Čech Soft Closure Spaces

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Abstract

The purpose of the present paper is to define and study the Čech soft closure spaces on the soft sets over the non-empty set X and discuss some of its properties.

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1 Introduction

Some theories such as theory of vague sets, theory of rough sets and etc., can be considered as mathematical tools for dealing with uncertainties. But all of these theories have their own difficulties. In 1999, D. Molodtsov [4] introduced the concept of soft sets in order to solve complicated problems in some sciences such as economics, engineering etc. In 2011, Shabir and Naz [5] introduced and studied the concepts of soft topological space and some related concepts such as soft interior, soft closed, soft subspace and soft separation axioms. Recent years, the theory of soft topological spaces is investigated by various new researchers.
Let X be a non-empty set and $P(X)$ be the power set of X. A mapping $c : P(X) \to P(X)$ is called a Čech closure operator provided it satisfies the following three axioms:

(C1) $c(\phi) = \phi$,

(C2) $A \subset c(A)$, for all $A \subset X$,

(C3) $c(A \cup B) = c(A) \cup c(B)$, for all $A, B \subset X$.

Then c, together with the underlying set X, is called a Čech closure space and is denoted by (X, c). If c also satisfies:

(C4) $c(c(A)) = c(A)$ for all $A \subset X$, then (X, c) is a topological space.

Čech closure spaces were introduced by E. Čech in [1]. In this paper, we introduced Čech soft closure spaces over the soft sets on a non-empty set X and we exhibit some results related to these concepts.

2 Soft sets

Definition 2.1. *[4] Let X be an initial universe and E be a set of parameters. Let $P(X)$ denote the power set of X and A be a non-empty subset of E. A soft set F_A on the universe X is defined by $F_A = \{(e, F(e))/e \in E, F(e) \in P(X)\}$, where $F : E \to P(X)$ such that $F(e) = \phi$ if $x \notin A$. Here, F is called an approximate function of the soft set F_A. The value of $F(e)$ may be arbitrary. Some of them may be empty and some of may have nonempty intersection. The class of all soft sets over X and the parameter set E is denoted by $S(X, E)$.*

Definition 2.2. *[3] For two soft sets F_A and G_B over a common universe X, we say that F_A is a soft subset of G_B if*

1. *$A \subseteq B$ and*
2. *For all $e \in A$, $F(e)$ and $G(e)$ are identical approximations.*

We write $F_A \tilde{\subseteq} G_B$ ($\tilde{\subseteq}$ denotes soft inclusion.)
Consequently, G_B is said to be a soft superset of F_A.

Definition 2.3. *[3] Two soft sets F_A and G_B over a common universe U are said to be soft equal if F_A is a soft subset of G_B and G_B is a soft subset of F_A.*

Definition 2.4. *[3] Let $E = \{e_1, e_2, \ldots, e_n\}$ be a set of parameters . The NOT set of E denoted by $\neg E$ is defined by $\neg E = \{\neg e_1, \neg e_2, ..., \neg e_n\}$ where $\neg e_i = note_i$ for all i.*

Definition 2.5. *[3] The complement of a soft set F_A is denoted by ${F_A}^c$ and is defined by ${F_A}^c = F_{\neg A}{}^{\neg}$, where $F^{\neg} : \neg A \to P(X)$ is a mapping given by $F^{\neg}(e) = X - F(e)$, for all $e \in \neg A$.*
Let us call $F^{\neg}$ to be the soft complement function of F. Clearly $(F^{\neg})^{\neg}$ is the same as F and $({F_A}^c)^c = F_A$.

Definition 2.6. *[3] A soft set F_A over X is said to be a NULL soft set denoted by ϕ if for all $e \in A$, $F(e) = \phi$ (null set).*

Definition 2.7. *[3] A soft set F_A over X is said to be an absolute soft set denoted by $\tilde{A}$ if for all $e \in A$, $F(e) = X$.*
Clearly $\tilde{A}^c = \phi$ and $\phi^c = \tilde{A}$.

Definition 2.8. *[3] The union of two soft sets F_A and G_B over a common universe X, denoted by the soft set H_C, where $C = A \cup B$ and for all $e \in C$,*

$$H(e) = \begin{cases} F(e) & \text{if } e \in A - B \\ G(e) & \text{if } e \in B - A \\ F(e) \cup G(e) & \text{if } e \in A \cap B \end{cases}$$

We write $F_A \tilde{\cup} G_B = H_C$.

Definition 2.9. *[3] The intersection H_C of two soft sets F_A and G_B over a common universe X, denoted by $F_A \tilde{\cap} G_B$, is defined as $C = A \cap B$, $H(e) = F(e) \cap G(e)$, for all $e \in C$.*

Definition 2.10. *[5] The difference H_E of two soft sets F_E and G_E over X, denoted by $F_E - G_E$, is defined as $H(\alpha) = F(\alpha) - G(\alpha)$, for all $\alpha \in E$.*

Definition 2.11. *[5] Let F_E be a soft set over X and $x \in X$. We say that $x \tilde{\in} F_E$ read as x belongs to the soft set F_E whenever $x \in F(\alpha)$ for all $\alpha \in E$.*
Note that for any $x \in X, x \tilde{\notin} F_E$, if $x \notin F(\alpha)$ for some $\alpha \in E$.

Definition 2.12. *[5] Let Y be a non-empty subset of X, then $\tilde{Y}$ denotes the soft set Y_E over X for which $Y(\alpha) = Y$, for all $\alpha \in E$.*
In particular , X_E will be denoted by $\tilde{X}$.

Definition 2.13. *[5] Let $x \in X$, then x_E denotes the soft set over X for which $x_E(\alpha) = \{x\}$, for all $\alpha \in E$.*

Definition 2.14. *[5] Let F_E be a soft set over X and Y be a non-empty subset of X. Then the sub soft set of F_E over Y denoted by ${}^Y F_E$, is defined as follows ${}^Y F(\alpha) = Y \cap F(\alpha)$, for all $\alpha \in E$.*
In other words ${}^Y F_E = \tilde{Y} \cap F_E$.

