On gr*-Closed Sets in a Topological Spaces

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Abstract - The aim of this paper is to introduce the concept of generalized regular star closed sets and study some of its properties.

Keywords: rg-closed set, gr-closed sets, gr*-closed set, gr*-open set.

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1. INTRODUCTION

In 1970, Levine [8] first introduced the concept of generalized closed (briefly, g-closed) sets were defined and investigated. Later on N. Palaniappan [15] studied the concept of regular generalized closed (briefly, rg-closed) in a topological space. In 2000, M.K.R.S.Veerakumar [17] introduced the concept of g*-closed set and in 2001, S. Bhattacharya [5] studied the concept of generalized regular closed sets (briefly, gr-closed) in a topological space. In this paper, the concept of generalised regular star closed sets is to be introduced and studied some of their properties.

2. PRELIMINARIES

Throughout this paper, (X,τ) (or X) represent a topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space X, cl(A) and int(A) denote the closure of A and the interior of A, respectively.

Definition 2.1. A subset A of a topological space (X,τ) is called

1) a pre-open set [12] if A ⊆ int (cl(A)) and a pre-closed set if cl(int(A)) ⊆ A.
2) a semi-open set [7] if A ⊆ cl (int(A)) and a semi-closed set if int(cl(A)) ⊆ A.
3) a semi-preopen set [1] if A ⊆ cl(int(cl(A))) and a semi-pre-closed set [2] if (int(cl(A))) ⊆ A.
4) a regular-open set [16] if A = int(cl(A)) and a regular-closed set if A = cl(int(A)).

Definition 2.2. [3] For any subset A of (X,τ), rcl(A)=∩{B:B⊇A, B is a regular closed subset of X}.

Definition 2.3. A subset A of a topological space (X,τ) is called

1) a generalized closed set (briefly g-closed) [8] if cl(A) ⊆ U whenever A ⊆ U and U is open in (X,τ).
2) a generalized semi-closed set (briefly gs-closed) [2] if scl(A) ⊆ U whenever A ⊆ U and U is open in (X,τ).
3) a semi generalized closed set (briefly sg-closed) [5] if scl(A) ⊆ U whenever A ⊆ U and U is semi-open in (X,τ).

4) an generalized α-closed set (briefly gα-closed) [10] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in (x,τ).
5) an α-generalized closed set (briefly αg-closed) [9] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (x,τ).
6) a generalized semi-preclosed set (briefly gsp-closed) [6] if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ).
7) a generalized preclosed set (briefly gp-closed) [11] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ).
8) a generalized closed set (briefly g^*-closed) [17] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in (X,τ).
9) a regular generalized closed set (briefly rg-closed) [15] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in (X,τ).
10) a generalized regular closed set (briefly gr-closed) [5] if $rcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ).

3. gr^*-CLOSED SET

Definition 3.1. A subset A of a topological space (X,τ) is called a generalized regular star closed set [briefly gr^*-closed] if $Rcl(A)\subseteq U$ whenever $A\subseteq U$ and U is g-open subset of X.

Theorem 3.2. ϕ and X are generalized regular star closed subset of X.

Theorem 3.3. Every closed set in X is gr^*-closed in X.

Proof: Let A be a closed set in X. Let U be a g-open such that $A\subseteq U$. Since A is closed, that is $cl(A)=A$, $cl(A)\subseteq U$. But $cl(A)\subseteq Rcl(A)\subseteq U$. Therefore $Rcl(A)\subseteq U$. Hence A is gr^*-closed set in (X,τ).

Theorem 3.4. Every r-closed set in X is gr^*-closed in X.

Proof: Let A be r-closed set in X. Let U be g-open such that $A\subseteq U$. Since A is r closed, we have $Rcl(A) = A\subseteq U$. Hence A is gr^*-closed set in (X,τ).

Example 3.5. The converses of the above need not be true as seen from the following example.

