On Normed Space Valued Total Paranormed Orlicz Space of Null Sequences and its Topological Structures

Narayan Prasad Pahari

Associate Professor,Central Department of Mathematics,

Tribhuvan University, Kirtipur, Kathmandu, Nepal .

ABSTRACT

In this paper,we introduce and study a new class $c_0(S, M, \bar{u})$ of normed space S valued sequences using Orlicz function M as a generalization of basic space of null complex sequences c_0. Beside the investigation pertaining to the containment relations of the class $c_0(S, M, \bar{u})$ for the various values of $\bar{u}$ our primary interest is to explore its linear topological structures when topologized it with suitable natural paranorm.

Keywords - **Paranormed Space, Sequence space, Orlicz Space, Normal space, GK-space.**

I. Introduction

Before proceeding with the main results, we recall some of the basic notations and definitions that are used in this paper.

The notion of paranormed space is closely related to linear metric space , see Wilansky [1]. The studies of paranorm on sequence spaces were initiated by Maddox [6] and many others. Srivastava et al [8, 9], Basariv and Altundag [10],Pahari [12,13], Parashar and Choudhary [16],Bhardwaj and Bala [18], and many others further studied various types of paranormed spaces of sequences and functions.

Definition 1: A paranormed space (S, G) is a linear space S with zero element θ together with a function $G : S \to \boldsymbol{R}_+$ (called a *paranorm* on S) which satisfies the following axioms:

PN_1: $G(\theta) = 0$;

PN_2: $G(s) = G(-s)$, for all $s \in S$;

PN_3: $G(s + t) \leq G(s) + G(t)$, for all $s, t \in S$; and

PN_4: Scalar multiplication is continuous i.e., if $<\gamma_n>$ is a sequence of scalars with $\gamma_n \to \gamma$ as $n \to \infty$ and $<s_n>$ a sequence of vectors with $G(s_n - s) \to 0$ as $n \to \infty$ then $G(\gamma_n s_n - \gamma s) \to 0$ as $n \to \infty$.

Note that the continuity of scalar multiplication i.e. PN_4 is equivalent to

(i) if $G(s_n) \to 0$ and $\gamma_n \to \gamma$ as $n \to \infty$, then $G(\gamma_n s_n) \to 0$ as $n \to \infty$;and

(ii) if $\gamma_n \to 0$ as $n \to \infty$ and s be any element in S, then $G(\gamma_n s) \to 0$, see Wilansky [1].

A paranorm is called total if $G(s) = 0$ implies $s = \theta$, see Wilansky [1].

Definition 2: Let S be a normed space over $\boldsymbol{C}$,the field of complex numbers. Let $\omega(S)$ denotes the linear space of all sequences $\bar{s} = <s_k>$ with $s_k \in S$, $k \geq 1$ with usual coordinate wise operations

i.e., $\bar{s} + \bar{t} = <s_k + t_k>$ and $\gamma \bar{s} = <\gamma s_k>$,for all $\bar{s}, \bar{t} \in \omega(S)$ and $\gamma \in \boldsymbol{C}$.

We shall denote $\omega(C)$ by ω . Any linear subspace of ω is called a *sequence space*.

Further if $\bar{\gamma} = <\gamma_k> \in \omega$ and $\bar{s} \in \omega(S)$, we shall write $\bar{\gamma}\,\bar{s} = <\gamma_k s_k>$.

Definition 3: By an Orlicz function we mean a continuous, non decreasing and convex function $M: [0,\infty) \to [0,\infty)$ satisfying

$$M(0) = 0,\ M(x) > 0 \text{ for } x > 0 \text{ and } M(x) \to \infty \text{ as } x \to \infty .$$

Note that an Orlicz function is always unbounded. An Orlicz function satisfies the inequality

$M(\gamma x) \le \gamma\ M(x)$ for all γ satisfying $0 < \gamma < 1$.

Note that an Orlicz function is always unbounded. For example, the function $\xi(x) = x^{p}$ is an Orlicz function if $p > 1$.

An Orlicz function M is said to satisfy Δ_2-condition for all values of $x \ge 0$, if there exists a constant $T > 0$ such that $M(2x) \le T\,M(x)$. The Δ_2-condition is equivalent to the satisfaction of inequality

$$M(r\,x) \le T\,r\,M(x) \text{ for all values of } x \text{ and for } r > 1,$$ see [11].

