# Numerical Solution of Fuzzy Differential Equations by Extended Runge-Kutta Method and the Dependency Problem 

K. Kanagarajan, S. Muthukumar, S. Indrakumar<br>Department of Mathematics<br>Sri Ramakrishna Mission Vidyalaya College of Arts and Science<br>Coimbatore-641 020, Tamilnadu, India


#### Abstract

In this paper we use extended Runge-Kutta-like formulae of order four (ERK4) and of order five (ERK5) by taking into account the dependency problem that arises in fuzzy setting. This method is adopted to solve the dependency problem in fuzzy computation. Examples are presented to illustrate the theory.


Keywords -The Extended Runge-Kutta method, Fuzzy initial value problem, Dependency problem in fuzzy computation.

## I. Introduction

Fuzzy Differential Equations (FDEs) are used in modeling problems in science and engineering. Most of the problems in science and engineering require the solutions of FDEs which are satisfied by fuzzy initial conditions, therefore a Fuzzy Initial Value Problem(FIVP) occurs and should be solved. Fuzzy set was first introduced by Zadeh [19]. Since then, the theory has been developed and it is now emerged as an independent branch of Applied Mathematics. The elementary fuzzy calculus based on the extension principle was studied by Dubois and Prade [11]. Seikkala and Kaleva [17, 13] have discussed FIVP. Buckley and Feuring [10] compared the solutions of FIVPs which where obtained using different derivatives. The numerical solutions of FIVP by Euler's method was studied by Ma et al. [15]. Abbasbandy and Allviranloo [1,2] proposed the Taylor method and the fourth order RungeKutta method for solving FIVPs. Palligkinis et al. [16] applied the Runge-Kutta method for more general problems and proved the convergence for $n$-stage Runge-Kutta method. Recently Ghazanfari and Shakerami [12] studied the numerical solutions of fuzzy differential equations by ERK4. The dependency problem in fuzzy computation was discussed by Ahmad and Hasan [4] and they used Euler's method based on Zadeh's extension principle for finding the numerical solution of FIVPs. Omar and Hasan [7], adopted the same computation method to derive the fourth order Runge-Kutta method for FIVP. In this paper we study the dependency problem in fuzzy computations by using ERK4 and ERK5.

## II. PRELIMINARY CONCEPTS

In this section, we give some basic definitions.
Definition 2.1. Subset $\tilde{A}$ of a universal set X is said to be a fuzzy set if a membership function $\mu_{\tilde{\mathrm{A}}}(\mathrm{x})$ takes each object in X onto the interval [ 0,1$]$. The function $\mu_{\tilde{\mathrm{A}}}(\mathrm{x})$ is the possibility degrees to which each object is compatible with the properties that characterized the group.
A fuzzy set $\tilde{A} \subseteq X$ can also be presented as a set of ordered pairs

$$
\begin{equation*}
\tilde{A}=\left\{\left(x, \mu_{\tilde{A}}(x)\right): x \in X\right\}, \tag{1}
\end{equation*}
$$

The support, the core and the height of A are respectively

$$
\begin{align*}
\operatorname{supp}(\tilde{A}) & =\left\{x \in X: x>\mu_{\tilde{A}}(x)\right\},  \tag{2}\\
\operatorname{core}(\tilde{A}) & =\left\{x \in X: \mu_{\tilde{A}}(x)=1\right\},  \tag{3}\\
\operatorname{hgt}(\tilde{A}) & =\sup _{x \in X} \mu_{\tilde{A}}(x) . \tag{4}
\end{align*}
$$

Definition 2.2. A fuzzy number is a convex fuzzy subset $A$ of $R$, for which the following conditions are satisfied:
(i) $\tilde{A}$ is normalized. i.e. $\operatorname{hgt}(\tilde{A})=1$;
(ii) $\mu_{\tilde{A}}(x)$ are upper semicontinuous;
(iii) $\left\{x \in R: \mu_{\tilde{A}}(x)=\alpha\right\}$ are compact sets for $0<\alpha \leq 1$, and
(iv) $\left\{x \in R: \mu_{\tilde{A}}(x)=\alpha\right\}$ are also compact sets for $0<\alpha \leq 1$.

Definition 2.3. If $F(R)$ is the set of all fuzzy numbers, and $\tilde{A} \in F(R)$, we can characterize $\tilde{A}$ by its $\alpha$-levels by the following closed-bounded intervals:

$$
\begin{align*}
{[\tilde{A}]^{\alpha} } & =\left\{x \in R: \mu_{\tilde{A}}(x) \geq \alpha\right\}=\left[a_{1}^{\alpha}, a_{2}^{\alpha}\right], & & 0<\alpha \leq 1  \tag{5}\\
{[\tilde{A}]^{\alpha} } & =\overline{\left\{x \in R: \mu_{\tilde{A}}(x) \geq \alpha\right\}}=\overline{\left[a_{1}^{\alpha}, a_{2}^{\alpha}\right],} & & 0<\alpha \leq 1 \tag{6}
\end{align*}
$$

Operations on fuzzy numbers can be described as follows: If $\tilde{A}, \tilde{B} \in F(R)$, then for $0<\alpha \leq 1$

