scale $\mathbb{T}$. For $x \in Z(\mathbb{T})$ we define

$$\|x\| = sup_{t\in\mathbb{T}}|x(t)|.$$

Definition 2.10. By $(x_1, x_2) \in X(\mathbb{T}_1) \times Y(\mathbb{T}_2)$ where X and Y are function spaces, we mean that $x_1 \in X(\mathbb{T}_1)$ and $x_2 \in Y(\mathbb{T}_2)$ with the product topology on $X(\mathbb{T}_1) \times Y(\mathbb{T}_2)$. For $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ we define

$$(x_1, x_2)(t_1, t_2) = (x_1(t_1), x_2(t_2)).$$

Definition 2.11. For $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$, we define

$$\|(t_1, t_2)\| = \|t_1\| + \|t_2\| = |t_1| + |t_2|.$$

Definition 2.12. For $(x_1, x_2) \in C(\mathbb{T}_1) \times C(\mathbb{T}_2)$, we define

$$\begin{aligned} \|(x_1, x_2)\| &= \|x_1\| + \|x_2\| \\ &= sup_{t_1\in\mathbb{T}_1}|x_1(t_1)| + sup_{t_2\in\mathbb{T}_2}|x_2(t_2)|. \end{aligned}$$

Definition 2.13. Let $(y_{11}, y_{12}) \in X(\mathbb{T}_1) \times Y(\mathbb{T}_2)$. We say that (y_{11}, y_{12}) is continuous on $\mathbb{T}_1 \times \mathbb{T}_2$ if for $\epsilon > 0$ there exists $\delta > 0$ such that for arbitrarily fixed $(t_{01}, t_{02}) \in \mathbb{T}_1 \times \mathbb{T}_2$ and $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ such that

$$\begin{aligned} \| & (t_1, t_2) - (t_{01}, t_{02})\| < \delta \\ \Rightarrow & \|(y_{11}, y_{12})(t_1, t_2) - (y_{11}, y_{12})(t_{01}, t_{02})\| < \epsilon. \end{aligned} \tag{2.1}$$

Definition 2.14. A sequence $(y_{n1}, y_{n2}) \in C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ is said to be cauchy sequence if for every $\epsilon > 0$ there exists N such that $\forall n1, n2, m1, m2 > N$ implies

$$\|(x_{n1}, x_{n2}) - (x_{m1}, x_{m2})\| < \epsilon.$$

Definition 2.15. A sequence $(y_{n1}, y_{n2}) \in X(\mathbb{T}_1) \times Y(\mathbb{T}_2)$ is said to be equicontinuous if for every $\epsilon > 0$ there is a $\delta > 0$, depending only on ϵ, such that for all (y_{n1}, y_{n2}) and all $(t_1, t_2), (t_1^{'}, t_2^{'}) \in \mathbb{T}_1 \times \mathbb{T}_2$ satisfying

$$\begin{aligned} \| & (t_1, t_2) - (t_1^{'}, t_2^{'})\| < \delta \\ \Rightarrow & \|(y_{n1}, y_{n2})(t_1, t_2) - (y_{n1}, y_{n2})(t_1^{'}, t_2^{'})\| < \epsilon. \end{aligned} \tag{2.2}$$

Definition 2.16. The space $X(\mathbb{T}_1) \times Y(\mathbb{T}_2)$ is said to convex if for every $(y_{11}, y_{12}), (y_{21}, y_{22}) \in X(\mathbb{T}_1) \times Y(\mathbb{T}_2)$, we have

$$\alpha(y_{11}, y_{12}) + (1 - \alpha)(y_{21}, y_{22}) \in X(\mathbb{T}_1) \times Y(\mathbb{T}_2) \text{ for } 0 < \alpha < 1.$$

3. Existence and Uniqueness of Solution for a Fourth Order Initial Value Problem associated with Nonlinear Singular Interface Problem

3.1. Definition of the Problem

Let $\mathbb{T}_1 = [0,a]_{\mathbb{T}}$(a time scale with end points 0 and a), $K_1 = [\sigma(a), l]_{\mathbb{T}}$(a time scale with end points $\sigma(a)$ and l), $\mathbb{T}_2 = {K_1}^{\kappa^4}$ where $a, \sigma(a), l < +\infty$. Also let (f_1, f_2) be nonlinear function tuple in $\mathcal{C}(\mathbb{T}_1 \times R^3) \times \mathcal{C}(\mathbb{T}_2 \times R^3)$. In this paper we consider the following IVP associated with singular interface problem(IVP-SIP).

$$\begin{aligned} y_1^{\Delta\Delta\Delta\Delta}(t) &= f_1(t, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta}),\ t \in \mathbb{T}_1 && (3.1)\\ y_2^{\Delta\Delta\Delta\Delta}(t) &= f_2(t, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta}),\ t \in {\mathbb{T}_2}^{\kappa^4} && (3.2) \end{aligned}$$

with the initial conditions

$$\begin{aligned} y_1(0) &= 0 && (3.3)\\ y_1^{\Delta}(0) &= 0 && (3.4)\\ y_1^{\Delta\Delta}(0) &= 0 && (3.5)\\ y_1^{\Delta\Delta\Delta}(0) &= 0 && (3.6) \end{aligned}$$

followed by the matching interface conditions

$$\begin{aligned} \rho_1 y_1(a) &= \rho_2 y_2(\sigma(a)), && (3.7)\\ \rho_3 y_1^{\Delta}(a) &= \rho_4 y_2^{\Delta}(\sigma(a)), && (3.8)\\ \rho_5 y_1^{\Delta\Delta}(a) &= \rho_6 y_2^{\Delta\Delta}(\sigma(a)), && (3.9)\\ \rho_7 y_1^{\Delta\Delta\Delta}(a) &= \rho_8 y_2^{\Delta\Delta\Delta}(\sigma(a)),\ \rho_i > 0,\ i = 1,2,3,4,5,6,7,8. && (3.10) \end{aligned}$$

3.2. Existence of Solution using Schauder's Fixed Point Theorem

In this section we prove the existence of solution for the IVP-SIP using Schauder's fixed point theorem.

Theorem 3.1. *If (f_1, f_2) is continuous and bounded, then there exists atleast one solution for the 4th order IVP-SIP.*(3.3)*-*(3.10)

Proof. Case I Let $t \in \mathbb{T}_1$

Then,

$$\begin{aligned}
y_1^{\Delta\Delta\Delta\Delta}(t) &= f_1(t, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta}) \\
y_1^{\Delta\Delta\Delta}(t) &= \int_0^t f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s + c_{11} \\
y_1^{\Delta\Delta}(t) &= \int_0^t \int_0^m f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s \Delta m + \int_0^t c_{11}\Delta s + c_{12} \\
y_1^{\Delta}(t) &= \int_0^t \int_0^m \int_0^r f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s \Delta r \Delta m \\
&+ \int_0^t \int_0^m c_{11}\Delta s \Delta m + \int_0^t c_{12}\Delta s + c_{13} \\
y_1(t) &= \int_0^t \int_0^m \int_0^r \int_0^d f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s \Delta d \Delta r \Delta m \\
&+ \int_0^t \int_0^m \int_0^r c_{11}\Delta s \Delta r \Delta m + \int_0^t \int_0^m c_{12}\Delta s \Delta m + \int_0^t c_{13}\Delta s + c_{14}
\end{aligned}$$

where c_{11} , c_{12} ,c_{13} and $c_{13}$are constants to be determined. By using the initial conditions (3.3),(3.4),(3.5)(3.6) we get

$$\begin{aligned}
y_1(0) = 0 &\Rightarrow c_{14} = 0 \\
y_1^{\Delta}(0) = 0 &\Rightarrow c_{13} = 0 \\
y_1^{\Delta\Delta}(0) = 0 &\Rightarrow c_{12} = 0 \\
y_1^{\Delta\Delta\Delta}(0) = 0 &\Rightarrow c_{11} = 0 \\
\Rightarrow y_1(t) &= \int_0^t \int_0^m \int_0^r \int_0^d f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s \Delta d \Delta r \Delta m.
\end{aligned}$$

Case II Let $t \in \mathbb{T}_2$

$$\begin{aligned}
y_2^{\Delta\Delta\Delta\Delta}(t) &= f_2(t, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta}) \\
y_2^{\Delta\Delta\Delta}(t) &= \int_{\sigma(a)}^{t} f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta})\Delta s + c_{21} \\
y_2^{\Delta\Delta}(t) &= \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m} f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta})\Delta s\Delta m + \int_{\sigma(a)}^{t} c_{21}\Delta s + c_{22} \\
y_2^{\Delta}(t) &= \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m}\int_{\sigma(a)}^{r} f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta})\Delta s\Delta r\Delta m \\
&+ \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m} c_{21}\Delta s\Delta m + \int_{\sigma(a)}^{t} c_{22}\Delta s + c_{23} \\
y_2(t) &= \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m}\int_{\sigma(a)}^{r}\int_{\sigma(a)}^{d} f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta})\Delta s\Delta d\Delta r\Delta m \\
&+ \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m}\int_{\sigma(a)}^{r} c_{21}\Delta s\Delta r\Delta m \\
&+ \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m} c_{22}\Delta s\Delta m + \int_{\sigma(a)}^{t} c_{23}\Delta s + c_{24}
\end{aligned}$$

where c_{21} , c_{22}, c_{23} and $c_{24}$are constants to be determined.