Lindenstrauss and Tzafriri (see, [7]) used the idea of Orlicz function to construct the sequence space ℓ_M of scalars $< s_k >$ such that $\sum_{k=1}^{\infty} M\left(\frac{||s_k||^{u_k}}{r}\right) < \infty$ for some $r > 0$. They proved that the space ℓ_M equipped with the norm defined by

$$||\,\overline{s}||_M = \inf\left\{ r > 0: \quad \sum_{k=1}^{\infty} M\left(\frac{|s_k|}{r}\right) \le 1 \right\}$$

becomes a Banach space. Clearly the space ℓ_M is closely related to the sequence space ℓ_p which is an Orlicz sequence space with $M(x) = x^{p}$, $1 \le p < \infty$. Subsequently various types of topological structures in sequence spaces using Orlicz function have been introduced and studied, for instances we refer a few [2], [3], [5], [8], [9], [12], [13], [15], [16], [17], [18], [19] and [20].

Definition 4: A sequence space S is said to be normal if $\overline{s} = < s_k > \in S$ and $\overline{\gamma} = < \gamma_k >$ a sequence of scalars with $|\gamma_k| \le 1$, for all $k \ge 1$, then $\overline{\gamma}\,\overline{s} = < \gamma_k\, s_k > \in S$.

In studying various properties of a vector valued sequence space (see, [14]) , we have the following definitions:

Definition 5: A normed space S - valued topological sequence space $V(S)$ equipped with the linear topology $\Im$ is said to be a GK-space if the map $\pi_k : V(S) \to S$, $\pi_k(\overline{s}) = s_k$, is continuous for each k.

Subsequently, various types of sequence spaces in normed space were introduced and studied in different directions ,for instances, see [2], [4] , [8] , [12] and [14].

II. MAIN RESULTS

Let $\overline{u} = < u_k >$ and $\overline{v} = < v_k >$ be any sequences of strictly positive real numbers.We now introduce the following class of Banach space S- valued sequences

$$c_0(S, M, \overline{u}) = \{\overline{s} = < s_k > : s_k \in S,\ k \ge 1 \text{ and } M\left(\frac{||\,s_k\,||}{r}\right)^{u_k} \to 0 \text{ as } k \to \infty, \text{ for some } r > 0\} \ldots(2.1)$$

Further when $u_k = 1$ for all k, then $c_0(S, M, \overline{u})$ will be denoted by $c_0(S, M)$.

Beside studying the class (2.1), we now introduce and study a new subclass $\overline{c_0}\,(S, M, \overline{u})$ of $c_0\,(S, M, \overline{u})$ as follows:

$$\overline{c_0}\,(S, M, \overline{u}) = \{\overline{s} = < s_k > : s_k \in S,\ M\left(\frac{||\,s_k\,||}{r}\right)^{u_k} \to 0 \text{ as } k \to \infty \text{, for every } r > 0\}. \ \ldots(2.2)$$

In this paper, we investigate some inclusion relations between the classes $(S, M, \overline{u})$ arising in terms of different $\overline{u}$ and then investigate some results that characterize the linear topological structures of the class $(S, M, \overline{u})$ by endowing it with a suitable natural paranorm.

Following inequality will also be used in this paper :

$$|\,s + t\,|^{u_k} \le Q\{|s|^{u_k} + |t|^{u_k}\},$$

where $s, t \in \mathbf{C}$, $0 < u_k \le \sup_k u_k = L$, and $\boldsymbol{Q} = \max(1, 2^{L-1})$ and write $z_k = \frac{v_k}{u_k}$, $k \ge 1$.

Theorem 2.1: If M satisfies Δ_2 - condition then $c_0\,(S, M, \overline{u}) = \overline{c_0}\,(S, M, \overline{u})$.

Proof:

To prove the theorem, it suffices to show that

$$c_0\,(S, M, \overline{u}) \subseteq \overline{c_0}\,(S, M, \overline{u}),$$

since its reverse inclusion is always true.