Definition 2.15. *[5] The relative complement of a soft set F_A is denoted by $(F_A)'$ and is defined by $(F_A)' = F'_A$ where $F' : A \to P(X)$ is a mapping given by $F'(\alpha) = X - F(\alpha)$ for all $\alpha \in A$.*

Definition 2.16. *[6] Let $S(X,E)$ and $S(Y,E^*)$ be the families of soft sets over X and Y, respectively. The mapping φ_ψ is called a soft mapping from X to Y, defined by $\varphi_\psi : S(X,E) \rightarrow S(Y,E^*)$, where $\varphi : X \rightarrow Y$ and $\psi : E \rightarrow E^*$ are two mappings.*

1. *Let $F_A \in S(X,E)$, then the image of F_A under the soft mapping φ_ψ is the soft set over Y denoted by $\varphi_\psi(F_A)_B, B = \psi(A)\tilde{\subseteq}E^*$ and defined by*
$$\varphi_\psi(F_A)(e^*) = \begin{cases} \bigcup_{e\in\psi^{-1}(e^*)\cap A} \varphi(F(e)), & if \psi^{-1}(e^*) \cap A \neq \phi \\ \phi, & otherwise \end{cases}$$
for $e^* \subseteq E^*$.

2. *Let $G_C \in S(Y,E^*)$, then the pre-image of G_C under the soft mapping φ_ψ is the soft set over X denoted by $\varphi_\psi^{-1}(G_C)_D, D = \psi^{-1}(C) \subseteq E$, where*
$$\varphi_\psi^{-1}(G_C)(e) = \begin{cases} \varphi^{-1}(G(\psi(e))), & if \psi(e) \in C \\ \phi, & otherwise \end{cases}$$
for $e \in D$.
The soft mapping φ_ψ is called injective, if φ and ψ are injective. The soft mapping φ_ψ is called surjective, if φ and ψ are surjective.

Definition 2.17. *Let $F_A \in S(X_1,E_1)$ and $G_B \in S(X_2,E_2)$. The cartesian product $F_A \times G_B$ is defined by $(F \times G)_{(A\times B)}$ where $(F \times G)_{(A\times B)}(e,k) = F_A(e) \times G_B(k)$, for all $(e,k) \in A \times B$.*

According to this definition, the soft set $F_A \times G_B$ is a soft set over $X_1 \times X_2$ and its parameter universe is $E_1 \times E_2$.
In the following, we give the definition of soft projection mappings.

Definition 2.18. *The soft mappings $(p_q)_i$, $i \in \{1,2\}$, is called soft projection mapping from $X_1 \times X_2$ to X_i and defined by $(p_q)_i((F_A)_1 \times (F_A)_2) = (p_q)_i((F_1 \times F_2)_{A_1\times A_2}) = p_i(F_1 \times F_2)_{q_i(A_1\times A_2)} = (F_A)_i$, where $(F_A)_1 \in S(X_1,E_1)$, $(F_A)_2 \in S(X_2,E_2)$ and $p_i : X_1 \times X_2 \rightarrow X_i$, $q_i : E_1 \times E_2 \rightarrow E_i$ are projection mappings in classical meaning.*

Definition 2.19. *[5] Let τ be a collection of soft sets over X, then τ is said to be a soft topology on X if*

(1) ϕ, $\tilde{X}$ belong to τ.

(2) The union of any number of soft sets in τ belongs to τ.

(3) The intersection of any two soft sets in τ belongs to τ.

The triplet $(X, \tilde{\tau}, E)$ is called a soft topological space over X.

Theorem 2.20. *[5] Let F_E and G_E be the soft sets over X. Then*

1. $(F_E \tilde{\cup} G_E)' = (F_E)' \tilde{\cap} (G_E)'$
2. $(F_E \tilde{\cap} G_E)' = (F_E)' \tilde{\cup} (G_E)'$

3 Čech Soft Closure Spaces

Definition 3.1. *An operator $\tilde{c} : S(X,E) \to S(X,E)$ is defined on the set of all soft sets $S(X,E)$ of a set X is called Čech soft closure operator on X if the following three axioms are satisfied:*

(C1) $\tilde{c}(\phi) = \phi$,

(C2) $F_A \tilde{\subset} \tilde{c}(F_A)$, *for all soft sets F_A over$X$,*

(C3) $\tilde{c}(F_A \tilde{\cup} F_B) = \tilde{c}(F_A) \tilde{\cup} \tilde{c}(F_B)$, *for all soft sets F_A, F_B over$X$.*

Then $\tilde{c}$, together with the underlying set X, is called a Čech soft closure space and is denoted by $(X, \tilde{c}, E)$. If $\tilde{c}$ also satisfies:

(C4) $\tilde{c}(\tilde{c}(F_A)) = \tilde{c}(F_A)$ *for all F_A over X, then $(X, \tilde{c}, E)$ is a soft topological space.*