1. $[\tilde{A}+\tilde{B}]^{\alpha}=\left[a_{1}^{\alpha}+b_{1}^{\alpha}, a_{2}^{\alpha}+b_{2}^{\alpha}\right]$;
2. $[\tilde{A}-\tilde{B}]^{\alpha}=\left[a_{1}^{\alpha}-b_{1}^{\alpha}, a_{2}^{\alpha}-b_{2}^{\alpha}\right]$;
3. $[\tilde{A} \cdot \tilde{B}]^{\alpha}=\left[\min \left\{a_{1}^{\alpha} \cdot b_{1}^{\alpha}, a_{1}^{\alpha} \cdot b_{2}^{\alpha}, a_{2}^{\alpha} \cdot b_{1}^{\alpha}, a_{2}^{\alpha} \cdot b_{2}^{\alpha}\right\}, \max \left\{a_{1}^{\alpha} \cdot b_{1}^{\alpha}, a_{1}^{\alpha} \cdot b_{2}^{\alpha}, a_{2}^{\alpha} \cdot b_{1}^{\alpha}, a_{2}^{\alpha} \cdot b_{2}^{\alpha}\right\}\right]$;
4. $\left[\frac{A}{B}\right]^{\alpha}=\left[\min \left\{\frac{a_{1}^{\alpha}}{b_{1}^{\alpha}}, \frac{a_{1}^{\alpha}}{b_{2}^{\alpha}}, \frac{,}{b_{1}^{\alpha}}, \frac{a_{2}^{\alpha}}{b_{2}^{\alpha}}\right\}, \max \left\{\frac{a_{1}^{\alpha}}{b_{1}^{\alpha}}, \frac{a_{1}^{\alpha}}{b_{2}^{\alpha}}, \frac{a_{2}^{\alpha}}{b_{1}^{\alpha}}, \frac{a_{2}^{\alpha}}{b_{2}^{\alpha}}\right\}\right]$ here $0 \notin[\tilde{B}]^{\alpha}$;
5. $[s \tilde{A}]^{\alpha}=s[\tilde{A}]^{\alpha}$ where $s$ is scalar and
6. $\left[a_{1}^{\alpha_{i}}, a_{2}^{\alpha_{i}}\right]=\left[a_{1}^{\alpha_{j}}, a_{2}^{\alpha_{j}}\right]$ for $0<\alpha_{i} \leq \alpha_{j}$.

Definition 2.4. A trapezoidal (triangular, gaussian) fuzzy number is a fuzzy number where its membership function has trapezoidal shape (triangular, gaussian). The membership function of a trapezoidal fuzzy number will be interpreted as follows:
$\mu_{\tilde{A}}(x)=\left\{\begin{aligned} 0, & \text { if } x<a_{1} ; \\ \frac{x-a_{1}}{a_{2}-a_{1}}, & \text { if } a_{1} \leq x \leq a_{2} ; \\ 1, & \text { if } a_{2} \leq x \leq a_{3} ; \\ \frac{a_{4}-x}{a_{4}-a_{3}}, & \text { if } a_{3} \leq x \leq a_{4} ; \\ 0, & \text { if } x>a_{4} .\end{aligned}\right.$
(when $a_{2}=a_{3}$ the equation (7) represents the membership function of triangular fuzzy number).
Definition 2.5. A fuzzy process is a mapping $\tilde{x}: I \rightarrow F(R)$, where $I$ is a real interval [17].
This process can be denoted as:

$$
\begin{equation*}
[\tilde{x}(t)]^{\alpha}=\left[x_{1}^{\alpha}(t), x_{2}^{\alpha}(t)\right], \quad t \in I \quad \text { and } \quad 0<\alpha \leq 1 . \tag{8}
\end{equation*}
$$

The fuzzy derivative of a fuzzy process $x(t)$ is defined by

$$
\begin{equation*}
[\tilde{x}(t)]^{\alpha}=\left[x_{1}^{\prime \alpha}(t), x_{2}^{\prime \alpha}(t)\right], \quad t \in I \quad \text { and } \quad 0<\alpha \leq 1 . \tag{9}
\end{equation*}
$$

## III. FUZZY INITIAL VALUE PROBLEM

The FIVP can be considered as follows

$$
\begin{equation*}
\frac{d x(t)}{d t}=f(t, x(t)), \quad x(0)=\tilde{X}_{0}, \tag{10}
\end{equation*}
$$

where $f: R_{+} \times R \rightarrow R$ is a continuous mapping and $\tilde{X}_{0} \in F(R)$ with $\alpha$-level interval

$$
\begin{equation*}
\left[\tilde{x}_{0}\right]^{\alpha}=\left[x_{01}^{\alpha}, x_{02}^{\alpha}\right] \quad 0<\alpha \leq 1 . \tag{11}
\end{equation*}
$$