Let $\bar{s} \in c_0(S, M, \bar{u})$. Then for some $r > 0$, $M\left(\frac{\| s_k \|}{r}\right)^{u_k} \to 0$ as $k \to \infty$.

Let us consider an arbitrary $r_1 > 0$. If $r \le r_1$, then obviously we have

$$M\left(\frac{\| s_k \|}{r_1}\right)^{u_k} \le M\left(\frac{\| s_k \|}{r}\right)^{u_k} \to 0 \text{ as } k \to \infty.$$

and hence we get $\bar{s} \in \bar{c}_0(S, M, \bar{u})$. But on the other hand, if $r > r_1$, so that $\frac{r}{r_1} > 1$ then by using Δ_2 condition of M, we get

$$M\left(\frac{\| s_k \|}{r_1}\right)^{u_k} = M\left(\frac{r}{r_1} \frac{\| s_k \|}{r}\right)^{u_k} \le T.\frac{r}{r_1} M\left(\frac{\| s_k \|}{r}\right)^{u_k} \to 0 \text{ as } k \to \infty,$$

where T is the number involved in Δ_2- condition . Hence $\bar{s} \in \bar{c}_0(S, M, \bar{u})$ and therefore

$c_0(S, M, \bar{u}) \subseteq \bar{c}_0(S, M, \bar{u})$. This completes the proof.

Theorem 2.2: If $c_0(S, M, \bar{u})$ forms a linear space over the field of complex numbers $\boldsymbol{C}$,then $< u_k >$ is bounded above.

Proof:

Assume that $c_0(S, M, \bar{u})$ is a linear space over $\boldsymbol{C}$ but $\sup_k u_k = \infty$. Then there exists a sequence $< k(n) >$ of positive integers satisfying $1 \le k(n) < k(n+1)$, $n \ge 1$, for which

$$u_{k(n)} > n \text{ ,for each } n \ge 1. \qquad \ldots(2.3)$$

Now, corresponding to $s \in S$ with $\| s \| = 1$, we define a sequence $\bar{s} = < s_k >$ by

$$s_k = \begin{cases} n^{-2/u_{k(n)}} s, \text{ for } k = k(n), n \ge 1 \\ \theta, \text{ otherwise.} \end{cases} \qquad \ldots(2.4)$$

Let $r > 0$. Then for each $k = k(n)$, $n \ge 1$, we have

$$M\left(\frac{\| s_k \|}{r}\right)^{u_k} = M\left(\frac{\| n^{-2/u_{k(n)}} s \|}{r}\right)^{u_{k(n)}} = M\left(\frac{\| s \|}{n^2 r}\right)^{u_{k(n)}}$$

$$\le \frac{1}{n^2} M\left(\frac{1}{r}\right)$$

and $M\left(\frac{\| s_k \|}{r}\right)^{u_k} = 0$, for $k \ne k(n)$, $n \ge 1$,

showing that $\bar{s} \in c_0(S, M, \bar{u})$. But on the other hand in view of (2.3) and (2.4) for $k = k(n)$, $n \ge 1$, $r > 0$ with scalar $\alpha = 4$ using non decreasing property of M ,we have

$$M\left(\frac{\|\alpha s_k\|}{r}\right)^{u_k} = M\left(\frac{\| 4 n^{-2/u_{k(n)}} s \|}{r}\right)^{u_{k(n)}}$$

$$\ge M\left(\frac{4^n}{n^2 r}\right) \ge M\left(\frac{1}{r}\right).$$

This shows that $\alpha \bar{s} \notin c_0(S, M, \bar{u})$,which contradicts our assumption and the theorem is proved.

Theorem 2.3: $c_0(S, M, \bar{u})$ forms a linear space over $\boldsymbol{C}$ if $< u_k >$ is bounded above.