Example 3.2. *Let $X = \{a, b\}$, $E = \{e_1, e_2\}$. Then $S(X,E) = \{F_{E_1} = \{(e_1, \{a\})\}$, $F_{E_2} = \{(e_1, \{b\})\}$, $F_{E_3} = \{(e_1, \{a, b\})\}$, $F_{E_4} = \{(e_2, \{a\})\}$, $F_{E_5} = \{(e_2, \{b\})\}$, $F_{E_6} = \{(e_2, \{a, b\})\}$, $F_{E_7} = \{(e_1, \{a\}), (e_2, \{a\})\}$, $F_{E_8} = \{(e_1, \{a\}), (e_2, \{b\})\}$, $F_{E_9} = \{(e_1, \{a\}), (e_2, \{a, b\})\}$, $F_{E_{10}} = \{(e_1, \{b\}), (e_2, \{a\})\}$, $F_{E_{11}} = \{(e_1, \{b\}), (e_2, \{b\})\}$, $F_{E_{12}} = \{(e_1, \{b\}), (e_2, \{a, b\})\}$, $F_{E_{13}} = \{(e_1, \{a, b\}), (e_2, \{a\})\}$, $F_{E_{14}} = \{(e_1, \{a, b\}), (e_2, \{b\})\}$, $F_{E_{15}} = \{(e_1, \{a, b\}), (e_2, \{a, b\})\}$, $F_{E_{16}} = \phi\}$. Define $\tilde{c}(F_{E_1}) = F_{E_3}$, $\tilde{c}(F_{E_2}) = F_{E_3}$, $\tilde{c}(F_{E_4}) = F_{E_6}$, $\tilde{c}(F_{E_5}) = F_{E_6}$, $\tilde{c}(F_{E_3}) = F_{E_3}$, $\tilde{c}(F_{E_6}) = F_{E_6}$ and for all other soft sets F_{E_i} over X,*

let $\tilde{c}(F_{E_i}) = \begin{cases} \phi & \text{if } F_{E_i} = \phi, \\ \tilde{\cup}\{\tilde{c}(e_i, F(e_i)) : (e_i, F(e_i)) \in F_{E_i}\} & \text{otherwise.} \end{cases}$

(C1) $\tilde{c}(\phi) = \phi$,

(C2) $F_{E_i} \tilde{\subset} \tilde{c}(F_{E_i})$, *for all soft sets F_{E_i} over$X$,*

(C3) Let F_{E_i} and F_{E_j} be soft sets over X. Then $\tilde{c}(F_{E_i} \tilde{\cup} F_{E_j}) = \tilde{\cup}\{\tilde{c}(e, F(e)) : (e, F(e)) \in F_{E_i} \tilde{\cup} F_{E_j}\} = (\tilde{\cup}\{\tilde{c}(e, F(e)) : (e, F(e)) \in F_{E_i}\}) \tilde{\cup} (\tilde{\cup}\{\tilde{c}(e, F(e)) : (e, F(e)) \in F_{E_j}\}) = \tilde{c}(F_{E_i}) \tilde{\cup} \tilde{c}(F_{E_j})$. Thus $(X, \tilde{c}, E)$ is a Čech soft closure space.

Example 3.3. *Let* $X = \{a, b, c\}$*,* $E = \{e_1, e_2\}$ *and let* $\tilde{c} : S(X, E) \to S(X, E)$ *be an operator defined by* $\tilde{c}(\{(e_1, \{a, c\}), (e_2, \{b, c\})\}) = \{(e_1, \{a, c\}), (e_2, \{b, c\})\}$*,*
$\tilde{c}(\{(e_1, \{a\}), (e_2, \{c\})\}) = \{(e_1, \{a\}), (e_2, \{c\})\}$*,*
$\tilde{c}(\{(e_1, \{c\}), (e_2, \{c\})\}) = \{(e_1, \{c\}), (e_2, \{c\})\}$*,*
$\tilde{c}(\{(e_1, \{a, c\}), (e_2, \{b\})\}) = \{(e_1, \{a, c\}), (e_2, \{b\})\}$*,*
$\tilde{c}(\phi) = \phi$*,* $\tilde{c}(\tilde{X}) = \tilde{X}$ *and for all other* F_{E_i}*, define* $\tilde{c}(F_{E_i}) = \tilde{X}$*.*
Then $\tilde{c}$ *is not a* $\check{C}$*ech soft closure operator on* X *because* $\tilde{c}(\{(e_1, \{a\}), (e_2, \{c\})\}$
$\tilde{\cup}\{(e_1, \{c\}), (e_2, \{c\})\}) = \tilde{c}(\{(e_1, \{a, c\}), (e_2, \{c\})\}) = \tilde{X}$*.*
But $\tilde{c}(\{(e_1, \{a\}), (e_2, \{c\})\})\tilde{\cup}\tilde{c}(\{(e_1, \{c\}), (e_2, \{c\})\}) = \{(e_1, \{a\}), (e_2, \{c\})\}\tilde{\cup}\{(e_1, \{c\}), (e_2, \{c\})\}$
$= \{(e_1, \{a, c\}), (e_2, \{c\})\} \neq \tilde{X}$*.*

Definition 3.4. *Let* $\tilde{c}$ *and* $\tilde{d}$ *be two* $\tilde{C}$*ech soft closure operators on a set* X*.* $\tilde{c}$ *is said to be coarser than* $\tilde{d}$*, or equivalently* $\tilde{d}$ *is finer than* $\tilde{c}$ *, if* $\tilde{d}(F_E) \tilde{\subset} \tilde{c}(F_E)$*, for each soft set* F_E *over* X*.*

Definition 3.5. *Let* $(X, \tilde{c}, E)$ *be a* $\tilde{C}$*ech soft closure space. A soft subset* F_E *over* X *is called soft closed provided* $F_E = \tilde{c}(F_E)$*. A soft subset* F_E *over* X *is called soft open provided its soft complement* $\tilde{X} - F_E$ *is soft closed.*

Definition 3.6. *Let* $(X, \tilde{c}, E)$ *be a* $\tilde{C}$*ech soft closure space.* G_E *be a soft set over* X *and* $x \in X$*. Then* x *is said to be a soft interior point of* G_E *if there exists a soft open set* F_E *such that* $x \tilde{\in} F_E \tilde{\subset} G_E$*.*

Definition 3.7. *Let* $(X, \tilde{c}, E)$ *be a* $\tilde{C}$*ech soft closure space.* G_E *be a soft set over* X *and* $x \in X$*. Then* G_E *is said to be a soft neighbourhood of* x *if there exists a soft open set* F_E *such that* $x \in F_E \tilde{\subset} G_E$*.*