Consider the topological space $X=\{a,b,c,d\}$ with the topology $\tau = \{\phi, \{d\}, \{b,c\}, \{b,c,d\},X\}$. Let $A=\{a,c,d\}$. Then A is gr^*-closed set but not closed set and r-closed set.

Theorem 3.6. For a topological space (X,τ), the following conditions are hold.

(i) Every gr^*-closed set is gs-closed set.
(ii) Every gr^*-closed set is sg-closed set.
(iii) Every gr^*-closed set is gp-closed set.
(iv) Every gr^*-closed set is gsp-closed set.
(v) Every gr^*-closed set is rg-closed set.
(vi) Every gr^*-closed set is gpr-closed set.
(vii) Every gr^*-closed set is αg-closed set.
(viii) Every gr^*-closed set is g-closed set.

Proof: i) Let A be gr^*-closed set in X. Let U be open set such that $A\subseteq U$. Since every open set is g-open and A is gr^*-closed, we have $scl(A)\subseteq rcl(A)\subseteq U$. Therefore A is gs-closed set in X.

ii) Let A be gr^*-closed set in X. Let U be a semi-open such that $A\subseteq U$. Since every semi-open set is g-open and A is gr^*-closed. We have $scl(A) \subseteq rcl(A)\subseteq U$. Therefore A is sg-closed set in X.

iii) Let A be gr^*-closed set in X. Let U be a open set such that $A\subseteq U$. Since every open set is g-open and A is gr^*-closed. We have $pcl(A) \subseteq rcl(A)\subseteq U$. Therefore A is gp-closed set in X.

iv) Let A be gr^*-closed set in X. Let U be a open set such that A⊆U. Since every open set is g-open and A is gr^*-closed. We have spcl(A) ⊆rcl(A)⊆U. Therefore A is gsp-closed set in X.
v) Let A be gr^*-closed set in X. Let U be a r-open set such that A⊆U. Since every r-open set is g-open and A is gr^*-closed. We have cl(A) ⊆rcl(A)⊆U. Therefore A is rg-closed set in X.
vi) Let A be gr^*-closed set in X. Let U be a r-open set such that A⊆U. Since every r-open set is s-open and A is gr^*-closed. We have pcl(A) ⊆rcl(A)⊆U. Therefore A is gpr-closed set in X.
vii) Let A be gr^*-closed set in X. Let U be a open set such that A⊆U. Since every open set is g-open and A is gr^*-closed. We have αcl(A) ⊆rcl(A)⊆U. Therefore A is αg-closed set in X.
viii) It is obvious.

The converses of the above need not be true as seen from the following example.

Example3.7. Consider the topological space X={a,b,c,d} with topology τ={ϕ,{c},{d},{c,d},{b,c,d},X}. Let A={b} is g-closed, gs-closed, sg-closed, gp-closed, gsp-closed, rg-closed, gpr-closed and αg-closed set but not gr^*-closed set in (X,τ).

Theorem 3.8. Every gr^*-closed set is gr-closed set but not converserly.
Prove: Let A be gr^*-closed set in X. Let U be a open set such that A⊆U. Since every open set is g-open and A is gr^*-closed. We have cl(A) ⊆ rcl(A) ⊆ U. Therefore A is gr-closed set in X.

Example 3.9. Consider the topological space X={a,b,c,d} with topology τ={ϕ,{a,b},{c,d},X}. Let A= {a} is gr-closed but not gr^*-closed set in (X,τ).

Remark 3.10. gr^*-closed sets and gα-closed sets are independent of each other. It is shown by following example.

Example 3.11. In Example: 3.5, The subset {b} is gα-closed but not gr^*-closed and the subset {a,d} is gr^*-closed but not gα-closed.

Remark 3.12. gr^*-closed sets and g^*s-closed sets are independent of each other. It is shown by following example.

Example 3.13. In Example: 3.7, The subset {a, c, d} is gr^*-closed set but not g^*s-closed set and the subset {b} is g^*s-closed but not gr^*-closed set.