Proof:

Assume that $\sup_k u_k = L < \infty$. Let $\bar{s} = < s_k >$, $\bar{t} = < t_k > \in c_0(S, M, \bar{u})$ and $\alpha, \beta \in \boldsymbol{C}$. Then there exist $r_1 > 0$ and $r_2 > 0$ such that

$$M\left(\frac{\|s_k\|}{r_1}\right)^{u_k} \to 0 \quad \text{and} \quad M\left(\frac{\|t_k\|}{r_2}\right)^{u_k} \to 0, \text{ as } k \to \infty .$$

We now choose $r_3 > 0$ such that $2\,Q\,r_1 \max(1, |\alpha|^L) \le r_3$ and $2\,Q\,r_2 \max(1, |\beta|^L) \le r_3$. For such r_3 , using non decreasing and convex properties of M , we have

$$M\left(\frac{||\alpha\, s_k+\beta t_k||^{u_k}}{r_3}\right) \le M\left(\frac{Q||\alpha s_k||^{u_k}+Q||\beta t_k||^{u_k}}{r_3}\right)$$
$$= M\left(\frac{Q|\alpha|^{u_k}||s_k||^{u_k}}{r_3}+\frac{Q|\beta|^{u_k}||t_k||^{u_k}}{r_3}\right)$$
$$\le M\left(\frac{Q\, max(1,|\alpha|^{L})\,||s_k||^{u_k}}{r_3}+\frac{Q\, max(1,|\beta|^{L})\,||t_k||^{u_k}}{r_3}\right)$$
$$\le M\left(\frac{1}{2\,r_1}||s_k||^{u_k}+\frac{1}{2\,r_2}||t_k||^{u_k}\right)$$
$$\le \frac{1}{2}M\left(\frac{||s_k||^{u_k}}{r_1}\right)+\frac{1}{2}M\left(\frac{||t_k||^{u_k}}{r_2}\right)\to 0,\text{ as } k\to\infty .$$

This implies that $c_0\,(S, M, \bar{u})$ forms a linear space over $\boldsymbol{C}$. This completes the proof.
After combining the theorem 2.2 and 2.3, we get

Theorem 2.4: $c_0\,(S, M, \bar{u})$ forms a linear space over C if and only if $< u_k >$ is bounded above.

Theorem 2.5: The space $c_0\,(S, M, \bar{u})$ forms a normal.

Proof:

Let $\bar{s} = < s_k > \in c_0(S, M, \bar{u})$. So that $M\left(\frac{||\,s_k\,||^{u_k}}{r}\right)\to 0$ as $k\to\infty$ for some $r > 0$.

Let $< \alpha_k >$ be a sequence of scalars such that $|\alpha_k| \le 1$ for all $k \ge 1$. Since M is non-decreasing, we have

$$M\left(\frac{||\alpha_k s_k||^{u_k}}{r}\right) = M\left(\frac{|\alpha_k|^{u_k}\,||\,s_k\,||^{u_k}}{r}\right) \le M\left(\frac{||\,s_k\,||^{u_k}}{r}\right) \to 0 \text{ as } k\to\infty,$$

and hence $<\alpha_k s_k> \in c_0(S, M, \bar{u})$. So $c_0(S, M, \bar{u})$ forms a normal.

Theorem 2.6: If $c_0(S, M, \bar{u}) \subseteq (S, M, \bar{v})$,then $< z_k >$ has positive limit inferior.

Proof:

Assume that $c_0(S, M, \bar{u}) \subseteq c_0(S, M, \bar{v})$ but $\lim\inf_k z_k = 0$. Then there exists a sequence $< k(n) >$ of positive integers such that $1 \le k(n) < k(n+1)$, for which

$$n\, v_{k(n)} < u_{k(n)}\ ,\text{ for each } n \ge 1. \qquad \ldots(2.5)$$

Now, corresponding to $s \in S$ with $||\,s\,|| = 1$, we define a sequence $\bar{s} = < s_k >$ by

$$s_k = \begin{cases} n^{-1/u_{k(n)}}\, s\ , \text{ for } k = k\,(n),\ n \ge 1 \\ \theta, \text{ otherwise.}\end{cases} \qquad \ldots(2.6)$$