When $x=x(t)$ is a fuzzy number, the extension principle of Zadeh leads to the following definition:

$$
\begin{equation*}
f(t, x)(s)=\sup \{x(\tau): s=f(t, \tau)\}, \quad s \in R \tag{12}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
[f(t, x(t))]^{\alpha}=\left[f_{1}^{\alpha}(t, x(t)), f_{2}^{\alpha}(t, x(t))\right], \quad 0<\alpha \leq 1, \tag{13}
\end{equation*}
$$

where

$$
\begin{array}{ll}
f_{1}^{\alpha}(t, x(t))=\min \left\{f(t, w): w \in\left[x_{1}^{\alpha}(t), x_{2}^{\alpha}(t)\right]\right\}, & 0<\alpha \leq 1, \\
f_{2}^{\alpha}(t, x(t))=\max \left\{f(t, w): w \in\left[x_{1}^{\alpha}(t), x_{2}^{\alpha}(t)\right]\right\}, & 0<\alpha \leq 1 . \tag{15}
\end{array}
$$

Theorem 3.1. Let $f$ satisfy

$$
\begin{equation*}
\left|f(t, x)-f\left(t, x^{*}\right)\right| \leq g\left(t,\left|x-x^{*}\right|\right), \quad t \geq 0 \quad x, x^{*} \in R \tag{16}
\end{equation*}
$$

where $g: R_{+} \times R_{+} \rightarrow R_{+}$is a continuous mapping such that $r \rightarrow g(t, r)$ is nondecreasing, the IVP

$$
\begin{equation*}
z^{\prime}(t)=g(t, z(t)), \quad z(0)=z_{0}, \tag{17}
\end{equation*}
$$

has a solution on $R_{+}$for $z_{0}>0$ and that $z(t) \equiv 0$ is the only solution of equation (17) for $z_{0}=0$. Then the FIVP (10) has a unique fuzzy solution.
Proof. See[17].
In the fuzzy computation, the dependency problem arises when we apply the straightforward fuzzy interval arithmetic and Zadeh's extension principle by computing the interval separately. For the dependency problem we refer [7].

## IV. The Extended Runge-Kutta method of order 4

We consider the IVP in equation (10) but with crisp initial condition $x\left(t_{0}\right)=x_{0} \in R$ and $t \in\left[t_{0}, T\right]$. The formula [12]

$$
\begin{equation*}
x_{r+1}=x_{r}+h \sum_{j=1}^{m} b_{j} k_{j}^{(1)}+h^{2} \sum_{j=1}^{m} c_{j} k_{j}^{(2)}, \tag{18}
\end{equation*}
$$

where
$k_{j}^{(1)}=f\left(t_{r}, y_{r}+h \sum_{s=1}^{j-1} a_{j s} k_{j}^{(1)}\right)$,
$k_{j}^{(2)}=f^{\prime}\left(t_{r}, y_{r}+h \sum_{s=1}^{j-1} b_{j s} k_{s}^{(1)}\right), \quad j=1,2, \ldots, m, \quad s=1,2, \ldots, j-1$,
with constant $h$ as the step-size of the iterations. The non-zero constants $b_{j}$ and $c_{j}$ in the extended Runge-Kutta method of order $4(n=4)$ are

$$
b_{1}=1, c_{1}=\frac{1}{6}, \quad c_{3}=\frac{1}{2}, \quad b_{32}=\frac{1}{2}, \quad a_{21}=\frac{1}{4}
$$

Therefore we have,
$k_{1}^{(1)}=f\left(t_{r}, x_{r}\right)$
$k_{1}^{(1)}=f\left(t_{r}, x_{r}+\frac{h}{4} k_{1}^{(1)}\right)$
$k_{1}^{(2)}=f^{\prime}\left(t_{r}, y_{r}\right)$
$k_{3}^{(2)}=f^{\prime}\left(t_{r}, x_{r}+\frac{h}{2} k_{2}^{(1)}\right)$
and

$$
\begin{equation*}
x_{r+1}=x_{r}+h k_{1}^{(1)}+\frac{1}{6} h^{2} k_{1}^{(2)}+\frac{1}{3} h^{2} k_{3}^{(2)} \tag{20}
\end{equation*}
$$

where $t_{0} \leq t_{1} \leq \cdots \leq t_{N}=T$ and $h=\frac{T-t_{0}}{N}=t_{r+1}-t_{r}, r=0,1, \cdots, N$.
For the fuzzy initial condition of equation (10), we modify the classical ERK4 by taking into account the dependency problem in fuzzy computation. We first consider the right-hand side of equation (20) as one function

$$
\begin{equation*}
V\left(t_{r}, k_{1}^{(1)}, k_{2}^{(1)}, k_{3}^{(1)}, k_{1}^{(2)}, k_{2}^{(2)}, k_{3}^{(2)}\right)=x_{r}+h k_{1}^{(1)}+\frac{1}{6} h^{2} k_{1}^{(2)}+\frac{1}{3} h^{2} k_{3}^{(2)}, \tag{21}
\end{equation*}
$$

or by the equivalent formula

$$
\begin{equation*}
V\left(t_{r}, h, x_{r}\right)=x_{r}+h f\left(t_{r}, x_{r}\right)+\frac{1}{6} h^{2} f^{\prime}\left(t_{r}, x_{r}\right)+\frac{1}{3} h^{2} f^{\prime}\left(t_{r}, x_{r}+\frac{1}{2} h f\left(t_{r}, x_{r}+\frac{1}{4} h f\left(t_{r}, x_{r}\right)\right)\right) . \tag{22}
\end{equation*}
$$