Remark 3.8. *For each* $\tilde{c}$*ech soft closure space, there exists an underlying topological space that can be defined in a natural way. If* $(X,\tilde{c}, E)$ *is a* $\tilde{C}$*ech soft closure space, we denote the associated topology on* X *by* $\tilde{\tau}(\tilde{c})$ *. That is* $\tilde{\tau}(\tilde{c}) = \{F'_E : \tilde{c}(F_E) = F_E\}$ *where* F'_E *denotes the relative complement of* F_E *. Members of* $\tilde{\tau}(\tilde{c})$ *are the soft open sets of* $(X, \tilde{c}, E)$ *and their complements are the soft closed sets.*

Theorem 3.9. *Let* $(X, \tilde{c}, E)$ *be a ech soft closure space and* $F_E \tilde{\subset} G_E \tilde{\subset} \tilde{X}$*. Then* $\tilde{c}(F_E)$ *is contained in* $\tilde{c}(G_E)$*.*

Proof. $\tilde{c}(F_E) \tilde{\subset} \tilde{c}(F_E) \tilde{\cup} \tilde{c}(G_E) = \tilde{c}(F_E \tilde{\cup} G_E) = \tilde{c}(G_E)$, since $F_E \tilde{\cup} G_E = G_E$. □

Theorem 3.10. *Let* $(X, \tilde{c}, E)$ *be a* $\tilde{C}$*ech soft closure space and* F_E *be a soft set over* X*. If* $\tilde{c}(F_E)$ *is contained in* F_E*, then* F_E *is soft closed.*

Theorem 3.11. *Let* $(X, \tilde{c}, E)$ *be a* $\tilde{C}$*ech soft closure space. Then* $\tilde{\tau}(\tilde{c})$ *is a soft topology on* X*.* $\tilde{\tau}(\tilde{c})$ *is called the underlying soft topology of* $(X, \tilde{c}, E)$*.*

Proof. Clearly, $\tilde{X}$ and ϕ are members of $\tilde{\tau}(\tilde{c})$. Suppose F_E and G_E are members of $\tilde{\tau}(\tilde{c})$. $\tilde{X}-(F_E\tilde{\cap}G_E)=(\tilde{X}-F_E)\tilde{\cup}(\tilde{X}-G_E)=\tilde{c}(\tilde{X}-F_E)\tilde{\cup}\tilde{c}(\tilde{X}-G_E)=\tilde{c}(\tilde{X}-F_E\,\tilde{\cup}\,\tilde{X}-G_E)=\tilde{c}(\tilde{X}-(F_E\tilde{\cap}G_E))$. Now consider an arbitrary collection of soft sets $\{F_{E_\alpha}:\alpha\in J\}$, each a member of $\tilde{\tau}(\tilde{c})$. For each $\alpha\in J$, $\tilde{X}-F_{E_\alpha}$ is soft closed and $\tilde{\cap}\{\tilde{X}-F_{E_\alpha}:\alpha\in J\}$ is contained in $\tilde{X}-F_{E_\alpha}$. Theorem 3.8, then implies that $\tilde{c}(\tilde{\cap}\{\tilde{X}-F_{E_\alpha}:\alpha\in J\})$ is contained in $\tilde{c}(\tilde{X}-F_{E_\alpha})=\tilde{X}-F_{E_\alpha}$, for every $\alpha\in J$. Hence $\tilde{c}(\tilde{\cap}\{\tilde{X}-F_{E_\alpha}:\alpha\in J\})$ is contained in $\tilde{\cap}\{\tilde{X}-F_{E_\alpha}:\alpha\in J\}$ and by Theorem 3.9, $\tilde{\cap}\{\tilde{X}-F_{E_\alpha}:\alpha\in J\}=\tilde{X}-\tilde{\cup}\{F_{E_\alpha}:\alpha\in J\}$ is soft closed. □

Theorem 3.12. *Let $\{F_{E_\alpha}\}_{\alpha\in J}$ be a collection of soft closed sets in a $\tilde{C}$ech soft closure space $(X,\tilde{c},E)$. Then*

1. *The intersection of any number of soft closed sets is a soft closed set over X.*

2. *The union of any two soft closed sets is a soft closed set over X.*

Proof. (i) Since $\tilde{\cap}F_{E_\alpha}\tilde{\subseteq}F_{E_\alpha}$ for each $\alpha\in J$, by Theorem 3.8, $\tilde{c}\tilde{\cap}_{\alpha\in J}F_{E_\alpha}\tilde{\subseteq}\tilde{c}F_{E_\alpha}$, for all $\alpha\in J$. Since $\{F_{E_\alpha}\}_{\alpha\in J}$ is a collection of soft closed sets, $\tilde{c}(F_{E_\alpha})=F_{E_\alpha}$ for all $\alpha\in J$. Hence $\tilde{c}(\tilde{\cap}_{\alpha\in J}F_{E_\alpha})\tilde{\subseteq}F_{E_\alpha}$, for each $\alpha\in J$. Thus, $\tilde{c}(\tilde{\cap}_{\alpha\in J}F_{E_\alpha})\tilde{\subseteq}\tilde{\cap}_{\alpha\in J}F_{E_\alpha}$. Since $\tilde{\cap}_{\alpha\in J}F_{E_\alpha}\tilde{\subseteq}\tilde{c}(\tilde{\cap}_{\alpha\in J}F_{E_\alpha})$, $\tilde{\cap}_{\alpha\in J}F_{E_\alpha}=\tilde{c}(\tilde{\cap}_{\alpha\in J}F_{E_\alpha})$. Therefore, $\tilde{\cap}_{\alpha\in J}F_{E_\alpha}$ is a soft closed set.
(ii) Follows from the definition of $\check{C}$ech soft closure space. □