Let $r > 0$. Then for each $k = k(n), n \ge 1$, we have

$$M\left(\frac{||\,s_k\,||^{u_k}}{r}\right) = M\left(\frac{||n^{-1/u_{k(n)}}\ s\,||^{u_{k(n)}}}{r}\right) = M\left(\frac{||\,s\,||^{u_{k(n)}}}{n\,r}\right) \le \frac{1}{n}\,M\left(\frac{1}{r}\right)$$

and $M\left(\frac{||\,s_k\,||^{u_k}}{r}\right) = 0$, for $k \ne k(n)$, $n \ge 1$,

showing that $\bar{s} \in c_0(S, M, \bar{u})$. But on the other hand for each $k = k(n)$, $n \ge 1$, in view of (2.5) and (2.6) using non decreasing property of M ,we have

$$M\left(\frac{||\,s_k\,||^{v_k}}{r}\right) = M\left(\frac{||n^{-1/\,u_{k(n)}}\ s\,||^{v_{k(n)}}}{r}\right) \ge M\left(\frac{1}{r\,n^{1/n}}\right) \ge M\left(\frac{1}{r\sqrt{e}}\right),$$

This shows that $\bar{s} \notin c_0(S, M, \bar{v})$, a contradiction. This completes the proof.

Theorem 2.7: $c_0(S, M, \bar{u}) \subseteq (S, M, \bar{v})$ if $< z_k >$ has positive limit inferior.

Proof:

Assume that $\liminf_k z_k > 0$. Then there exists a $m > 0$ such that $v_k > m\, u_k$, for all sufficiently large values of k.

Let $\bar{s} = < s_k > \in c_0(S, M, \bar{u})$. Then for some $r > 0$,

$$M\left(\frac{\| s_k \|^{u_k}}{r}\right) \to 0 \text{ as } k \to \infty.$$

Hence for a given $\varepsilon > 0$, if we choose $0 < \eta < 1$ satisfying $\eta^{m}\, M\left(\frac{1}{r}\right) < \varepsilon$, then we have

$$M\left(\frac{\| s_k \|^{u_k}}{r}\right) < M\left(\frac{\eta}{r}\right), \quad \text{for all sufficiently large values of } k.$$

Since M is non decreasing, therefore for all large values of k

$\| s_k\|^{u_k} < \eta < 1$ and so $\| s_k\| < 1$.

Hence using the convexity of M, we have

$$M\left(\frac{\| s_k \|^{v_k}}{r}\right) \le M\left(\frac{[\| s_k \|^{u_k}]^m}{r}\right) \le M\left(\frac{\eta^{m}}{r}\right) \le \eta^{m}\, M\left(\frac{1}{r}\right) < \varepsilon,$$

for all sufficiently large values of k. This implies that $\bar{s} \in c_0(S, M, \bar{v})$.

Hence $c_0(S, M, \bar{u}) \subseteq c_0(S, M, \bar{v})$. This completes the proof.

After combining the theorem 2.6 and 2.7, we get

Theorem 2.8: $c_0(S, M, \bar{u}) \subseteq (S, M, \bar{v})$ if and only if $< z_k >$ has positive limit inferior.

In the following example, we show that $c_0(S, M, \bar{u})$ may strictly be contained in $c_0(S, M, \bar{v})$ in spite of the satisfaction of the condition of Theorem 2.8 .

Example: 2.9

Let S be a Banach space and consider a sequence $\bar{s} = < s_k >$ defined by

$s_k = k^{-k} s$, if $k = 1, 2, 3, \ldots$,where $s \in S$ such that $\| s\| = 1$. … (2.7)

Further, let $u_k = k^{-1}$, if k is odd integer, $u_k = k^{-2}$, if k is even integer, $v_k = k^{-1}$ for all values of k,

Further, $z_k = \dfrac{v_k}{u_k} = 1$, if k is odd integers, $z_k = k$, if k is even integers. Therefore $\liminf_k z_k > 0$. Hence the condition of Theorem 2.8 is satisfied. Let $r > 0$ and in view of (2.7) , we have

$$M\left(\frac{\| s_k \|^{v_k}}{r}\right) = M\left(\frac{\| k^{-k} s \|^{1/k}}{r}\right) \le \frac{1}{k}\, M\left(\frac{1}{r}\right) \to 0 \text{ as } k \to \infty,$$

showing that $\bar{s} \in c_0(S, M, \bar{v})$. But for even integer k,

$$M\left(\frac{\| s_k \|^{u_k}}{r}\right) = M\left(\frac{\| k^{-k} s\|^{1/k^2}}{r}\right) = M\left(\frac{(1/k)^{1/k}}{r}\right) > M\left(\frac{1}{2k\, r}\right).$$

This implies that $\bar{s} = < s_k > \notin c_0(S, M, \bar{u})$. Thus the containment of $c_0(S, M, \bar{u})$ in $c_0(S, M, \bar{v})$ is strict inspite of the satisfaction of the condition of the Theorem 2.8.