Now, let $\tilde{X} \in F(R)$, the formula

$$
V\left(t_{r}, h, \tilde{X}_{r}\right)\left(v_{r}\right)=\left\{\begin{array}{r}
\sup _{x_{r} \in V^{-1}\left(t_{r}, h, v_{r}\right)} \tilde{X}_{r}\left(x_{r}\right), \text { if } v_{r} \in \operatorname{range}(V) ;  \tag{23}\\
0, \text { if } v_{r} \notin \operatorname{range}(V)
\end{array}\right.
$$

can extend equation (22) in the fuzzy setting.
Let $\left[\tilde{X}_{r}\right]^{\alpha}=\left[x_{r, 1}^{\alpha}, x_{r, 2}^{\alpha}\right]$ represent the $\alpha$-level of the fuzzy number defined in equation (23). We rewrite equation (23) using the $\alpha$-level as follows:

$$
\begin{equation*}
V\left(t_{r}, h,\left[\tilde{X}_{r}\right]^{\alpha}\right)=\left[\min \left\{t_{r}, h, x \mid x \in x_{r, 1}^{\alpha}, x_{r, 2}^{\alpha}\right\}, \max \left\{t_{r}, h, x \mid x \in x_{r, 1}^{\alpha}, x_{r, 2}^{\alpha}\right\}\right] \tag{24}
\end{equation*}
$$

By applying equation (24) in (20) we get

$$
\begin{equation*}
\left[\tilde{X}_{r+1}\right]^{\alpha}=\left[x_{r+1,1}^{\alpha}, x_{r+1,2}^{\alpha}\right] \tag{25}
\end{equation*}
$$

where

$$
\begin{align*}
x_{r+1,1}^{\alpha} & =\min \left\{V\left(t_{r}, h, x\right) \mid x \in x_{r, 1}^{\alpha}, x_{r, 2}^{\alpha}\right\},  \tag{26}\\
x_{r+1,2}^{\alpha} & =\max \left\{V\left(t_{r}, h, x\right) \mid x \in x_{r, 1}^{\alpha}, x_{r, 2}^{\alpha}\right\} . \tag{27}
\end{align*}
$$

Therefore

$$
\begin{align*}
x_{r+1,1}^{\alpha} & =\min \left\{\left.x_{r}+h k_{1}^{(1)}+\frac{1}{6} h^{2} k_{1}^{(2)}+\frac{1}{3} h^{2} k_{3}^{(2)} \right\rvert\, x \in\left[x_{r, 1}^{\alpha}, x_{r, 2}^{\alpha}\right]\right\},  \tag{28}\\
x_{r+1,2}^{\alpha} & =\max \left\{\left.x_{r}+h k_{1}^{(1)}+\frac{1}{6} h^{2} k_{1}^{(2)}+\frac{1}{3} h^{2} k_{3}^{(2)} \right\rvert\, x \in\left[x_{r, 1}^{\alpha}, x_{r, 2}^{\alpha}\right]\right\} . \tag{29}
\end{align*}
$$

By using the computational method proposed in [5], we compute the minimum and maximum in equations (28), (29) as follows

$$
\begin{align*}
& x_{r+1,1}^{\alpha_{i}}=\min \left[\min _{x \in\left[x_{r, 1}^{\alpha_{i}} x_{r, 1}^{\alpha_{i+1}}\right]} V(t, h, x), \cdots, \min _{x \in\left[x_{r, 1}^{\alpha_{n}} x_{r, 2}^{\alpha_{n}}\right]} V(t, h, x), \cdots, \min _{x \in\left[x_{r, 2}^{\alpha_{i+1}}, x_{r, 2}\right]} V(t, h, x)\right] \text {, } \tag{30}
\end{align*}
$$

## V. The Extended Runge-Kutta method of order 5

Consider the IVP in equation (10) but with crisp initial condition $x\left(t_{0}\right)=x_{0} \in R$ and $t \in\left[t_{0}, T\right]$. The formula [12], Now we consider the equation (18) and (19), we get the nonzero constants $b_{j}$ and $c_{j}$ in the extended Runge-Kutta method of order $5(\mathrm{n}=5)$ are

$$
\begin{aligned}
& b_{1}=1, c_{1}=\frac{3-\sqrt{11}}{6}, c_{2}=\frac{50+3 \sqrt{11}}{168}, \quad c_{4}=\frac{25(-2+\sqrt{11})}{168}, \\
& b_{21}=\frac{5-\sqrt{11}}{14}, b_{31}=1-\frac{\sqrt{11}}{5}-\frac{14}{25}, a_{32}=\frac{14}{25}, \quad b_{43}=\frac{3+\sqrt{11}}{10} .
\end{aligned}
$$