Theorem 3.13. *Let $\{F_{E_\alpha}\}_{\alpha\in J}$ be a collection of soft open sets in a $\tilde{C}$ech soft closure space $(X,\tilde{c},E)$. Then*

1. *The union of any number of soft open sets is a soft open set over X.*

2. *The intersection of any two soft open sets is a soft open set over X.*

Proof. Follows from the Theorem 3.12, and De-Morgan's laws for soft sets which are given in Theorem 2.20. □

Definition 3.14. *Let $(X,\tilde{c},E)$ be a $\tilde{C}$ech soft closure space.If $\tilde{c}(F_E)=F_E$ for every soft set F_E contained in $\tilde{X}$, $\tilde{c}$ is called the soft discrete closure operator on X. If $\tilde{c}(F_E)=\tilde{X}$ for every soft set F_E contained in $\tilde{X}$, $\tilde{c}$ is called the trivial $\check{C}$ech soft closure operator on X.*

we extend the notions of a soft subspace, a soft sum and a product to $\check{C}$ech soft closure spaces.
One may easily verify the following theorem.

Theorem 3.15. *Let $(X, \tilde{c}, E)$ be Čech soft closure space and let Y be an arbitrary subset of X. The operator $\tilde{c}_Y : S(Y, E) \to S(Y, E)$ defined by $\tilde{c}_Y(F_E) = \tilde{Y}\tilde{\cap}\tilde{c}(F_E)$ is a Čech soft closure operator on Y.*

Definition 3.16. *Let $(X, \tilde{c}, E)$ be Čech soft closure space and let Y be an arbitrary subset of X. The Čech soft closure operator $\tilde{c}_Y$(defined above) is called the relative Čech soft closure operator on Y induced by $\tilde{c}$. The triple $(Y, \tilde{c}_Y, E)$ is said to be a Čech soft closure subspace of $(X, \tilde{c}, E)$, it is a soft closed(resp. soft open) subspace if $\tilde{c}(\tilde{Y}) = \tilde{Y}$ (resp. $\tilde{c}(\tilde{X} - \tilde{Y}) = \tilde{X} - \tilde{Y}$.*

If $(X, \tilde{c}, E)$ is a soft topological space under its closure operator, the definition of a Čech soft closure subspace is reduced to the corresponding definition of a subspace. It is well known that if $(X, \tilde{c}, E)$ is a soft toplogical space under its Čech soft closure operator, then the two soft topologies defined on a subset Y of X, viz., the soft relative topology $(\tilde{\tau}(\tilde{c}))_Y$ on Y induced by $\tilde{\tau}(\tilde{c})$ and the soft topology $\tilde{\tau}(\tilde{c}_Y)$ are identical. But in general, we have only the following result.

Theorem 3.17. *Let Y be an arbitrary subset of a Čech soft closure space $(X, \tilde{c}, E)$. The relative soft topology $(\tilde{\tau}(\tilde{c}))_Y$ on Y induced by $\tilde{\tau}(\tilde{c})$ is coarser than the soft topology $\tilde{\tau}(\tilde{c}_Y)$.*

Proof. Let F_E be a $(\tilde{\tau}(\tilde{c}))_Y$-soft closed set over Y. Then $F_E = \tilde{Y}\tilde{\cap}F_F$ for some $\tilde{\tau}(\tilde{c})$-soft closed set in F_F over X. But $F_E\tilde{\subseteq}F_F$ implies that $\tilde{c}F_E\tilde{\subseteq}\tilde{c}F_F = F_F$ Hence, $\tilde{c}_Y(F_E) = \tilde{Y}\tilde{\cap}\tilde{c}F_E\tilde{\subseteq}\tilde{Y}\tilde{\cap}F_F = F_E$. Consequently $\tilde{c}_Y(F_E) = F_E$ and F_E in $\tilde{\tau}(\tilde{c}_Y)$-soft closed. □

Theorem 3.18. *Let $\{(X_i, \tilde{c}_i, E_i) : i \in I\}$ be a family of pairwise disjoint Čech soft closure spaces. If $X = \cup X_i$ and $E = \cup E_i$, then the operator $\oplus\tilde{c}_i : S(X, E) \to S(X, E)$ defined by, $\oplus\tilde{c}_i(F_E) = \tilde{\cup}\tilde{c}_i(\tilde{X}_i\tilde{\cap}F_E)$, is a Čech soft closure operator on X.*

Proof. (C1) $\oplus\tilde{c}_i(\phi) = \tilde{\cup}\tilde{c}_i(\tilde{X}_i\tilde{\cap}\phi) = \tilde{\cup}\tilde{c}_i(\phi) = \phi$.

(C2) $\oplus\tilde{c}_i(F_E) = \tilde{\cup}\tilde{c}_i(\tilde{X}_i\tilde{\cap}F_E)\tilde{\supseteq}\tilde{\cup}(\tilde{X}_i\tilde{\cap}F_E) = F_E$.

(C3) $\oplus\tilde{c}_i(F_E\tilde{\cup}G_E) = \tilde{\cup}\tilde{c}_i(\tilde{X}_i\tilde{\cap}(F_E\tilde{\cup}G_E)) = \tilde{\cup}\tilde{c}_i((\tilde{X}_i\tilde{\cap}F_E)\tilde{\cup}(\tilde{X}_i\tilde{\cap}G_E)) = \tilde{\cup}(\tilde{c}_i(\tilde{X}_i\tilde{\cap}F_E)\tilde{\cup}\tilde{c}_i(\tilde{X}_i\tilde{\cap}G_E)) = (\tilde{\cup}\tilde{c}_i(\tilde{X}_i\tilde{\cap}F_E))\tilde{\cup}(\tilde{\cup}\tilde{c}_i(\tilde{X}_i\tilde{\cap}G_E)) = \oplus\tilde{c}_i(F_E)\tilde{\cup} \oplus \tilde{c}_i(G_E)$.