Analogous to the proof of the Theorem 2.8, we have

Theorem 2.10: $c_0(S, M, \bar{v}) \subseteq c_0(S, M, \bar{u})$ if and only if $< z_k >$ has finite limit superior.

On combining the Theorems 2.8 and 2. 10, one obtain

Theorem 2.11: $c_0(S, M, \bar{u}) = c_0(S, M, \bar{v})$ if and only if $0 < \liminf_k z_k \leq \limsup_k z_k < \infty$.

In what follows we shall take $< u_k >$ as bounded, $\sup_k u_k = L < \infty$ and $\inf_k u_k = l > 0$.

Denote $w_k = \dfrac{u_k}{L}$ and consider a set

$$A(\bar{s}) = \{ r > 0 : \sup_k M\left(\frac{||s_k||}{r}^{w_k}\right) \leq 1\}, \text{ for } \bar{s} = < s_k > \in c_0(S, M, \bar{u}). \quad \ldots (2.8)$$

Consider a real valued function G on $c_0(S, M, \bar{u})$ defined by

$$G(\bar{s}) = \inf \{ r > 0 : \sup_k M\left(\frac{||s_k||}{r}^{w_k}\right) \leq 1 \}, \bar{s} = < s_k > \in c_0(S, M, \bar{u}). \quad \ldots (2.9)$$

We prove below that $c_0(S, M, \bar{u})$ with respect to G forms a paranormed space.

Theorem 2.12: $c_0(S, M, \bar{u})$ forms a paranormed space with respect to G.

Proof:

For $\bar{s} = < s_k >, \bar{t} = < t_k > \in c_0(S, M, \bar{u})$, obviously $M(\bar{\theta}) = 0$ and $M(-\bar{s}) = M(\bar{s})$ follows.

Now in view of (2.8), for $\bar{s}, \bar{t} \in c_0(S, M, \bar{u})$, consider $r_1 \in A(\bar{s})$ and $r_2 \in A(\bar{t})$ and $r_3 = r_1 + r_2$. Then clearly by the convexity of M we have

$$M\left(\frac{||s_k + t_k||}{r_3}^{w_k}\right) \leq \frac{r_1}{r_3} \sup_k M\left(\frac{||s_k||}{r_1}^{w_k}\right) + \frac{r_2}{r_3} \sup_k M\left(\frac{||t_k||}{r_2}^{w_k}\right) \leq 1.$$

This shows that $r_3 = r_1 + r_2 \in A(\bar{s} + \bar{t})$. Thus $G(\bar{s} + \bar{t}) \leq r_1 + r_2$ for each $r_1 \in A(\bar{s})$ and $r_2 \in A(\bar{t})$ implies that $G(\bar{s} + \bar{t}) \leq G(\bar{s}) + G(\bar{t})$.

Finally we show the continuity of scalar multiplication. Let $\bar{s}^{(n)} = < s_k^{(n)} >$ be a sequence in $c_0(S, M, \bar{u})$ such that $G(\bar{s}^{(n)}) \to 0$ as $n \to \infty$ and $< \alpha_n >$ a sequence of scalars such that $\alpha_n \to \alpha$. Then we have

$$G(\alpha_n \bar{s}^{(n)}) = \inf\left\{ r : \sup_k M\left(\frac{||\alpha_n\ s_k^{(n)}||}{r}^{w_k}\right) \leq 1\right\}$$

$$\leq \inf\left\{ r : \sup_k M\left(\frac{H^{w_k} ||\ s_k^{(n)}||}{r}^{w_k}\right) \leq 1\right\}$$

where $H = \sup_n|\alpha_n|$ and $|\alpha_n|^{w_k} \leq H^{w_k}$ is used. Thus for $t = \max(1, H)$, we get

$$G(\alpha_n \bar{s}^{(n)}) \leq \inf\left\{ r : \sup_k M\left(\frac{t\,||\ s_k^{(n)}||}{r}^{w_k}\right) \leq 1\right\}.$$

Now if we take $r = tz$, then

$$G(\alpha_n \bar{s}^{(n)}) \leq \inf\left\{ tz : \sup_k M\left(\frac{||s_k^{(n)}||}{t}^{w_k}\right) \leq 1\right\} = z \times G(\bar{s}^{(n)})$$

implies that $G(\alpha_n \bar{s}^{(n)}) \to 0$, as $G(\bar{s}^{(n)}) \to 0$ as $n \to \infty$.