Now, by (3.7), we get

$$\begin{aligned}
\rho_1 y_1(a) &= \rho_2 y_2(\sigma(a)) \\
\Rightarrow c_{24} &= \frac{\rho_1}{\rho_2}\left(\int_0^a\int_0^m\int_0^r\int_0^d f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta d\Delta r\Delta m\right).
\end{aligned}$$

Also, by (3.8), we get

$$\begin{aligned}
\rho_3 y_1^{\Delta}(a) &= \rho_4 y_2^{\Delta}(\sigma(a)) \\
\Rightarrow c_{23} &= \frac{\rho_3}{\rho_4}\left(\int_0^a\int_0^m\int_0^r f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta r\Delta m\right).
\end{aligned}$$

Also,by (3.9),we get

$$\begin{aligned}
\rho_5 y_1^{\Delta\Delta}(a) &= \rho_6 y_2^{\Delta\Delta}(\sigma(a)) \\
\Rightarrow c_{22} &= \frac{\rho_5}{\rho_6}\left(\int_0^a\int_0^m f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta m\right).
\end{aligned}$$

Also,by (3.10),we get

$$\begin{aligned}
\rho_7 y_1^{\Delta\Delta}(a) &= \rho_8 y_2^{\Delta\Delta}(\sigma(a)) \\
\Rightarrow c_{21} &= \frac{\rho_7}{\rho_8}\left(\int_0^a f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\right).
\end{aligned}$$

Hence,

$$\begin{aligned} y_2(t) &= \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m}\int_{\sigma(a)}^{r}\int_{\sigma(a)}^{d} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta d\Delta r\Delta m \\ &+ \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m}\int_{\sigma(a)}^{r}\frac{\rho_7}{\rho_8}\left(\int_0^a f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\right)\Delta s'\Delta r\Delta m \\ &+ \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m}\frac{\rho_5}{\rho_6}\left(\int_0^a\int_0^m f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta m\right)\Delta s'\Delta m' \\ &+ \int_{\sigma(a)}^{t}\frac{\rho_3}{\rho_4}\left(\int_0^a\int_0^m\int_0^r f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta r\Delta m\right)\Delta s' \\ &+ \frac{\rho_1}{\rho_2}\left(\int_0^a\int_0^m\int_0^r\int_0^d f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta d\Delta r\Delta m\right). \end{aligned}$$

We now define the integral operator $T: C(\mathbb{T}_1)\times C(\mathbb{T}_2)\to C(\mathbb{T}_1)\times C(\mathbb{T}_2)$.

$$\begin{aligned} T(y_1,y_2) &= \Bigg(\int_0^{t_1}\int_0^m\int_0^r\int_0^d f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta d\Delta r\Delta m \\ &, \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\int_{\sigma(a)}^{d'} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta d'\Delta r'\Delta m' \\ &+ \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\frac{\rho_7}{\rho_8}\left(\int_0^a f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\right)\Delta s'\Delta r'\Delta m' \\ &+ \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\frac{\rho_5}{\rho_6}\left(\int_0^a\int_0^{m'} f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta m'\right)\Delta s'\Delta m' \\ &+ \int_{\sigma(a)}^{t_2}\frac{\rho_3}{\rho_4}\left(\int_0^a\int_0^{m'}\int_0^{r'} f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta r'\Delta m'\right)\Delta s' \\ &+ \frac{\rho_1}{\rho_2}\left(\int_0^a\int_0^{m'}\int_0^{r'}\int_0^{d'} f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta d'\Delta r'\Delta m'\right)\Bigg) \end{aligned}$$

where $t_1, m, r, d \in \mathbb{T}_1$ and $t_2, m', r', d' \in \mathbb{T}_2$. It is clear that (y_1, y_2) is a solution of IVP-SIP iff (y_1, y_2) solves the operator equation $(y_1, y_2) = T(y_1, y_2)$. In other words a fixed point for the operator $(y_1, y_2) = T(y_1, y_2)$ is a solution for the IVP-SIP.

We use Schauder's fixed point theorem to show the existence of a solution.

Claim 3.2. The space $C(\mathbb{T}_1)\times C(\mathbb{T}_2)$ is convex.

Let $(y_{11}, y_{12}), (y_{21}, y_{22}) \in C(\mathbb{T}_1)\times C(\mathbb{T}_2)$. For the space to be convex we need to show that

$$\alpha(y_{11}, y_{12}) + (1-\alpha)(y_{21}, y_{22}) \in C(\mathbb{T}_1)\times C(\mathbb{T}_2) \text{ for } \alpha < 1.$$

Since $(y_{11}, y_{12}) \in C(\mathbb{T}_1)\times C(\mathbb{T}_2)$(Definition(2.13)) for fixed $(t_{01}, t_{02}) \in \mathbb{T}_1\times\mathbb{T}_2$ and $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ such that $\forall \epsilon > 0\ \exists\ \delta > 0$ such that whenever

$$\begin{aligned}
\| \quad & (t_1, t_2) - (t_{01}, t_{02})\| < \delta(> 0) \\
i.e., \quad & |t_1 - t_{01}| + |t_2 - t_{02}| < \delta \\
\Rightarrow \quad & \|(y_{11}(t_1), y_{12}(t_2)) - (y_{11}(t_{01}), y_{12}(t_{02}))\| < \frac{\epsilon}{2\alpha} \\
i.e., \quad & sup_{t_1\in\mathbb{T}_1}|y_{11}(t_1) - y_{11}(t_{01})| + sup_{t_2\in\mathbb{T}_2}|y_{12}(t_2) - y_{12}(t_{02})| < \frac{\epsilon}{2\alpha}. \quad (3.11)
\end{aligned}$$

Similarly we can show that

$$sup_{t_1\in\mathbb{T}_1}|y_{21}(t_1) - y_{21}(t_{01})| + sup_{t_2\in\mathbb{T}_2}|y_{22}(t_2) - y_{22}(t_{02})| < \frac{\epsilon}{2\alpha}. \quad (3.12)$$

Now, let us consider

$$\begin{aligned}
&\Big[\alpha(y_{11}, y_{12}) + (1-\alpha)(y_{21}, y_{22})\Big] \\
&= \Big[(\alpha y_{11}, \alpha y_{12}) + ((1-\alpha)y_{21}, (1-\alpha)y_{22})\Big] \\
&= \Big[(\alpha y_{11} + (1-\alpha)y_{21}, \alpha y_{12} + (1-\alpha)y_{22})\Big]
\end{aligned}$$

We see that $\|\{(\alpha y_{11} + (1-\alpha)y_{21})(t_1), (\alpha y_{12} + (1-\alpha)y_{22})(t_2)\}$
$-\{(\alpha y_{11} + (1-\alpha)y_{21})(t_{01}), (\alpha y_{12} + (1-\alpha)y_{22})(t_{02})\}\|$

$$\begin{aligned}
\leq \quad & sup_{t_1\in\mathbb{T}_1}\Big|(\alpha y_{11} + (1-\alpha)y_{21})(t_1) - (\alpha y_{11} + (1-\alpha)y_{21})(t_{01})\Big| \\
+ \quad & sup_{t_2\in\mathbb{T}_2}\Big|(\alpha y_{12} + (1-\alpha)y_{22})(t_2) - (\alpha y_{12} + (1-\alpha)y_{22})(t_{02})\Big| \\
\leq \quad & sup_{t_1\in\mathbb{T}_1}\alpha|y_{11}(t_1) - y_{11}(t_{01})| + sup_{t_2\in\mathbb{T}_2}\alpha|y_{12}(t_2) - y_{12}(t_{02})| \\
+ \quad & sup_{t_1\in\mathbb{T}_1}(1-\alpha)|y_{21}(t_1) - y_{21}(t_{01})| + sup_{t_2\in\mathbb{T}_2}(1-\alpha)|y_{22}(t_2) - y_{22}(t_{02})| \\
< \quad & \alpha\frac{\epsilon}{2\alpha} + (1-\alpha)\frac{\epsilon}{2(1-\alpha)} = \epsilon
\end{aligned}$$

whenever, $|t_1 - t_{01}| + |t_2 - t_{02}| < \delta$. Hence $C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ is convex.