□

Definition 3.19. *Let $\Im = \{(X_i, \tilde{c}_i, E_i) : i \in I\}$ be a family of pairwise disjoint Čech soft closure spaces. The Čech soft closure operator $\oplus\tilde{c}_i$ (defined above) is called the sum Čech soft closure operator on $X = \cup X_i$ and the triple $(X, \oplus\tilde{c}_i, E)$, where $E = \cup E_i$ is the sum Čech soft closure space of the family $\Im$.*

Remark 3.20. *If $\{(X_i, \tilde{\tau}(\tilde{c}_i), E_i) : i \in I\}$ are soft topological spaces under their Čech soft closure operators, the definition of the sum Čech soft closure space is reduced to the corresponding definition of sum soft space.*

Remark 3.21. *One may notice that in a sum Čech soft closure space $(X, \oplus\tilde{c}_i, E)$, $x \in \oplus\tilde{c}_i(F_E)$ if and only if $x \in \tilde{c}_i(\tilde{X}_i \tilde{\cap} F_E)$ for a unique i.*

We shall define the product Čech soft closure space. Given a cartesian product of sets $X = \pi X_i$ and the parameter set $E = \pi E_i$, let $p_i : X \to X_i$, and $q_i : E \to E_i$ be projection maps in classical meaning. Then the soft map $p_{q_i} : S(X, E) \to S(X_i, E_i)$ is called soft projection map, where $p_{q_i}(F_E) = F_{E_i}$ for F_E over πX_i.

Theorem 3.22. *Let $\{(X_i, \tilde{c}_i, E) : i \in I\}$ be a family of Čech soft closure spaces and let $X = \pi X_i$. Define an operator $\otimes\tilde{c}_i : S(X, E) \to S(X, E)$ as follows:*
For a soft subset F_E over X and $x \in X$; $x \in \otimes\tilde{c}_i(F_E)$ if the following condition is satisfied:
$F_E = F_{E_1} \tilde{\cup} F_{E_2} \tilde{\cup} ... \tilde{\cup} F_{E_n} (F_{E_j} \tilde{\subset} \tilde{X})$ implies there is j such that $x_i \in \tilde{c}_i(p_{q_i}(F_{E_j}))$ for all i. The operator $\otimes\tilde{c}_i$ is a Čech soft closure operator on X.

Proof. (C1) Clearly $\otimes\tilde{c}_i(\phi) = \phi$.

(C2) Let $x \in F_E = F_{E_1} \tilde{\cup} F_{E_2} \tilde{\cup} ... \tilde{\cup} F_{E_n}$, $(F_{E_i} \tilde{\subset} \tilde{X})$. There is j such that $x \in F_{E_i}$. Thus $x_i \in p_{q_i}(F_{E_i}) \tilde{\subset} \tilde{c}_i(p_{q_i}(F_{E_i}))$ for all i. By definition, $x \in \otimes\tilde{c}_i(F_E)$, that is $F_E \tilde{\subset} \otimes \tilde{c}_i(F_E)$.

(C3) Clearly, if $F_E \tilde{\subset} G_E (F_E, F_E \tilde{\subset} \tilde{X})$, then $\otimes\tilde{c}_i(F_E) \tilde{\subset} \otimes \tilde{c}_i(G_E)$. Thus we need only show that $\otimes\tilde{c}_i(F_E \tilde{\cup} G_E) \tilde{\subset} \otimes \tilde{c}_i(F_E) \tilde{\cup} \otimes \tilde{c}_i(G_E)$. Let $x \notin \otimes\tilde{c}_i(F_E) \tilde{\cup} \otimes \tilde{c}_i(G_E)$. Then there exist covering F_{E_1},...F_{E_m} of F_E and $F_{E_{m+1}}$,...F_{E_n} of G_E together with incies $i_1, ..., i_n$ such that $x_j \notin \tilde{c}_j(p_j(F_{E_i}))$ for $j = i_k$, where $k = 1, ...n$. Since F_{E_1},... F_{E_n} covers $F_E \tilde{\cup} G_E$; $x \notin \otimes\tilde{c}_i(F_E \tilde{\cup} G_E)$.

□

Definition 3.23. *Let $\Im = \{(X_i, \tilde{c}_i, E) : i \in J\}$ be a family of Čech soft closure spaces. The soft closure operator $\tilde{c}_i$ on $X = \pi X_i$, defined above, is called the product Čech soft closure operator on X. The triplet $(X, \otimes\tilde{c}_i, E)$ is said to be the product Čech soft closure space of the family $\Im$.*

4 Soft continuous

Definition 4.1. *Let $(X, \tilde{c}, E)$ and $(Y, \tilde{d}, E*)$ be two Čech soft closure spaces. A soft mapping $\varphi_\psi : (X, \tilde{c}, E) \to (Y, \tilde{d}, E*)$ is said to be soft continuous if $\varphi_\psi(\tilde{c}(F_A)) \tilde{\subseteq} \tilde{d}(\varphi_\psi(F_A))$, for every soft subset $F_A \tilde{\subseteq} \tilde{X}$.*

Theorem 4.2. *$(X, \tilde{c}, E)$ and $(Y, \tilde{d}, E*)$ be two Čech soft closure spaces. $\varphi_\psi : (X, \tilde{c}, E) \to (Y, \tilde{d}, E*)$ is soft continuous, then $\tilde{c}(\varphi_\psi^{-1}(F_B)) \tilde{\subseteq} \varphi_\psi^{-1}(\tilde{d}(F_B))$ for every soft subset $F_B \tilde{\subseteq} \tilde{Y}$.*