Therefore we have,

$$
\begin{aligned}
& k_{1}^{(1)}=f\left(t_{r}, x_{r}\right), \\
& k_{2}^{(1)}=f\left(t_{r}, x_{r}+\left[\frac{5-\sqrt{11}}{14}\right] h k_{1}^{(1)}\right), \\
& k_{3}^{(1)}=f\left(t_{r}, x_{r}+\left[1-\frac{\sqrt{11}}{5}-\frac{14}{25}\right] h k_{1}^{(1)}+\left[\frac{14}{25}\right] h k_{2}^{(1)}\right), \\
& k_{1}^{(2)}=f^{\prime}\left(t_{r}, y_{r}\right), \\
& k_{2}^{(2)}=f^{\prime}\left(t_{r}, x_{r}+\left[\frac{5-\sqrt{11}}{14}\right] h k_{1}^{(1)}\right), \\
& k_{4}^{(2)}=f^{\prime}\left(t_{r}, x_{r}+\left[\frac{3+\sqrt{11}}{10}\right] h k_{3}^{(1)}\right),
\end{aligned}
$$

and

$$
\begin{equation*}
x_{r+1}=x_{r}+h k_{1}^{(1)}+h^{2}\left(\frac{3-\sqrt{11}}{6} k_{1}^{(2)}+\frac{50+3 \sqrt{11}}{168} k_{2}^{(2)}+\frac{25(2+\sqrt{11})}{6} k_{4}^{(2)}\right), \tag{32}
\end{equation*}
$$

where $t_{0} \leq t_{1} \leq \cdots \leq t_{N}=T$ and $h=\frac{T-t_{0}}{N}=t_{r+1}-t_{r}, r=0,1, \cdots, N$.
For the fuzzy initial condition of equation (10), we modify the classical ERK4 by taking into account the dependency problem in fuzzy computation. We first consider the right-hand side of equation (32) as one function

$$
\begin{align*}
& V\left(t_{r}, k_{1}^{(1)}, k_{2}^{(1)}, k_{3}^{(1)}, k_{4}^{(1)}, k_{1}^{(2)}, k_{2}^{(2)}, k_{3}^{(2)}, k_{4}^{(2)}\right) \\
& \quad=x_{r}+h k_{1}^{(1)}+h^{2}\left(\frac{3-\sqrt{11}}{6} k_{1}^{(2)}+\frac{50+3 \sqrt{11}}{168} k_{2}^{(2)}+\frac{25(-2+\sqrt{11})}{168} k_{4}^{(2)}\right), \tag{33}
\end{align*}
$$

or by the equivalent formula
$V\left(t_{r}, h, x_{r}\right)$

$$
=x_{r}+\left\{\begin{array}{l}
h f\left(t_{r}, x_{r}\right)  \tag{34}\\
+h^{2}\binom{\left[\frac{3-\sqrt{11}}{6}\right] f^{\prime}\left(t_{r}, x_{r}\right)+\left[\frac{50+3 \sqrt{11}}{168}\right] f^{\prime}\left(t_{r}, x_{r}+h\left[\frac{5-\sqrt{11}}{14}\right] f\left(t_{r}, x_{r}\right)\right)}{+\left[\frac{25(-2+\sqrt{11})}{168}\right]^{\prime}\left(f_{r}, x_{r}+\left[\frac{3+\sqrt{11}}{10}\right] h f\binom{t_{r}, x_{r}+\left[1-\frac{\sqrt{11}}{5}-\frac{14}{25}\right] h f\left(t_{r}, x_{r}\right)}{+\left[\frac{14}{25}\right] h f\left(t_{r}, x_{r}+\left[\frac{5-\sqrt{11}}{14}\right] h f\left(t_{r}, x_{r}\right)\right.}\right.}
\end{array}\right)
$$

Now, let $\tilde{X} \in F(R)$, the formula

$$
V\left(t_{r}, h, \tilde{X}_{r}\right)\left(v_{r}\right)=\left\{\begin{array}{r}
\sup _{x_{r} \in V^{-1}\left(t_{r}, h, v_{r}\right)} \tilde{X}_{r}\left(x_{r}\right), \text { if } v_{r} \in \operatorname{range}(V) ;  \tag{35}\\
0, \text { if } v_{r} \notin \operatorname{range}(V)
\end{array}\right.
$$

can extend equation (34) in the fuzzy setting.

Let $\left[\tilde{X}_{r}\right]^{\alpha}=\left[x_{r, 1}^{\alpha}, x_{r, 2}^{\alpha}\right]$ represent the $\alpha$-level of the fuzzy number defined in equation (37). We rewrite equation (35) using the $\alpha$-level as follows:

$$
\begin{equation*}
V\left(t_{r}, h,\left[\tilde{X}_{r}\right]^{\alpha}\right)=\left[\min \left\{V\left(t_{r}, h, x\right) \mid x \in x_{r, 1}^{\alpha}, x_{r, 2}^{\alpha}\right\}, \max \left\{V\left(t_{r}, h, x\right) \mid x \in x_{r, 1}^{\alpha}, x_{r, 2}^{\alpha}\right\}\right] . \tag{36}
\end{equation*}
$$

By applying equation (36) in (32) we get

$$
\begin{equation*}
\left[\tilde{X}_{r+1}\right]^{\alpha}=\left[x_{r+1,1}^{\alpha}, x_{r+1,2}^{\alpha}\right], \tag{37}
\end{equation*}
$$

where

$$
\begin{align*}
x_{r+1,1}^{\alpha} & =\min \left\{V\left(t_{r}, h, x\right) \mid x \in x_{r, 1}^{\alpha}, x_{r, 2}^{\alpha}\right\},  \tag{38}\\
x_{r+1,2}^{\alpha} & =\max \left\{V\left(t_{r}, h, x\right) \mid x \in x_{r, 1}^{\alpha}, x_{r, 2}^{\alpha}\right\} . \tag{39}
\end{align*}
$$