Claim 3.3. T is a completely continuous map.

We first show that T is continuous. We prove it by showing that T preserves convergence.

Indeed let (y_{n1}, y_{n2}) be a sequence of functions in $\mathcal{C}(\mathbb{T}_1) \times \mathcal{C}(\mathbb{T}_2)$ such that

$$lim_{n\to\infty}\|(y_{n1}, y_{n2}) - (y_1, y_2)\| \to 0.$$

The above equation implies that

$$\begin{aligned} lim_{n\to\infty}\|(y_{n1}-y_1, y_{n2}-y_2)\| &\to 0 \\ i.e., lim_{n\to\infty} sup_{t_1\in\mathbb{T}_1}|(y_{n1}-y_1)(t_1)| &\to 0 \\ \text{and } lim_{n\to\infty} sup_{t_2\in\mathbb{T}_2}|(y_{n2}-y_2)(t_2)| &\to 0. \end{aligned}$$

Let us consider
$\|T(y_{n1}, y_{n2}) - T(y_1, y_2)\|$

$$\begin{aligned}
&= sup_{t_1\in\mathbb{T}_1}\bigg|\int_0^{t_1}\int_0^{m}\int_0^{r}\int_0^{d} f_1(s, y_{n1}, y_{n1}^{\Delta}, y_{n1}^{\Delta\Delta}, y_{n1}^{\Delta\Delta\Delta})\Delta s\Delta d\Delta r\Delta m \\
&- \int_0^{t_1}\int_0^{m}\int_0^{r}\int_0^{d} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta d\Delta r\Delta m\bigg| \\
&+ sup_{t_2\in\mathbb{T}_2}\bigg|\bigg(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\int_{\sigma(a)}^{d'} f_2(s, y_{n2}, y_{n2}^{\Delta}, y_{n2}^{\Delta\Delta}, y_{n2}^{\Delta\Delta\Delta})\Delta s\Delta d'\Delta r'\Delta m' \\
&- \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\int_{\sigma(a)}^{d'} f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta})\Delta s\Delta d'\Delta r'\Delta m' \\
&+ \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'} \frac{\rho_7}{\rho_8}\bigg(\int_0^{a} f_1(s, y_{n1}, y_{n1}^{\Delta}, y_{n1}^{\Delta\Delta}, y_{n1}^{\Delta\Delta\Delta})\Delta s\bigg)\Delta s'\Delta r'\Delta m' \\
&- \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'} \frac{\rho_7}{\rho_8}\bigg(\int_0^{a} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\bigg)\Delta s'\Delta r'\Delta m' \\
&+ \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'} \frac{\rho_5}{\rho_6}\bigg(\int_0^{a}\int_0^{m'} f_1(s, y_{n1}, y_{n1}^{\Delta}, y_{n1}^{\Delta\Delta}, y_{n1}^{\Delta\Delta\Delta})\Delta s\Delta m\bigg)\Delta s'\Delta m' \\
&- \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'} \frac{\rho_5}{\rho_6}\bigg(\int_0^{a}\int_0^{m'} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta m\bigg)\Delta s'\Delta m' \\
&+ \int_{\sigma(a)}^{t_2} \frac{\rho_3}{\rho_4}\bigg(\int_0^{a}\int_0^{m'}\int_0^{r'} f_1(s, y_{n1}, y_{n1}^{\Delta}, y_{n1}^{\Delta\Delta}, y_{n1}^{\Delta\Delta\Delta})\Delta s\Delta r'\Delta m'\bigg)\Delta s' \\
&- \int_{\sigma(a)}^{t_2} \frac{\rho_3}{\rho_4}\bigg(\int_0^{a}\int_0^{m'}\int_0^{r'} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta r'\Delta m'\bigg)\Delta s' \\
&+ \frac{\rho_1}{\rho_2}\bigg(\int_0^{a}\int_0^{m'}\int_0^{r'}\int_0^{d'} f_1(s, y_{n1}, y_{n1}^{\Delta}, y_{n1}^{\Delta\Delta}, y_{n1}^{\Delta\Delta\Delta})\Delta s\Delta d'\Delta r'\Delta m'\bigg) \\
&- \frac{\rho_1}{\rho_2}\bigg(\int_0^{a}\int_0^{m'}\int_0^{r'}\int_0^{d'} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta d'\Delta r'\Delta m'\bigg)\bigg)\bigg|.
\end{aligned}$$

Since (f_1, f_2) is continuous on $\mathcal{C}(\mathbb{T}_1) \times \mathcal{C}(\mathbb{T}_2)$ we have

$$\begin{aligned} lim_{n\to\infty}|f_1(s, y_{n1}, y_{n1}^{\Delta}, y_{n1}^{\Delta\Delta}, y_{n1}^{\Delta\Delta\Delta}) - f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})| &\to 0, \\ lim_{n\to\infty}|f_2(s, y_{n2}, y_{n2}^{\Delta}, y_{n2}^{\Delta\Delta}, y_{n2}^{\Delta\Delta\Delta}) - f_1(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta})| &\to 0. \end{aligned}$$

Now $\|T(y_{n1}, y_{n2}) - T(y_1, y_2)\|$

$$\begin{aligned} \leq\ & sup_{t_1\in\mathbb{T}_1} \int_0^{t_1}\int_0^{m}\int_0^{r}\int_0^{d} |f_1(s, y_{n1}, y_{n1}^{\Delta}, y_{n1}^{\Delta\Delta}, y_{n1}^{\Delta\Delta\Delta}) \\ -\ & f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})|\Delta s\Delta d\Delta r\Delta m \\ +\ & sup_{t_2\in\mathbb{T}_2} \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\int_{\sigma(a)}^{d'} |f_2(s, y_{n2}, y_{n2}^{\Delta}, y_{n2}^{\Delta\Delta}, y_{n2}^{\Delta\Delta\Delta}) \\ -\ & f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta})|\Delta s\Delta d'\Delta r'\Delta m' \\ +\ & \left(\frac{\rho_7}{\rho_8}\right) sup_{t_2\in\mathbb{T}_2} \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\int_0^{a} |f_1(s, y_{n1}, y_{n1}^{\Delta}, y_{n1}^{\Delta\Delta}, y_{n1}^{\Delta\Delta\Delta}) \\ -\ & f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})|\Delta s\Delta s'\Delta r'\Delta m' \\ +\ & \left(\frac{\rho_5}{\rho_6}\right) sup_{t_2\in\mathbb{T}_2} \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^{a}\int_0^{m'} |f_1(s, y_{n1}, y_{n1}^{\Delta}, y_{n1}^{\Delta\Delta}, y_{n1}^{\Delta\Delta\Delta}) \\ -\ & f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})|\Delta s\Delta m\Delta s'\Delta m' \\ +\ & \left(\frac{\rho_3}{\rho_4}\right) sup_{t_2\in\mathbb{T}_2} \int_{\sigma(a)}^{t_2}\int_0^{a}\int_0^{m'}\int_0^{r'} |f_1(s, y_{n1}, y_{n1}^{\Delta}, y_{n1}^{\Delta\Delta}, y_{n1}^{\Delta\Delta\Delta}) \\ -\ & f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})|\Delta s\Delta r'\Delta m'\Delta s' \\ +\ & \left(\frac{\rho_1}{\rho_2}\right) \int_0^{a}\int_0^{m'}\int_0^{r'}\int_0^{d'} |(f_1(s, y_{n1}, y_{n1}^{\Delta}, y_{n1}^{\Delta\Delta}, y_{n1}^{\Delta\Delta\Delta}) \\ -\ & f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})|\Delta s\Delta d'\Delta r'\Delta m'. \end{aligned}$$

Hence, $lim_{n\to\infty}\|T(y_{n1}, y_{n2}) - T(y_1, y_2)\| \to 0$ proving that T is continuous.