Proof. Let $F_B \tilde{\subseteq} \tilde{Y}$. Then $\varphi_\psi^{-1}(F_B) \tilde{\subseteq} \tilde{X}$. Since φ_ψ is soft continuous, we have $\varphi_\psi(\tilde{c}(\varphi_\psi^{-1}(F_B))) \tilde{\subseteq} \tilde{d}(\varphi_\psi(\varphi_\psi^{-1}(F_B))) \tilde{\subseteq} \tilde{d}(F_B)$. Therefore, $\varphi_\psi^{-1}(\varphi_\psi(\tilde{c}(\varphi_\psi^{-1}(F_B)))) \tilde{\subseteq} \varphi_\psi^{-1}(\tilde{d}(F_B))$. Hence $\tilde{c}(\varphi_\psi^{-1}(F_B)) \tilde{\subseteq} \varphi_\psi^{-1}(\tilde{d}(F_B))$. □

Clearly, if $\varphi_\psi : (X, \tilde{c}, E) \to (Y, \tilde{d}, E*)$ is soft continuous, then $\varphi_\psi^{-1}(F_F)$ is a soft closed subset of $(X, \tilde{c}, E)$ for every soft closed subst F_F of $(Y, \tilde{d}, E*)$. The following statement is evident.

Theorem 4.3. *Let $(X, \tilde{c}, E)$ and $(Y, \tilde{d}, E*)$ be Čech soft closure spaces. If $\varphi_\psi : (X, \tilde{c}, E) \to (Y, \tilde{d}, E*)$ is soft continuous, then $\varphi_\psi^{-1}(F_G)$ is a soft open subset of $(X, \tilde{c}, E)$ for every soft open subset F_G of $(Y, \tilde{d}, E*)$.*

Theorem 4.4. *Let $(X, \tilde{c}, E)$, $(Y, \tilde{d}, E*)$ and $(Z, \tilde{e}, E**)$ be Čech soft closure spaces. If $\varphi_\psi : (X, \tilde{c}, E) \to (Y, \tilde{d}, E*)$ and $\gamma_\delta : (Y, \tilde{d}, E*) \to (Z, \tilde{e}, E**)$ are soft continuous, then $\gamma_\delta \tilde{\circ} \varphi_\psi : (X, \tilde{c}, E) \to (Z, \tilde{e}, E**)$ is soft continuous.*

Proof. Let $F_A \tilde{\subseteq} \tilde{X}$. Since $\gamma_\delta \tilde{\circ} \varphi_\psi(\tilde{c}(F_A)) = \gamma_\delta(\varphi_\psi(\tilde{c}(F_A)))$ and φ_ψ is soft continuous , $\gamma_\delta(\varphi_\psi(\tilde{c}(F_A))) \tilde{\subseteq} \gamma_\delta(\tilde{d}(\varphi_\psi(F_A)))$. As γ_δ soft continuous, we get $\gamma_\delta(\tilde{d}(\varphi_\psi(F_A))) \tilde{\subseteq} \tilde{e}(\gamma_\delta(\varphi_\psi(F_A)))$. Consequently, $\gamma_\delta \tilde{\circ} \varphi_\psi(\tilde{c}(F_A)) \tilde{\subseteq} \tilde{e}(\gamma_\delta(\varphi_\psi(F_A)))$. Hence $\gamma_\delta \tilde{\circ} \varphi_\psi$ is soft continuous. □

Definition 4.5. *Let $(X, \tilde{c}, E)$ and $(Y, \tilde{d}, E*)$ be Čech soft closure spaces. A soft mapping $\varphi_\psi : (X, \tilde{c}, E) \to (Y, \tilde{d}, E*)$ is said to be soft closed (resp. soft open) if $\varphi_\psi(F_F)$ is a soft closed (resp. soft open) subset of $(Y, \tilde{d}, E*)$ whenever F_F is soft closed (resp. soft open) subset of $(X, \tilde{c}, E)$.*

Theorem 4.6. *A soft mapping $\varphi_\psi : (X, \tilde{c}, E) \to (Y, \tilde{d}, E*)$ is soft closed if and only if , for each soft subset F_B of $\tilde{Y}$ and each soft open subset F_G of $(X, \tilde{c}, E)$ containing $\varphi_\psi^{-1}(F_B)$, there is a soft open subset F_U of $(Y, \tilde{d}, E*)$ such that $F_U \tilde{\subseteq} F_B$ and $\varphi_\psi^{-1}(F_U) \tilde{\subseteq} F_G$.*

Proof. Suppose that φ_ψ is soft closed let F_B be a soft subset of $\tilde{Y}$ and F_G be a soft open subset of $(X, \tilde{c}, E)$ such that $\varphi_\psi^{-1}(F_B)\tilde{\subseteq} F_G$. Then $\varphi_\psi(\tilde{X} - F_G)$ is a soft closed subset of $(Y, \tilde{d}, E*)$. Let $F_U = \tilde{Y} - \varphi_\psi(\tilde{X} - F_G)$. Then F_U is a soft open subset of $(Y, \tilde{d}, E*)$ and $\varphi_\psi^{-1} F_U = \varphi_\psi^{-1}(\tilde{Y} - \varphi_\psi(\tilde{X} - F_G)) = \tilde{X} - \varphi_\psi^{-1}(\varphi_\psi(\tilde{X} - F_G))\tilde{\subseteq}\tilde{X} - (\tilde{X} - F_G) = F_G$. Therefore, F_U is a soft open subset of $(Y, \tilde{d}, E*)$ containing F_B such that $\varphi_\psi^{-1}(F_U)\tilde{\subseteq} F_G$. Conversely, suppose that F_F is a soft closed subset of $(X, \tilde{c}, E)$. Then $\varphi_\psi^{-1}(\tilde{Y} - \varphi_\psi(F_F)\tilde{\subseteq}\tilde{X} - F_F$ and $\tilde{X} - F_F$ is a soft open subset of $(X, \tilde{c}, E)$. By hypothesis, there is a soft open subset F_U of $(Y, \tilde{d}, E*)$ such that $\tilde{Y} - \varphi_\psi(F_F)\tilde{\subseteq} F_U$ and $\varphi_\psi^{-1}(F_U)\tilde{\subseteq}\tilde{X} - F_F$. Therefore $F_F\tilde{\subseteq}\tilde{X} - \varphi_\psi^{-1}(F_U)$. Consequently, $\tilde{Y} - F_U\tilde{\subseteq}\varphi_\psi(F_F)\tilde{\subseteq}\varphi_\psi(\tilde{X} - \varphi_\psi^{-1}(F_U))\tilde{\subseteq}\tilde{Y} - F_U$, which implies that $\varphi_\psi(F_F) = \tilde{Y} - F_U$. Thus, $\varphi_\psi(F_F)$ is a soft closed subset of $(Y, \tilde{d}, E*)$. Hence φ_ψ is soft closed. □