Therefore

$$
\begin{align*}
& x_{r+1,1}^{\alpha}=\min \left\{x_{r}+h k_{1}^{(1)}+h^{2}\left(\frac{3-\sqrt{11}}{6} k_{1}^{(2)}+\frac{50+3 \sqrt{11}}{168} k_{2}^{(2)}+\frac{25(-2+\sqrt{11})}{168} k_{4}^{(2)}\right) x \in\left[x_{r, 1}^{\alpha}, x_{r, 2}^{\alpha}\right]\right\},  \tag{40}\\
& x_{r+1,2}^{\alpha}=\max \left\{\left.x_{r}+h k_{1}^{(1)}+h^{2}\left(\frac{3-\sqrt{11}}{6} k_{1}^{(2)}+\frac{50+3 \sqrt{11}}{168} k_{2}^{(2)}+\frac{25(-2+\sqrt{11})}{168} k_{4}^{(2)}\right) \right\rvert\, x \in\left[x_{r, 1}^{\alpha}, x_{r, 2}^{\alpha}\right]\right\}, \tag{41}
\end{align*}
$$

By using the computational method proposed in [5], we compute the minimum and maximum in equations (40), (41) as follows:

$$
\begin{align*}
& x_{r+1,1}^{\alpha_{i}}=\min \left[\min _{x \in\left[x_{r, 1}^{\alpha_{i}}, x_{r, 1}^{\alpha_{i+1}}\right]} V(t, h, x), \cdots, \min _{x \in\left[x_{r, 1}^{\alpha_{n}}, x_{r, 2}^{\alpha_{n+1}}\right]} V(t, h, x), \cdots, \min _{x \in\left[\begin{array}{l}
\left.x_{r, 2}^{\alpha_{i+1}}, x_{r, 2}^{\alpha_{i}}\right] \\
\alpha_{i} \\
\end{array}(t, h, x)\right],}\right.  \tag{42}\\
& x_{r+1,2}^{\alpha_{i}}=\max \left[\max _{x \in\left[x_{r, 1}^{\alpha_{i}} x_{r, 1}^{\alpha_{i+1}}\right]} V(t, h, x), \cdots, \max _{x \in\left[x_{r, 1}^{\alpha_{n}} x_{r, 2}^{\alpha_{n}}\right]} V(t, h, x), \cdots, \max _{x \in\left[x_{r, 2}^{\alpha_{i+1}}, x_{r, 2}\right]} V(t, h, x)\right] . \tag{43}
\end{align*}
$$

## VI. Numerical Examples

In this section, we present some numerical examples including linear and nonlinear FIVPs.
Example 5.1. Consider the following FIVP.

$$
\left\{\begin{array}{cl}
x^{\prime}(t)=x(1-2 t), & t \in[0,2]  \tag{44}\\
\tilde{X}^{0}(w)=\left\{\begin{array}{cl}
0, & \text { if } w<-0.5 \\
1-4 w^{2}, & \text { if }-0.5 \leq w \leq 0.5 \\
0, & \text { if } w>0.5
\end{array}\right.
\end{array}\right.
$$

The exact solution of equation (44) is given by

$$
\begin{equation*}
[X(t)]^{\alpha}=\left[\left(-\frac{\sqrt{(1-\alpha)}}{2}\right) e^{t-t^{2}},\left(\frac{\sqrt{(1-\alpha)}}{2}\right) e^{t-t^{2}}\right] . \tag{45}
\end{equation*}
$$

The approximate fuzzy solution by ERK5 for $\mathrm{h}=0.1$, plotted in figures (1) and (2) and the numerical values with error are shown in Table 1.
Fig.2. (a) The graph of the comparison between the fuzzy exact solution and ERK5 and ERK4 fuzzy approximated solutions at $t_{10}=1$ and $\mathrm{h}=0.1$; (b) The graph of the comparision between the ERK5 and ERK4 with fuzzy approximated solutions and fuzzy exact solution at $t_{10}=1$ and $\mathrm{h}=0.1$;

For this example, the comparison of the absolute local error between ERK5, ERK4 with the fuzzy exact solution is given in Table 1 for various values of $\alpha$-level $(\alpha=0,0.1, \cdots, 1)$ and
fixed value of $h\left(t_{20}=2\right)$. The grapical comparision between ERK5, ERK4 with exactsolutions at fixed $h\left(t_{10}=1\right)$. In fig(a) clearly show that ERK5 more accurate result than ERK4. On the other hand, the result of the comparison between the fuzzy solution of ERK5 and the one ERK4 shows decresing of the fuzzy interval of ERK5.