Let

$$\begin{aligned} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta}) &\leq M_1, \text{ for some } M_1 > 0, \forall s \in \mathbb{T}_1, \\ f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta}) &\leq M_2, \text{ for some } M_2 > 0, \forall s \in \mathbb{T}_2. \end{aligned}$$

We now show that $T(\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2))$ is bounded and equicontinuous subset of $\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2)$. Let us assume that $\|(y_1,y_2)\|\leq M$. Then

$$\begin{aligned}
\|T(y_1,y_2)\| &\leq sup_{t_1\in\mathbb{T}_1}\int_0^{t_1}\int_0^{m}\int_0^{r}\int_0^{d}|f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})|\Delta s\Delta d\Delta r\Delta m\\
&+ sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_{\sigma(a)}^{r^{'}}\int_{\sigma(a)}^{d^{'}}|f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})|\Delta s\Delta d^{'}\Delta r^{'}\Delta m^{'}\\
&+ sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_{\sigma(a)}^{r^{'}}\frac{\rho_7}{\rho_8}\bigg(\int_0^{a}|f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},\\
&y_1^{\Delta\Delta\Delta})|\Delta s\bigg)\Delta s^{'}\Delta r^{'}\Delta m^{'}\\
&+ sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\frac{\rho_5}{\rho_6}\bigg(\int_0^{a}\int_0^{m'}|f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},\\
&y_1^{\Delta\Delta\Delta})|\Delta s\Delta m\bigg)\Delta s^{'}\Delta m^{'}\\
&+ sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\frac{\rho_3}{\rho_4}\bigg(\int_0^{a}\int_0^{m'}\int_0^{r'}|f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},\\
&y_1^{\Delta\Delta\Delta})|\Delta s\Delta r'\Delta m^{'}\bigg)\Delta s^{'}\\
&+ \frac{\rho_1}{\rho_2}\bigg(\int_0^{a}\int_0^{m^{'}}\int_0^{r^{'}}\int_0^{d^{'}}|f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})|\Delta s\Delta d'\Delta r^{'}\Delta m^{'}\bigg).
\end{aligned}$$

Since (f_1,f_2) is bounded we can conclude that there exists a $K>0$ independent of choice of (y_1,y_2) such that $\|T(y_1,y_2)\|\leq K$. Hence, $T(\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2))$ is bounded.

We next prove that $T(\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2))$ is equicontinuous subset of $\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2)$. We need to show that $\forall\epsilon>0\ \exists\ \delta>0$ such that whenever

$$\begin{aligned}
\| &\quad (t_1,t_2)-(t_1^{'},t_2^{'})\|<\delta\\
\Rightarrow &\quad \|T(y_1(t_1),y_2(t_2))-T(y_1(t_1^{'}),y_2(t_2^{'}))\|<\epsilon.
\end{aligned}$$

Let us assume that $|t_1 - t_1^{'}| + |t_2 - t_2^{'}| < \delta$. We see that $T(y_1(t_1), y_2(t_2)) - T(y_1(t_1^{'}), y_2(t_2^{'}))$

$$
\begin{aligned}
= \quad & \Bigg(\int_0^{t_1}\int_0^{m}\int_0^{r}\int_0^{d} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta d\Delta r\Delta m \\
- & \int_0^{t_1^{'}}\int_0^{m}\int_0^{r}\int_0^{d} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta d\Delta r\Delta m, \\
& \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_{\sigma(a)}^{r^{'}}\int_{\sigma(a)}^{d^{'}} f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta})\Delta s\Delta d'\Delta r^{'}\Delta m^{'} \\
- & \int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m^{'}}\int_{\sigma(a)}^{r^{'}}\int_{\sigma(a)}^{d^{'}} f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta})\Delta s\Delta d'\Delta r^{'}\Delta m^{'} \\
+ & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'} \frac{\rho_7}{\rho_8}\Big(\int_0^{a} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Big)\Delta s^{'}\Delta r'\Delta m^{'} \\
- & \int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'} \frac{\rho_7}{\rho_8}\Big(\int_0^{a} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Big)\Delta s^{'}\Delta r'\Delta m^{'} \\
. \ + & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'} \frac{\rho_5}{\rho_6}\Big(\int_0^{a}\int_0^{m'} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta m'\Big)\Delta s^{'}\Delta m^{'} \\
- & \int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m'} \frac{\rho_5}{\rho_6}\Big(\int_0^{a}\int_0^{m'} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta m'\Big)\Delta s^{'}\Delta m^{'} \\
. \ + & \int_{\sigma(a)}^{t_2} \frac{\rho_3}{\rho_4}\Big(\int_0^{a}\int_0^{m'}\int_0^{r'} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta r'\Delta m'\Big)\Delta s^{'} \\
- & \int_{\sigma(a)}^{t_2^{'}} \frac{\rho_3}{\rho_4}\Big(\int_0^{a}\int_0^{m'}\int_0^{r'} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta r'\Delta m'\Big)\Delta s^{'}\Bigg).
\end{aligned}
$$

Now

$$
\begin{aligned}
& \int_0^{t_1}\int_0^{m}\int_0^{r}\int_0^{d} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta d\Delta r\Delta m \\
- & \int_0^{t_1^{'}}\int_0^{m}\int_0^{r}\int_0^{d} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta d\Delta r\Delta m
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_0^{t_1}\int_0^m\int_0^r\int_0^d M_1\Delta s\Delta d\Delta r\Delta m - \int_0^{t_1'}\int_0^m\int_0^r\int_0^d M_1\Delta s\Delta d\Delta r\Delta m \\
&= \int_0^{t_1}\int_0^m\int_0^r M_1 d\Delta d\Delta r\Delta m - \int_0^{t_1'}\int_0^m\int_0^r M_1 d\Delta d\Delta r\Delta m \\
&= \int_0^{t_1}\int_0^m M_1(\frac{r^2}{2})\Delta r\Delta m - \int_0^{t_1'}\int_0^m M_1(\frac{r^2}{2})\Delta r\Delta m \\
&= \int_0^{t_1} M_1\frac{m^3}{6}\Delta m - \int_0^{t_1'} M_1\frac{m^3}{6}\Delta m \\
&= \frac{M_1}{24}\left(t_1^4 - t_1'^4\right) \\
&= \frac{M_1}{24}\left((t_1^2 - t_1'^2)(t_1^2 + t_1'^2)\right) \\
. &= \frac{M_1}{24}(t_1 - t_1')\left(1.(t_1 + t_1').(t_1^2 + t_1'^2)\right)
\end{aligned}
$$

Hence, whenever $|t_1 - t_1'| < \delta$ we have

$$
\begin{aligned}
\Bigg| &\int_0^{t_1}\int_0^m\int_0^r\int_0^d f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta d\Delta r\Delta m \\
- &\int_0^{t_1'}\int_0^m\int_0^r\int_0^d f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta d\Delta r\Delta m\Bigg| < \frac{\epsilon}{5}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\int_{\sigma(a)}^{d'} f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta})\Delta s\Delta d'\Delta r'\Delta m' \\
- &\int_{\sigma(a)}^{t_2'}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\int_{\sigma(a)}^{d'} f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta})\Delta s\Delta d'\Delta r'\Delta m'
\end{aligned}
$$

$$
\begin{aligned}
\leq & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_{\sigma(a)}^{r^{'}}\int_{\sigma(a)}^{d^{'}} M_2 \Delta s \Delta d^{'} \Delta r^{'} \Delta m^{'} \\
- & \int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m^{'}}\int_{\sigma(a)}^{r^{'}}\int_{\sigma(a)}^{d^{'}} M_2 \Delta s \Delta d^{'} \Delta r' \Delta m^{'} \\
= & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_{\sigma(a)}^{r^{'}} M_2 (d^{'} - \sigma(a)) \Delta d^{'} \Delta r^{'} \Delta m^{'} \\
- & \int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m^{'}}\int_{\sigma(a)}^{r^{'}} M_2 (d^{'} - \sigma(a)) \Delta d^{'} \Delta r^{'} \Delta m^{'} \\
= & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}} M_2 (\frac{r'^2}{2} - \sigma(a) r^{'} + \frac{\sigma(a)^2}{2}) \Delta r^{'} \Delta m^{'} \\
- & \int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m^{'}} M_2 (\frac{r'^2}{2} - \sigma(a) r^{'} + \frac{\sigma(a)^2}{2}) \Delta r^{'} \Delta m^{'} \\
= & \int_{\sigma(a)}^{t_2} M_2 (\frac{m'^3}{6} + \frac{\sigma(a)^2 m'}{2} - \frac{\sigma(a) m'^2}{2} - \frac{\sigma(a)^3}{6}) \Delta m^{'} \\
- & \int_{\sigma(a)}^{t_2^{'}} M_2 (\frac{m'^3}{6} + \frac{\sigma(a)^2 m'}{2} - \frac{\sigma(a) m'^2}{2} - \frac{\sigma(a)^3}{6}) \Delta m^{'}
\end{aligned}
$$