Theorem 4.7. *Let $(X, \tilde{c}, E)$, $(Y, \tilde{d}, E*)$ and $(Z, \tilde{e}, E**)$ be Čech soft closure spaces. Let $\varphi_\psi : (X, \tilde{c}, E) \to (Y, \tilde{d}, E*)$ and $\gamma_\delta : (Y, \tilde{d}, E*) \to (Z, \tilde{e}, E**)$ soft mappings. Then*

1. *If φ_ψ and γ_δ are soft closed, then $\gamma_\delta \tilde{\circ} \varphi_\psi$.*
2. *If $\gamma_\delta \tilde{\circ} \varphi_\psi$ is soft closed and φ_ψ is soft continuous and surjection, then γ_δ is soft closed.*
3. *If $\gamma_\delta \tilde{\circ} \varphi_\psi$ is soft closed and γ_δ is soft continuous and injection, then φ_ψ is soft closed.*

Proof. (i)Let F_F be a soft closed subset of $(X, \tilde{c}, E)$. Since φ_ψ is soft closed, $\varphi_\psi(F_F)$ is soft closed in $(Y, \tilde{d}, E*)$. Hence $\gamma_\delta(\varphi_\psi(F_G))$ is soft closed in $(Z, \tilde{e}, E**)$. Thus $\gamma_\delta \tilde{\circ} \varphi_\psi$ is soft closed .
(ii) Let F_F be a soft closed subset of $(Y, \tilde{d}, E*)$. Since φ_ψ is a soft continuous map, $\varphi_\psi^{-1}(F_F)$ is soft closed in $(X, \tilde{c}, E)$. Since $\gamma_\delta \tilde{\circ} \varphi_\psi$ is soft closed , $\gamma_\delta \tilde{\circ} \varphi_\psi(\varphi_\psi^{-1}(F_G)) = \gamma_\delta(\varphi_\psi(\varphi_\psi^{-1}(F_G)))$ is soft closed in $(Z, \tilde{e}, E**)$. But φ_ψ is surjection, so that $\gamma_\delta \tilde{\circ} \varphi_\psi(\varphi_\psi^{-1}(F_G)) = \gamma_\delta(\varphi_\psi(\varphi_\psi^{-1}(F_G))) = \gamma_\delta(F_G)$. Hence, $\gamma_\delta(F_G)$ soft closed in $(Z, \tilde{e}, E**)$. Therefore γ_δ is soft closed.
(iii) Let F_F be a soft closed subset of $(X, \tilde{c}, E)$. Since $\gamma_\delta \tilde{\circ} \varphi_\psi$ is soft closed, $\gamma_\delta(\varphi_\psi(F_F)$ is soft closed in $(Z, \tilde{e}, E**)$. As γ_δ is soft continuous,$\gamma_\delta^{-1}(\gamma_\delta(\varphi_\psi(F_F)))$ is soft closed in $(Y, \tilde{d}, E*)$. But γ_δ is injective, so that $\gamma_\delta^{-1}(\gamma_\delta(\varphi_\psi(F_F))) = \varphi_\psi(F_F)$ is soft closed in $(Y, \tilde{d}, E*)$. Therefore, φ_ψ is soft closed. □

Theorem 4.8. *The product Čech soft closure space is the largest Čech soft closure space for which every soft projection is soft continuous.*

Proof. Let $\Im = \{(X_i, \tilde{c}_i, E_i) : i \in I\}$ be a family of Čech soft closure spaces and let $(X, \otimes\tilde{c}_i, E)$ be the product Čech soft closure space of $\Im$. The Čech soft continuity of each soft projection follows from Definitions and. Now, given any Čech soft closure operator $\tilde{c}$ for which each soft projection is Čech soft continuous, consider any point $x \in \tilde{c}(F_A)$ and let $F_A = F_{A_1}\tilde{\cup}F_{A_2}\tilde{\cup}F_{A_3}\tilde{\cup}...\tilde{\cup}F_{A_n}(F_{A_i}\tilde{\subset}\tilde{X})$. By (C3), there exists j such that $x \in \tilde{c}(F_{A_j})$. Since each p_{q_i} is Čech soft continuous, it follows that $p_{q_i}(c(F_{A_j}))\tilde{\subset}\tilde{c}_i(p_{q_i}(F_{A_j}))$ for all i. This implies that $x \in \otimes\tilde{c}_i(F_{A_j})\tilde{\subset} \otimes \tilde{c}_i(F_A)$. □

Conclusion 4.9. *In this paper, we have studied Čech soft closure operators which are defined in the set of all soft sets over a non-empty set and a fixed set of parameters. The notions of soft closed set, soft open set, soft interior points, soft neighbourhood of a soft point are also studied. Then we have studied for each Čech soft closure space there exists an underlying soft topological space that can be defined in a natural way. We have defined sum Čech soft closure space and product Čech soft closure space from a family of Čech soft closure spaces. Finally, we have defined Čech soft continuous function and studied some of its properties.*

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