TABLE 1.
The error of the obtained results with the exact solution at $\mathrm{t}=2$.

| r | ERK4 <br> Approximation |  | ERK5 <br> Approximation |  | Exact |  | Error ERK4 |  | Error ERK5 |  |
| :---: | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | $x_{1}\left(t_{i} ; r\right)$ | $x_{2}\left(t_{i} ; r\right)$ | $x_{1}\left(t_{i} ; r\right)$ | $x_{2}\left(t_{i} ; r\right)$ | $x_{1}\left(t_{i} ; r\right)$ | $x_{2}\left(t_{i} ; r\right)$ | $x_{1}\left(t_{i} ; r\right)$ | $x_{2}\left(t_{i} ; r\right)$ | $x_{1}\left(t_{i} ; r\right)$ | $x_{2}\left(t_{i} ; r\right)$ |
| 0.0 | -0.0667 | 0.0667 | -0.0667 | 0.0667 | -0.0677 | 0.0677 | $-9.57 \mathrm{e}-4$ | $9.57 \mathrm{e}-4$ | $-9.59 \mathrm{e}-4$ | $9.59 \mathrm{e}-4$ |
| 0.1 | -0.0633 | 0.0633 | -0.0633 | 0.0633 | -0.0642 | 0.0642 | $-9.08 \mathrm{e}-4$ | $9.08 \mathrm{e}-4$ | $-9.10 \mathrm{e}-4$ | $9.10 \mathrm{e}-4$ |
| 0.2 | -0.0597 | 0.0597 | -0.0597 | 0.0597 | -0.0605 | 0.0605 | $-8.56 \mathrm{e}-4$ | $8.56 \mathrm{e}-4$ | $-8.58 \mathrm{e}-4$ | $8.58 \mathrm{e}-4$ |
| 0.3 | -0.0558 | 0.0558 | -0.0558 | 0.0558 | -0.0566 | 0.0566 | $-8.00 \mathrm{e}-4$ | $8.00 \mathrm{e}-4$ | $-8.02 \mathrm{e}-4$ | $8.02 \mathrm{e}-4$ |
| 0.4 | -0.0517 | 0.0517 | -0.0517 | 0.0517 | -0.0524 | 0.0524 | $-7.41 \mathrm{e}-4$ | $7.41 \mathrm{e}-4$ | $-7.43 \mathrm{e}-4$ | $7.43 \mathrm{e}-4$ |
| 0.5 | -0.0472 | 0.0472 | -0.0472 | 0.0472 | -0.0478 | 0.0478 | $-6.76 \mathrm{e}-4$ | $6.76 \mathrm{e}-4$ | $-6.78 \mathrm{e}-4$ | $6.78 \mathrm{e}-4$ |
| 0.6 | -0.0422 | 0.0422 | -0.0422 | 0.0422 | -0.0428 | 0.0428 | $-6.05 \mathrm{e}-4$ | $6.05 \mathrm{e}-4$ | $-6.06 \mathrm{e}-4$ | $6.06 \mathrm{e}-4$ |
| 0.7 | -0.0365 | 0.0365 | -0.0365 | 0.0365 | -0.0371 | 0.0371 | $-5.24 \mathrm{e}-4$ | $5.24 \mathrm{e}-4$ | $-5.25 \mathrm{e}-4$ | $5.25 \mathrm{e}-4$ |
| 0.8 | -0.0298 | 0.0298 | -0.0298 | 0.0298 | -0.0303 | 0.0303 | $-4.24 \mathrm{e}-4$ | $4.28 \mathrm{e}-4$ | $-4.29 \mathrm{e}-4$ | $4.29 \mathrm{e}-4$ |
| 0.9 | -0.0211 | 0.0211 | -0.0211 | 0.0211 | -0.0214 | 0.0214 | $-3.02 \mathrm{e}-4$ | $3.02 \mathrm{e}-4$ | $-3.03 \mathrm{e}-4$ | $3.03 \mathrm{e}-4$ |
| 1.0 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |



Figure 1 The approximation of fuzzy solution by ERK5 (h=0.1)


Figure 2 Comparision between the exact and the approximation of fuzzy solution by ERK5
Example 5.2. Consider the following FIVP

$$
\left\{\begin{align*}
x^{\prime}(t) & =x \cos (t+x) & & t \in[0,10]  \tag{46}\\
\tilde{X}_{0}^{\alpha} & =[0.5+0.5 \alpha, 1.5-0.5 \alpha] & & 0<\alpha \leq 1 .
\end{align*}\right.
$$

The absolute results of the numerical fuzzy ERK5 approximated solutions at $t_{100}=10$. See Figure 3.

In this example, we compare the solution obtained by ERK5 method with the exact solution and the solution obtained by ERK4. We have given the numerical values in Table 2 fixed value of $t\left(t_{20}=2\right)$ and for different values of $\alpha$.


Figure 3 The approximation of fuzzy solution by ERK5 (h=0.1)


Figure 4 The approximation of fuzzy solution by ERK5 and ERK4 (h=0.1)
TABLE 2.
The error difference between the results obtained at $\mathrm{t}=2$.