Hence, whenever $|t_2 - t_2^{'}| < \delta$ we have

$$
\begin{aligned}
\Bigg| & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_{\sigma(a)}^{r^{'}}\int_{\sigma(a)}^{d^{'}} f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta}) \Delta s \Delta d^{'} \Delta r^{'} \Delta m^{'} \\
- & \int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m^{'}}\int_{\sigma(a)}^{r^{'}}\int_{\sigma(a)}^{d^{'}} f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta}) \Delta s \Delta d^{'} \Delta r^{'} \Delta m^{'} \Bigg| < \frac{\epsilon}{5}.
\end{aligned}
$$

Now

$$
\begin{aligned}
& \int_{\sigma(a)}^{t_2} \frac{\rho_3}{\rho_4} \Bigg(\int_0^a \int_0^{m'} \int_0^{r'} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta}) \Delta s \Delta r' \Delta m^{'} \Bigg) \Delta s^{'} \\
- & \int_{\sigma(a)}^{t_2^{'}} \frac{\rho_3}{\rho_4} \Bigg(\int_0^a \int_0^{m'} \int_0^{r'} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta}) \Delta s \Delta r' \Delta m^{'} \Bigg) \Delta s^{'}
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{\rho_3}{\rho_4} \int_{\sigma(a)}^{t_2} \int_0^a M_1 \Bigg(\frac{m'^2}{2} \Delta m' \Bigg) \Delta s^{'} - \frac{\rho_3}{\rho_4} \int_{\sigma(a)}^{t_2'} \int_0^a M_1 \Bigg(\frac{m'^2}{2} \Delta m' \Bigg) \Delta s^{'} \\
= & \frac{\rho_3}{\rho_4} M_1 (\frac{a^3}{6})(t_2 - t_2^{'})
\end{aligned}
$$

Hence, whenever $|t_2 - t_2^{'}| < \delta$ we have

$$\begin{aligned} \Big| & \int_{\sigma(a)}^{t_2} \frac{\rho_3}{\rho_4}\Big(\int_0^a \int_0^{m'} \int_0^{r'} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s \Delta r' \Delta m^{'}\Big)\Delta s^{'} \\ - & \int_{\sigma(a)}^{t_2^{'}} \frac{\rho_3}{\rho_4}\Big(\int_0^a \int_0^{m'} \int_0^{r'} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s \Delta r' \Delta m^{'}\Big)\Delta s^{'}\Big| < \frac{\epsilon}{5}. \end{aligned}$$

Now

$$\begin{aligned} & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'} \frac{\rho_5}{\rho_6}\Big(\int_0^a \int_0^{m'} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s \Delta m\Big)\Delta s^{'}\Delta m^{'} \\ - & \int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m'} \frac{\rho_5}{\rho_6}\Big(\int_0^a \int_0^{m'} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s \Delta m\Big)\Delta s^{'}\Delta m^{'} \end{aligned}$$

$$\begin{aligned} &= \frac{\rho_5}{\rho_6}M_1\frac{a^2}{2}\Big(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\Delta m^{'}\Delta s^{'} - \int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m'}\Delta m^{'}\Delta s^{'}\Big) \\ &= \frac{\rho_5}{\rho_6}M_1\frac{a^2}{2}(t_2 - t_2{'})\Big(\frac{t_2 + t_2{'}}{2} - \sigma(a)(1)\Big) \end{aligned}$$

Hence, whenever $|t_2 - t_2^{'}| < \delta$ we have

$$\begin{aligned} \Big| & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'} \frac{\rho_5}{\rho_6}\Big(\int_0^a \int_0^{m'} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s \Delta m\Big)\Delta s^{'}\Delta m^{'} \\ - & \int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m'} \frac{\rho_5}{\rho_6}\Big(\int_0^a \int_0^{m'} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s \Delta m\Big)\Delta s^{'}\Delta m^{'}\Big| < \frac{\epsilon}{5}. \end{aligned}$$

Now

$$\begin{aligned} & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'} \frac{\rho_7}{\rho_8}\Big(\int_0^a f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Big)\Delta s^{'}\Delta r^{'}\Delta m^{'} \\ - & \int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'} \frac{\rho_7}{\rho_8}\Big(\int_0^a f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Big)\Delta s^{'}\Delta r^{'}\Delta m^{'} \end{aligned}$$

$$\begin{aligned} &= \frac{\rho_7}{\rho_8}M_1 a\Big(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\Delta s^{'}\Delta r^{'}\Delta m^{'} - \int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\Delta s^{'}\Delta r^{'}\Delta m^{'}\Big) \\ &= \frac{\rho_7}{\rho_8}M_1 a(t_2 - t_2^{'})\Big(\frac{1}{6}(t_2^2 + t_2 t_2^{'} + t_2'^2) + \frac{\sigma(a)}{2}(-1)(t_2 + t_2^{'}) + \frac{\sigma(a)^2}{2}(1)\Big) \end{aligned}$$

Hence, whenever $|t_2 - t_2^{'}| < \delta$ we have

$$\Bigg| \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'} \frac{\rho_7}{\rho_8}\Big(\int_0^a f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Big)\Delta s^{'}\Delta r^{'}\Delta m^{'}$$
$$- \int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'} \frac{\rho_7}{\rho_8}\Big(\int_0^a f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Big)\Delta s^{'}\Delta r^{'}\Delta m^{'}\Bigg| < \frac{\epsilon}{5}.$$

So, we see that

$$\|T(y_1(t_1), y_2(t_2)) - T(y_1(t_1^{'}), y_2(t_2^{'}))\| < \frac{\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} = \epsilon$$

whenever

$$\begin{aligned} \| \quad & (t_1, t_2) - (t_1^{'}, t_2^{'})\| < \delta \\ i.e., \quad & |t_1 - t_1^{'}| + |t_2 - t_2^{'}| < \delta. \end{aligned}$$

So, $T(\mathcal{C}(\mathbb{T}_1) \times \mathcal{C}(\mathbb{T}_2))$ is equicontinuous subset of $\mathcal{C}(\mathbb{T}_1) \times \mathcal{C}(\mathbb{T}_2)$.

Condition(iii) of Arzela-Acoli theorem can be seen from that fact that any $g_i(s, z_i) \in \mathcal{C}(\mathbb{T}_i \times \mathcal{C}(\mathbb{T}_i)),\ i = 1, 2$, is uniformly continuous on $\mathbb{T}_i$ as $\mathbb{T}_i$ is compact(since closed and bounded).

Thus T is compact by Arzela-Ascoli theorem. So from Schauder's fixed point theorem(2.6) a fixed point exists for the operator equation $(y_1, y_2) = Ty$. Hence a solution exists for the IVP-SIP. □

3.3. Existence and Uniqueness of Solution using Banach's Fixed Point Theorem

In this section, we prove the existence of a unique solution of IVP-SIP with certain restricted conditions on the interface constants. We use the Banach contraction principle.