Next let $\alpha_n \to 0$ as $n \to \infty$ and $\bar{s} = < s_k >$ be any element in $c_0(S, M, \bar{u})$. We show that $G(\alpha_n \bar{s}) \to 0$. Now for $0 < \varepsilon < 1$, we can find a positive integer N such that $|\alpha_n| \leq \varepsilon$ for all $n \geq N$. In view of $\inf_k u_k = l > 0$, we get $|\alpha_n|^{w_k} \leq \varepsilon^{l/L}$. This shows that for each $n \geq N$, we have

$$M\left(\frac{||\ \alpha_n s_k||}{r}^{w_k}\right) \leq M\left(\frac{\varepsilon^{l/L}\ ||\ s_k||}{r}^{w_k}\right)$$

and consequently we get $A(\varepsilon^{l/L} \bar{s}) \subseteq A(\alpha_n \bar{s})$. Hence we have

$$\inf \{ r : r \in A(\alpha_n \bar{s}) \} \leq \inf \{ r : r \in A(\varepsilon^{l/L} \bar{s}) \}$$

$$= \varepsilon^{l/L} \inf\{ r : r \in A(\overline{s})\}$$

which shows that $G(\alpha_n \overline{s}) \leq \varepsilon^{l/L} G(\overline{s})$ for all $n \geq N$, i.e., $G(\alpha_n \overline{s}) \to 0$ as $n \to \infty$.

Hence $c_0(S, M, \overline{u})$ forms a paranormed space. This completes the poof.

Theorem 2.13: $c_0(S, M, \overline{u})$ forms a total paranormed space with respect to G.

Proof:

For $\overline{s} = < s_k > \in c_0(S, M, \overline{u})$,suppose that $G(\overline{s}) = 0$. Then for every $\varepsilon > 0$, there exists some r_ε $(0 < r_\varepsilon < \varepsilon)$, such that

$$\sup_k M\left(\frac{||s_k||^{w_k}}{r_\varepsilon}\right) \leq 1 .$$

This shows that

$$\sup_k M\left(\frac{||s_k||^{w_k}}{\varepsilon}\right) \leq 1, \text{ for every } \varepsilon > 0.$$

This is possible only when $||s_k||^{w_k} = 0$ for each $k \geq 1$. Hence $\overline{s} = \mathbf{0}$.

Hence in view of Theorem 2.12, $(c_0(S, M, \overline{u}), G)$ forms a total paranormed space.

Theorem 2.14: Let $\overline{u} = < u_k >$ such that $\sup_k u_k < \infty$ and S be a normed space. Then $(c_0(S, M, \overline{u}), G)$ forms a GK–space.

Proof:

Let $\overline{s} = < s_k > \in c_0(S, M, \overline{u})$.Then by definition of paranorm G defined in (2.9), we see that

$$\sup_k M\left(\frac{||s_k||^{w_n}}{G(\overline{s})}\right) \leq 1 \quad \text{and hence } M\left(\frac{||s_k||^{w_n}}{G(\overline{s})}\right) \leq 1 .$$

Let k_0 be a fixed positive real number such that $M(k_0) \geq 1$, then $M\left(\frac{||s_k||^{w_k}}{G(\overline{s})}\right) \leq M(k_0)$.

Since M is non-decreasing therefore $||s_k||^{w_k} < k_0\, G(\overline{s})$

or $||s_k|| < [k_0\ G(\overline{s})]^{1/w_k}$

and so $||\pi_k(\overline{s})|| = ||s_k|| < [k_0\, G(\overline{s})]^{1/w_k}$

shows that $\pi_k : c_0(S, M, \overline{u}) \to S$, where $\pi_k(\overline{s}) = s_k$ for each $s_k \in S$, $k \geq 1$,is continuous and hence $(c_0(S, M, \overline{u}), G)$ forms a GK- space. This completes the poof.