| r | ERK4 Approximation |  | ERK5 Approximation |  | Diff ERK5 \& ERK4 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $x_{1}\left(t_{i} ; r\right)$ | $x_{2}\left(t_{i} ; r\right)$ | $x_{1}\left(t_{i} ; r\right)$ | $x_{2}\left(t_{i} ; r\right)$ | $x_{1}\left(t_{i} ; r\right)$ | $x_{2}\left(t_{i} ; r\right)$ |
| 0.0 | 0.4405 | 0.5641 | 0.4405 | 0.5641 | $-3.09 \mathrm{e}-7$ | $-1.82 \mathrm{e}-6$ |
| 0.1 | 0.4533 | 0.5611 | 0.4533 | 0.5611 | $-3.63 \mathrm{e}-7$ | $-1.73 \mathrm{e}-6$ |
| 0.2 | 0.4659 | 0.5581 | 0.4659 | 0.5581 | $-4.24 \mathrm{e}-7$ | $-1.64 \mathrm{e}-6$ |
| 0.3 | 0.4761 | 0.5551 | 0.4761 | 0.5551 | $-4.87 \mathrm{e}-7$ | $-1.55 \mathrm{e}-6$ |
| 0.4 | 0.4858 | 0.5516 | 0.4858 | 0.5516 | $-5.52 \mathrm{e}-7$ | $-1.47 \mathrm{e}-6$ |
| 0.5 | 0.4943 | 0.5482 | 0.4943 | 0.5482 | $-6.26 \mathrm{e}-7$ | $-1.39 \mathrm{e}-6$ |
| 0.6 | 0.5018 | 0.5444 | 0.5018 | 0.5444 | $-6.94 \mathrm{e}-7$ | $-1.31 \mathrm{e}-6$ |
| 0.7 | 0.5091 | 0.5404 | 0.5091 | 0.5404 | $-7.71 \mathrm{e}-7$ | $-1.23 \mathrm{e}-6$ |
| 0.8 | 0.5151 | 0.5362 | 0.5151 | 0.5362 | $-8.49 \mathrm{e}-7$ | $-1.15 \mathrm{e}-6$ |
| 0.9 | 0.5209 | 0.5314 | 0.5209 | 0.5314 | $-9.24 \mathrm{e}-7$ | $-1.07 \mathrm{e}-6$ |
| 1.0 | 0.5267 | 0.5267 | 0.5267 | 0.5267 | $-9.99 \mathrm{e}-7$ | $-9.99 \mathrm{e}-7$ |

VII. Conclusion

In this paper, we have studied the numerical solution of differential equations with fuzzy initial values. We used extended Runge-Kutta-like formula of order five by taking into account the dependency problem in fuzzy computation. We solved some linear and nonlinear differential equations with fuzzy initial values by the proposed method.

## REFERENCES

[1] S. Abbasbandy, T. Allahviranloo, Numerical solutions of fuzzy differential equations by Taylor Method, Computational Methods in Applied Mathematics, 2 (2002), 113-124.
[2] S. Abbasbandy, T. Allahviranloo, Numerical solution of Fuzzy differential equation by RungeKutta method, Nonlinear Studies, 11, (2004), 117-129.
[3] M. Ahamad, M. Hasen, A new approach to incorporate uncertainity into Euler method, Applied Mathematical Sciences, 4(51), (2010), 2509-2520.
[4] M. Ahamed, M. Hasan, A new fuzzy version of Euler's method for solving diffrential equations with fuzzy initial values, Sians Malaysiana, 40, (2011), 651-657.
[5] M. Ahmad, M. Hasan, Incorporating optimization technique into Zadeh's extension principle for computing non-monotone functions with fuzzy variable, Sains Malaysiana, 40, (2011) 643-650.
[6] N. Z. Ahmad, H. K. Hasan, B. De Baets, A new method for computing continuous function with fuzzy variable, Journal of Applied Sciences, 11(7) ,(2011), 1143-1149.
[7] A. H. Alsonosi Omar, Y. Abu Hasan, Numerical solution of fuzzy differential equations and the dependency problem, Applied Mathematics and Computation, 219, (2012), 1263-1272.
[8] A. Bonarini, G. Bontempi, A Qualitative simulation approach for fuzzy dynamical models, ACM Trans. Model. Comput. Simulat, 4, (1994), 285-313.
[9] R. Brent, Algorithms for Minimization without Derivatives, Dover Pubns, 2002.
[10] J. J. Buckley, T. Feuring, Fuzzy differential equations, Fuzzy Sets and Systems, 110, (2000) 4354.
[11] D. Dubois, H. Prade, Towards fuzzy differential calculus part 3: differentiation, Fuzzy Sets and Systems, 8, (1982), 225-233.
[12] B. Ghazanfari, A. Shakerami, Numerical solutions of fuzzy differential equations by extended Runge-Kutta like formulae of order 4, Fuzzy Sets and Systems, 189, (2011), 74-91.
[13] Kaleva Osmo, Fuzzy differential equations, Fuzzy Sets and Systems, 24, (1987), 301-317.
[14] A. Karimi Dizicheh, S. Salahshour, Fudzaih Bt. Ismail, A note on '`Numerical solutions of fuzzy differential equations by extended Runge-Kutta-like formulae of order 4", Fuzzy ets and systems, (23), (2013), 96-100.
[15] M. Ma, M. Friedman, A. Kandel, Numerical solutions of fuzzy differential equations, Fuzzy Sets and Systems, 105, (1999), 133-138.
[16] S. Palligkinis, G. Papageorgiou, I. Famelis, Runge-Kutta methods for fuzzy differential equations, Applied Mathematics and Computation, 209, (2009), 97-105.
[17] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems 24(3) (1987) 319-330.
[18] Xinyuan Wu, Jianlin Xia, Extended Runge-Kutta-like formula, Applied Numerical Mathematics 56 (2006) 1584-1605.
[19] L. A. Zadeh, Fuzzy Sets, Information and Control 8(1965) 338-353.