Theorem 3.4. *Let $\int_{\mathbb{T}_i} f_i\Delta t < \infty$ and $f_i,\ i = 1, 2$ satisfy*

$$\begin{aligned} |f_1(t, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta}) - f_1(t, z_1, z_1^{\Delta}, z_1^{\Delta\Delta}, z_1^{\Delta\Delta\Delta})| \quad &\leq \quad |y_1 - z_1| \text{ for all } t \in \mathbb{T}_1 \\ & y_1(t), z_1(t) \in \mathbb{R} \qquad (3.13) \\ |f_2(t, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta}) - f_2(t, z_2, z_2^{\Delta}, z_2^{\Delta\Delta}, z_2^{\Delta\Delta\Delta})| \quad &\leq \quad |y_2 - z_2| \text{ for all } t \in \mathbb{T}_2 \\ & y_2(t), z_2(t) \in \mathbb{R} \qquad (3.14) \end{aligned}$$

and

$$\begin{aligned} \frac{\rho_1}{\rho_2}(\frac{a^4}{24}) + \frac{\rho_3}{\rho_4}(\frac{a^2 l}{6} - \frac{a^3\sigma(a)}{6}) + \frac{\rho_5}{\rho_6}(\frac{a^2 l^2}{4} - \frac{a^2\sigma(a)^2}{2} + \frac{a^2\sigma(a)^2}{4}) & \\ + \frac{\rho_7}{\rho_8}(\frac{al^3}{6} - \frac{a\sigma(a)^3}{2} + \frac{al\sigma(a)^2}{2} - \frac{a\sigma(a)^3}{6}) + \frac{a^4}{24} &< 1 \quad (3.15) \\ \frac{l^4}{24} - \frac{5\sigma(a)^4}{24} + \frac{l^2\sigma(a)^2}{4} &< 1 \quad (3.16) \end{aligned}$$

Then the IVP-SIP has a unique solution.

Proof. We use Banach Contraction Mapping Theorem (2.5) to prove the existence of unique fixed point for the operator equation.

Claim 3.5. The space $C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ is complete.

Let (x_{n1}, x_{n2}) be a cauchy sequence in $C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ and let $\epsilon > 0$ be given. Let $(t_{01}, t_{o2}) \in \mathbb{T}_1 \times \mathbb{T}_2$ be fixed. From the definition of continuity (2.13) for $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ we have

$$\|(t_1, t_2) - (t_{01}, t_{o2})\| < \delta(> 0)$$

implies that

$$\|(x_{n1}(t_1), x_{n2}(t_2)) - (x_{n1}(t_{01}), x_{n2}(t_{02}))\| < \frac{\epsilon}{3}.$$

i.e.,

$$\begin{aligned} \| &\quad (t_1 - t_{01}, t_2 - t_{02})\| \\ = &\quad |t_1 - t_{01}| + |t_2 - t_{02}| < \delta \end{aligned}$$

$$\begin{aligned} \Rightarrow &\quad \|(x_{n1}(t_1) - x_{n1}(t_{01}), x_{n2}(t_2) - x_{n2}(t_{02}))\| \\ = &\quad sup_{t_1 \in \mathbb{T}_1} |x_{n1}(t_1) - x_{n1}(t_{01})| \\ + &\quad sup_{t_2 \in \mathbb{T}_2} |x_{n2}(t_2) - x_{n2}(t_{02})| < \frac{\epsilon}{3}. \end{aligned} \tag{3.17}$$

Now we let $lim_{n \to \infty}(x_{n1}, x_{n2}) \to (x_1, x_2)$. Then $\exists\, \mathcal{N} > 0$ such that $\forall n1, n2 > \mathcal{N}$ we have

$$\begin{aligned} \| &\quad (x_{n1}, x_{n2}) - (x_1, x_2)\| \\ = &\quad \|(x_{n1} - x_1, x_{n2} - x_{n2})\| \\ = &\quad sup_{t_1 \in \mathbb{T}_1} |(x_{n1} - x_1)(t_1)| \\ + &\quad sup_{t_2 \in \mathbb{T}_2} |(x_{n2} - x_2)(t_2)| < \frac{\epsilon}{3}. \end{aligned} \tag{3.18}$$

For the space $C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ to be complete we need to show that (x_1, x_2) belongs to the space. That is

$$\begin{aligned} \| &\quad (t_1, t_2) - (t_{01}, t_{02})\| < \delta \\ \Rightarrow &\quad \|(x_1(t_1), x_2(t_2)) - (x_1(t_{01}), x_2(t_{02}))\| < \epsilon. \end{aligned}$$

Now let us consider $\|(x_1(t_1), x_2(t_2)) - (x_1(t_{01}), x_2(t_{02}))\|$

$$\begin{aligned} = &\quad \|(x_1(t_1) - x_1(t_{01}), x_2(t_2) - x_2(t_{02}))\| \\ = &\quad sup_{t_1 \in \mathbb{T}_1} |x_1(t_1) - x_1(t_{01})| + sup_{t_2 \in \mathbb{T}_2} |x_2(t_2) - x_2(t_{02})| \\ = &\quad sup_{t_1 \in \mathbb{T}_1} |x_1(t_1) - x_{n1}(t_1) + x_{n1}(t_1) - x_{n1}(t_{01}) + x_{n1}(t_{01}) - x_1(t_{01})| \\ + &\quad sup_{t_2 \in \mathbb{T}_2} |x_2(t_2) - x_{n2}(t_2) + x_{n2}(t_2) - x_{n2}(t_{02}) + x_{n2}(t_{02}) - x_2(t_{02})| \\ \leq &\quad sup_{t_1 \in \mathbb{T}_1} |x_1(t_1) - x_{n1}(t_1)| + sup_{t_2 \in \mathbb{T}_2} |x_2(t_2) - x_{n2}(t_2)| \\ + &\quad sup_{t_1 \in \mathbb{T}_1} |x_{n1}(t_1) - x_{n1}(t_{01})| + sup_{t_2 \in \mathbb{T}_2} |x_{n2}(t_2) - x_{n2}(t_{02})| \\ + &\quad sup_{t_1 \in \mathbb{T}_1} |(x_{n1} - x_1)(t_{01})| + sup_{t_2 \in \mathbb{T}_2} |(x_{n2} - x_2)(t_{02})| \\ < &\quad \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \text{ (since (3.17), (3.18))} \\ = &\quad \epsilon. \end{aligned}$$

Hence, $C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ is complete.

Claim 3.6. The map T is a contraction.

We see that
$\|T(x_1,x_2)-T(z_1,z_2)\|$

$$
\begin{aligned}
= \quad & sup_{t_1\in\mathbb{T}_1}\Bigg|\int_0^{t_1}\int_0^{m}\int_0^{r}\int_0^{d} f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta d\Delta r\Delta m \\
- \quad & \int_0^{t_1}\int_0^{m}\int_0^{r}\int_0^{d} f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta d\Delta r\Delta m\Bigg| \\
+ \quad & sup_{t_2\in\mathbb{T}_2}\Bigg|\Bigg(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\int_{\sigma(a)}^{d'} f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\Delta s\Delta d'\Delta r'\Delta m' \\
- \quad & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\int_{\sigma(a)}^{d'} f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta d'\Delta r'\Delta m' \\
+ \quad & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\frac{\rho_7}{\rho_8}\Bigg(\int_0^{a} f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Bigg)\Delta s'\Delta r'\Delta m' \\
- \quad & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\frac{\rho_7}{\rho_8}\Bigg(\int_0^{a} f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Bigg)\Delta s'\Delta r'\Delta m' \\
+ \quad & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\frac{\rho_5}{\rho_6}\Bigg(\int_0^{a}\int_0^{m'} f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta m\Bigg)\Delta s'\Delta m' \\
- \quad & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\frac{\rho_5}{\rho_6}\Bigg(\int_0^{a}\int_0^{m'} f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta m\Bigg)\Delta s'\Delta m' \\
+ \quad & \int_{\sigma(a)}^{t_2}\frac{\rho_3}{\rho_4}\Bigg(\int_0^{a}\int_0^{m'}\int_0^{r'} f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta r'\Delta m'\Bigg)\Delta s' \\
- \quad & \int_{\sigma(a)}^{t_2}\frac{\rho_3}{\rho_4}\Bigg(\int_0^{a}\int_0^{m'}\int_0^{r'} f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta r'\Delta m'\Bigg)\Delta s' \\
+ \quad & \frac{\rho_1}{\rho_2}\Bigg(\int_0^{a}\int_0^{m'}\int_0^{r'}\int_0^{d'} f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta d'\Delta r'\Delta m'\Bigg) \\
- \quad & \frac{\rho_1}{\rho_2}\Bigg(\int_0^{a}\int_0^{m'}\int_0^{r'}\int_0^{d'} f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta d'\Delta r'\Delta m'\Bigg)\Bigg)\Bigg|.
\end{aligned}
$$