III. Conclusions

In this paper, we have examined some conditions that characterize the linear topological structures and containment relations on the Orlicz Space of Null Sequences. In fact, these results can be used for further generalization to investigate other properties of the spaces of null sequence using Orlicz function.

References

[1] A. Wilansky, Modern methods in topological vector spaces , Mc Graw_Hill Book Co.Inc.New York , 1978.

[2] C.A. Bektas, and Y. Altin , Sequence space $l_M(p, q, s)$ on semi normed spaces, Indian J. Pure Appl. Math. **34** (4) , (2003),529–534.

[3] D. Ghosh, and P.D. Srivastava, On some vector valued sequence spaces using Orlicz function, Glasnik Matematicki, **34** (54), (1999), 253–261.

[4] E. Kolk, Topologies in generalized Orlicz sequence spaces, Filomat, **25**(4), (2011), 191–211.

[5] E. Savas, and F. Patterson, An Orlicz extension of some new sequence spaces, Rend. Instit. Mat. Univ. Trieste , **37**, (2005), 145–154.

[6] I.J. Maddox,. Some properties of paranormed sequence spaces, London, J. Math. Soc. **2**(1), (1969), 316–322.

[7] J. Lindenstrauss, and L.Tzafriri, Classical Banach spaces, Springer-Verlag, New York , 1977.

[8] J.K. Srivastava, and N.P. Pahari, On Banach space valued sequence space $c_0(X, M, \bar{\lambda}, \overline{p}, L)$ defined by Orlicz function, Jour. of Rajasthan Academy of Physical Sciences, **11**(2) , (2011), 103-116.

[9] J.K. Srivastava, and N.P. Pahari, On 2-Banach space valued paranormed sequence space $c_0(X, M, ||.,.||, \bar{\lambda}, \overline{p})$ defined by Orlicz function, Jour. of Rajasthan Academy of Physical Sciences, **12**(3) , (2013), 319-338.

[10] M. Basariv, and S. Altundag, On generalized paranormed statistically convergent sequence spaces defined by Orlicz function, Handawi. Pub. Cor., J. of Inequality and Applications, **2009** (2009), 1-13.

[11] M.A. Krasnosel'skiî, and Y.B. Rutickiî, Convex functions and Orlicz spaces, P. Noordhoff Ltd-Groningen-The Netherlands, 1961.

[12] N.P. Pahari, On certain topological structures of paranormed Orlicz space $(S((X, ||.||), \Phi, \overline{\alpha}, \overline{u}), F)$ of vector valued sequences, International Jour. of Mathematical Archive , **4** (11), (2011), 231-241.

[13] N.P. Pahari , On certain topological structures of normed space valued generalized Orlicz function space. International Jour. of Scientific Engineering and Research (IJSER), **2(1),** (2014), 61–66.

[14] P.K. Kamthan, and M. Gupta, Sequence spaces and series , Lecture notes in pure and applied mathematics, Marcel Dekker, New York, 1981.

[15] Rao, K. C. & Subremanina, N.. The Orlicz space of entire sequences, IJMass, **68**, (2004), 3755– 3764.

[16] S.D. Parashar, and B. Choudhary, Sequence spaces defined by Orlicz functions, Indian J. Pure Appl. Math., **25**(4), (1994), 419–428.

[17] V.A. Khan, On a new sequence space defined by Orlicz functions, Common. Fac. Sci. Univ. Ank-series, **57**(2), (2008), 25–33.

[18] V.N. Bhardwaj, and I. Bala, Banach space valued sequence space $\ell_M(X, p)$, Int. J. of Pure and Appl. Maths., **41**(5), (2007), 617–626.

[19] V. Karakaya, Some new sequence spaces defined by a sequence of Orlicz functions, *Taiwanese J. of Maths*. **9**(4),(2005),617–627.

[20] W.H. Ruckle, Sequence spaces , Pitman Advanced Publishing Programme, 1981.