Now let us consider each of the term in the above equation separately.

$$
\begin{aligned}
& sup_{t_1\in\mathbb{T}_1}\Bigg|\int_0^{t_1}\int_0^{m}\int_0^{r}\int_0^{d} f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta d\Delta r\Delta m \\
- \quad & \int_0^{t_1}\int_0^{m}\int_0^{r}\int_0^{d} f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta d\Delta r\Delta m\Bigg|
\end{aligned}
$$

$$\begin{aligned}
&\leq & sup_{t_1\in\mathbb{T}_1}\int_0^{t_1}\int_0^m\int_0^r\int_0^d |f_1(s,x_1,x_1^\Delta,x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\\
&- & f_1(s,z_1,z_1^\Delta,z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})|\Delta s\Delta d\Delta r\Delta m\\
&\leq & sup_{t_1\in\mathbb{T}_1}\int_0^{t_1}\int_0^m\int_0^r\int_0^d |x_1-z_1|\Delta s\Delta d\Delta r\Delta m \text{ (since (3.13))}\\
&\leq & sup_{t_1\in\mathbb{T}_1}\left(sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\right)\int_0^{t_1}\int_0^m\int_0^r\int_0^d \Delta s\Delta d\Delta r\Delta m\\
&= & sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\ sup_{t_1\in\mathbb{T}_1}\left(\int_0^{t_1}\int_0^m \frac{r^2}{2}\Delta r\Delta m\right)\\
&= & sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\ sup_{t_1\in\mathbb{T}_1}\left(\int_0^{t_1}\frac{m^3}{6}\Delta m\right)\\
&= & sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\ sup_{t_1\in\mathbb{T}_1}\left(\frac{t_1^4}{24}\right).
\end{aligned}$$

So we have

$$\begin{aligned}
sup_{t_1\in\mathbb{T}_1}\bigg|&\int_0^{t_1}\int_0^m\int_0^r\int_0^d f_1(s,x_1,x_1^\Delta,x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta d\Delta r\Delta m\\
&-\int_0^{t_1}\int_0^m\int_0^r\int_0^d f_1(s,z_1,z_1^\Delta,z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta d\Delta r\Delta m\bigg|\\
&\leq sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\ sup_{t_1\in\mathbb{T}_1}\frac{{t_1}^4}{24}. \qquad (3.19)
\end{aligned}$$

Also,

$$\begin{aligned}
sup_{t_2\in\mathbb{T}_2}\bigg|\bigg(&\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\int_{\sigma(a)}^{d'} f_2(s,x_2,x_2^\Delta,x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\Delta s\Delta r'\Delta m'\\
- &\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\int_{\sigma(a)}^{d'} f_2(s,z_2,z_2^\Delta,z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta d'\Delta r'\Delta m'\bigg)\bigg|
\end{aligned}$$

$$\begin{aligned}
&\leq sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\int_{\sigma(a)}^{d'}|f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\\
&- f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})|\Delta s\Delta d'\Delta r'\Delta m'\\
&\leq sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\int_{\sigma(a)}^{d'}|x_2-z_2|\Delta s\Delta d'\Delta r'\Delta m' \text{ (since (3.14))}\\
&\leq sup_{t_2\in\mathbb{T}_2}\left(sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\right)\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\int_{\sigma(a)}^{d'}\Delta s\Delta d'\Delta r'\Delta m'\\
&= sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\ sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}(d'-\sigma(a))\Delta d'\Delta r'\Delta m'.\\
&= sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\ sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\left(\frac{r'^2}{2}-\sigma(a)r'+\frac{\sigma(a)^2}{2}\right)\Delta r'\Delta m'.\\
&= sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\ sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\left(\frac{m'^3}{6}-\frac{\sigma(a)m'^2}{2}+\frac{\sigma(a)^2m'}{2}-\frac{\sigma(a)^3}{6}\right)\Delta m'.\\
&= sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\ sup_{t_2\in\mathbb{T}_2}\left(\frac{t_2^4}{24}-\frac{\sigma(a)t_2^3}{6}+\frac{\sigma(a)^2t_2^2}{4}-\frac{\sigma(a)^3t_2}{6}+\frac{\sigma(a)^4}{24}\right)
\end{aligned}$$

Hence we have

$$\begin{aligned}
&sup_{t_2\in\mathbb{T}_2}\Bigg|\Bigg(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\int_{\sigma(a)}^{d'}f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\Delta s\Delta r'\Delta m'\\
&-\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\int_{\sigma(a)}^{d'}f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta d'\Delta r'\Delta m'\Bigg)\Bigg|\\
&\leq sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\ sup_{t_2\in\mathbb{T}_2}\left(\frac{t_2^4}{24}-\frac{\sigma(a)t_2^3}{6}+\frac{\sigma(a)^2t_2^2}{4}-\frac{\sigma(a)^3t_2}{6}+\frac{\sigma(a)^4}{24}\right). \quad (3.20)
\end{aligned}$$

Now

$$\begin{aligned}
&sup_{t_2\in\mathbb{T}_2}\Bigg|\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\frac{\rho_7}{\rho_8}\left(\int_0^a f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\right)\Delta s'\Delta r'\Delta m'\\
&-\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\frac{\rho_7}{\rho_8}\left(\int_0^a f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\right)\Delta s'\Delta r'\Delta m'\Bigg|
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{\rho_7}{\rho_8}\right) sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\int_0^a |f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\\
&- f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})|\Delta s\Delta s'\Delta r'\Delta m'\\
&\leq \left(\frac{\rho_7}{\rho_8}\right) sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\int_0^a |x_1-z_1|\Delta s\Delta s'\Delta r'\Delta m'\\
&\leq sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\left(\frac{\rho_7}{\rho_8}\right) sup_{t_2\in\mathbb{T}_2}(a)\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\Delta s'\Delta r'\Delta m'\\
&= sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\left(\frac{\rho_7}{\rho_8}\right) sup_{t_2\in\mathbb{T}_2}\left(\frac{at_2^3}{6}-\frac{a\sigma(a)t_2^2}{2}+\frac{a\sigma(a)^2t_2}{2}-\frac{a\sigma(a)^3}{6}\right)
\end{aligned}$$

which implies that

$$\begin{aligned}
& sup_{t_2\in\mathbb{T}_2}\left|\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\frac{\rho_7}{\rho_8}\left(\int_0^a f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\right)\Delta s'\Delta r'\Delta m'\right.\\
&- \left.\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{r'}\frac{\rho_7}{\rho_8}\left(\int_0^a f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\right)\Delta s'\Delta r'\Delta m'\right|
\end{aligned}$$

$$\leq sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\left(\frac{\rho_7}{\rho_8}\right) sup_{t_2\in\mathbb{T}_2}\left(\frac{at_2^3}{6}-\frac{a\sigma(a)t_2^2}{2}+\frac{a\sigma(a)^2t_2}{2}-\frac{a\sigma(a)^3}{6}\right). \quad (3.21)$$

Now

$$\begin{aligned}
& sup_{t_2\in\mathbb{T}_2}\left|\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\frac{\rho_5}{\rho_6}\left(\int_0^a\int_0^{m'} f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta m\right)\Delta s'\Delta m'\right.\\
&- \left.\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\frac{\rho_5}{\rho_6}\left(\int_0^a\int_0^{m'} f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\right)\Delta s'\Delta m'\right|
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\frac{\rho_5}{\rho_6}\right) sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a\int_0^{m'} |f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\\
&- f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})|\Delta s\Delta m\Delta s'\Delta m'\\
&\leq \left(\frac{\rho_5}{\rho_6}\right) sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a\int_0^{m'} |x_1-z_1|\Delta s\Delta m\Delta s'\Delta m'\\
&\leq sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\left(\frac{\rho_5}{\rho_6}\right) sup_{t_2\in\mathbb{T}_2}\frac{a^2}{2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\Delta m'\Delta s'\\
&= sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\left(\frac{\rho_5}{\rho_6}\right) sup_{t_2\in\mathbb{T}_2}\left(\frac{a^2t_2^2}{4}-\frac{a^2\sigma(a)t_2}{2}+\frac{a^2\sigma(a)^2}{4}\right)
\end{aligned}$$

which implies that

$$sup_{t_2\in\mathbb{T}_2}\Bigg|\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\frac{\rho_5}{\rho_6}\Bigg(\int_0^a\int_0^{m'} f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta m\Bigg)\Delta s^{'}\Delta m^{'} - \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\frac{\rho_5}{\rho_6}\Bigg(\int_0^a\int_0^{m'} f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Bigg)\Delta s^{'}\Delta m^{'}\Bigg|$$
$$\leq sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\left(\frac{\rho_5}{\rho_6}\right)sup_{t_2\in\mathbb{T}_2}\left(\frac{a^2t_2^2}{4}-\frac{a^2\sigma(a)t_2}{2}+\frac{a^2\sigma(a)^2}{4}\right). \quad (3.22)$$

Now

$$sup_{t_2\in\mathbb{T}_2}\Bigg|\int_{\sigma(a)}^{t_2}\frac{\rho_3}{\rho_4}\Bigg(\int_0^a\int_0^{m'}\int_0^{r^{'}} f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta r^{'}\Delta m'\Bigg)\Delta s^{'} - \int_{\sigma(a)}^{t_2}\frac{\rho_3}{\rho_4}\Bigg(\int_0^a\int_0^{m'}\int_0^{r^{'}} f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta r'\Delta m^{'}\Bigg)\Delta s^{'}\Bigg|$$

$$\begin{aligned}
&\leq \left(\frac{\rho_3}{\rho_4}\right)sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_0^a\int_0^{m'}\int_0^{r^{'}}|f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\\
&- f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})|\Delta s\Delta r'\Delta m^{'}\Delta s^{'}\\
&\leq \left(\frac{\rho_3}{\rho_4}\right)sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_0^a\int_0^{m'}\int_0^{r^{'}}|x_1-z_1|\Delta s\Delta r'\Delta m^{'}\Delta s^{'}\\
&\leq sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\left(\frac{\rho_3}{\rho_4}\right)sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_0^a\int_0^{m'}\int_0^{r^{'}}\Delta s\Delta r'\Delta m^{'}\Delta s^{'}\\
&= sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\left(\frac{\rho_3}{\rho_4}\right)sup_{t_2\in\mathbb{T}_2}\left(\frac{t_2a^3}{6}-\frac{\sigma(a)a^3}{6}\right)
\end{aligned}$$

which implies that

$$sup_{t_2\in\mathbb{T}_2}\Bigg|\int_{\sigma(a)}^{t_2}\frac{\rho_3}{\rho_4}\Bigg(\int_0^a\int_0^{m'}\int_0^{r^{'}} f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta r^{'}\Delta m'\Bigg)\Delta s^{'} - \int_{\sigma(a)}^{t_2}\frac{\rho_3}{\rho_4}\Bigg(\int_0^a\int_0^{m'}\int_0^{r^{'}} f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta r'\Delta m^{'}\Bigg)\Delta s^{'}\Bigg|$$
$$\leq sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\left(\frac{\rho_3}{\rho_4}\right)sup_{t_2\in\mathbb{T}_2}\left(\frac{t_2a^3}{6}-\frac{\sigma(a)a^3}{6}\right). \quad (3.23)$$

Finally we see that

$$\Bigg|\frac{\rho_1}{\rho_2}\Bigg(\int_0^a\int_0^{m^{'}}\int_0^{r^{'}}\int_0^{d^{'}} f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta d'\Delta r^{'}\Delta m^{'}\Bigg) - \frac{\rho_1}{\rho_2}\Bigg(\int_0^a\int_0^{m^{'}}\int_0^{r^{'}}\int_0^{d^{'}} f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta d'\Delta r^{'}\Delta m^{'}\Bigg)\Bigg|$$

$$\leq \left(\frac{\rho_1}{\rho_2}\right) sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)| \int_0^a\int_0^{m^{'}}\int_0^{r^{'}}\int_0^{d^{'}} \Delta s\Delta d'\Delta r^{'}\Delta m^{'}.$$

So we have

$$\left|\frac{\rho_1}{\rho_2}\left(\int_0^a\int_0^{m^{'}}\int_0^{r^{'}}\int_0^{d^{'}} f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta d'\Delta r'\Delta m^{'}\right)\right.$$
$$\left.-\frac{\rho_1}{\rho_2}\left(\int_0^a\int_0^{m^{'}}\int_0^{r^{'}}\int_0^{d^{'}} f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta d'\Delta r^{'}\Delta m^{'}\right)\right|$$
$$\leq sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\left(\frac{\rho_1}{\rho_2}\right)\frac{a^4}{24}. \quad (3.24)$$

From the equations (3.19)-(3.24) we can conclude that
$\|T(x_1,x_2)-T(z_1,z_2)\|$

$$\begin{aligned}
&\leq sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\ sup_{t_1\in\mathbb{T}_1}\frac{{t_1}^4}{24}\\
&+ sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\ sup_{t_2\in\mathbb{T}_2}\left(\frac{t_2^4}{24}-\frac{\sigma(a)t_2^3}{6}+\frac{\sigma(a)^2t_2^2}{4}-\frac{\sigma(a)^3t_2}{6}+\frac{\sigma(a)^4}{24}\right).\\
&+ sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\left(\frac{\rho_7}{\rho_8}\right)sup_{t_2\in\mathbb{T}_2}\left(\frac{at_2^3}{6}-\frac{a\sigma(a)t_2^2}{2}+\frac{a\sigma(a)^2t_2}{2}-\frac{a\sigma(a)^3}{6}\right)\\
&+ sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\left(\frac{\rho_5}{\rho_6}\right)sup_{t_2\in\mathbb{T}_2}\left(\frac{a^2t_2^2}{4}-\frac{a^2\sigma(a)t_2}{2}+\frac{a^2\sigma(a)^2}{4}\right).\\
&+ sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\left(\frac{\rho_3}{\rho_4}\right)sup_{t_2\in\mathbb{T}_2}\left(\frac{t_2a^3}{6}-\frac{\sigma(a)a^3}{6}\right)\\
&+ sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\left(\frac{\rho_1}{\rho_2}\right)\frac{a^4}{24}.
\end{aligned}$$

$$\Rightarrow \|T(x_1,x_2)-T(z_1,z_2)\| \leq K_1\left(sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\right)+K_2\left(sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\right)$$

where

$$\begin{aligned}
K_1 &= sup_{t_1\in\mathbb{T}_1}\frac{{t_1}^4}{24}+\left(\frac{\rho_3}{\rho_4}\right)sup_{t_2\in\mathbb{T}_2}\left(\frac{t_2a^3}{6}-\frac{\sigma(a)a^3}{6}\right).\\
&+ \left(\frac{\rho_5}{\rho_6}\right)sup_{t_2\in\mathbb{T}_2}\left(\frac{a^2t_2^2}{4}-\frac{a^2\sigma(a)t_2}{2}+\frac{a^2\sigma(a)^2}{4}\right).\\
&+ \left(\frac{\rho_7}{\rho_8}\right)sup_{t_2\in\mathbb{T}_2}\left(\frac{at_2^3}{6}-\frac{a\sigma(a)t_2^2}{2}+\frac{a\sigma(a)^2t_2}{2}-\frac{a\sigma(a)^3}{6}\right).\\
&+ sup_{t_2\in\mathbb{T}_2}\left(\frac{\rho_1}{\rho_2}\right)\frac{a^4}{24},\\
K_2 &= sup_{t_2\in\mathbb{T}_2}\left(\frac{t_2^4}{24}-\frac{\sigma(a)t_2^3}{6}+\frac{\sigma(a)^2t_2^2}{4}-\frac{\sigma(a)^3t_2}{6}+\frac{\sigma(a)^4}{24}\right).
\end{aligned}$$

Let $K = max\{K_1, K_2\}$. Then $K < 1$(since (3.15)-(3.16)). We now have

$$\begin{aligned}
&\|T(x_1,x_2) - T(z_1,z_2)\| \\
&\qquad \leq \; K\Big(sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)| + sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\Big) \\
&\qquad = \; K\|(x_1-z_1, x_2-z_2)\| \\
&\qquad = \; K\|(x_1,x_2)-(z_1,z_2)\|.
\end{aligned}$$

Since $K < 1$, by Banach Contraction mapping theorem (2.5) we have a unique fixed point for $(y_1, y_2) = T(y_1, y_2)$. Hence a unique solution exists for the IVP-SIP. □

Remark 3.7. The above theorems can be proved for IVPs for the Interface II and Interface III with suitable changes in the notations.

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