Theorem 3.14. Let A be a g-open subset of (X, τ). Then A is r-closed set if A is gr^*-closed set.

Theorem 3.15. A subset A of (X,τ) is a generalized regular star closed set if it is a regular
closed set.

Remark 3.16. The converse of the above theorem need not be true which follows from the example 3.5. Let A={a}. Then A is generalized regular star closed set but not a regular closed set.

Theorem 3.17. The finite union of the gr^*-closed sets is gr^*-closed.
Proof: Let A and B be gr^*-closed sets in X. Let U be a g-open in X such that A∪B⊆U. Then A⊆U and B⊆U. Since A and B are gr^*-closed set closed, rcl(A)⊆U and rcl(B)⊆U. Hence rcl(A∪B) = rcl(A) ∪ rcl(B) ⊆ U. Therefore A∪B is g^*r-closed.

Theorem 3.18. The finite intersection of two gr^*-closed sets is gr^*-closed.
Proof: The proof is obvious.

Theorem 3.19. The intersection of a generalized regular star closed set and a closed set is a generalized closed set.

Proof: Let A be a generalized regular star closed subset of X and F is a closed set. If U is an g-open subset of X with $A \cap F \subseteq U$ then $A \subseteq U \cup (X \backslash F)$. So, $Rcl(A) \subseteq U \cup (X \backslash F)$. Then $cl(A \cap F) = cl(A) \cap cl(F) \subseteq Rcl(A) \cap cl(F) = Rcl(A) \cap F \subseteq U$. So $A \cap F$ is a generalized closed set.

Remark 3.20. The intersection of a generalized regular star closed set and a regular closed set is a generalized regular star closed set i.e, the intersection of two regular closed set is a generalized regular star closed set.

Theorem 3.21. Let $A \subseteq B \subseteq Rcl(A)$ and A is a generalized regular star closed subset of (X,τ) then B is also a generalized regular star closed subset of (X,τ).

Proof: Since A is a generalized regular star closed subset of (X,τ). So, $Rcl(A) \subseteq U$, whenever $A \subseteq U$, U being an g-open subset of X. Let $A \subseteq B \subseteq Rcl(A)$.i.e $Rcl(A)=Rcl(B)$. Let if possible, there exists an open subset V of X such that $B \subseteq V$. So, $A \subseteq V$ and B being generalized regular star closed subset of X, $Rcl(A) \subseteq V$ i.e $Rcl(B) \subseteq V$. Hence B is also a gr^*-closed subset of X.

Theorem 3.22. Let $A \subseteq B \subseteq X$, where A is g-open in X. If A is gr^*-closed in X, then A is gr^*-closed in B.

Proof: Let $A \subseteq U$, where U is g-open set of X. Since $U=V \cap B$ for some g-open set V of X and B is g-open in X. Using assumption A if gr^*-closed in X. We have $Rcl(A) \subseteq U$ and so $Rcl(A)=cl(A) \cap B \subseteq U \cap B \subseteq U$. Hence A is gr^*-closed in B.

Theorem 3.23. Let $A \subseteq B \subseteq X$, where B is g-open and gr^*-closed in X. If A is gr^*-closed in B then A is gr^*-closed in X.

Proof: Let U be a g-open set of X such that $A \subseteq U$. Since $A \subseteq U \cap B$, where $U \cap B$ is g-open in B and A is gr^*-closed in B, $Rcl(A) \subseteq U \cap B$ holds. we have $Rcl(A) \cap B \subseteq U \cap B$. Since $A \subseteq B$ we have $Rcl(A) \subseteq Rcl(B)$. Since B is g-open and gr^*-closed in X, by Theorem 3.14 B is r-closed. Therefore $Rcl(B)=B$. Thus $Rcl(A) \subseteq B$ implies $Rcl(A)=Rcl(A) \cap B \subseteq U \cap B \subseteq U$. Hence A is gr^*-closed in X.

Theorem 3.24. A subset A of X is generalized regular star closed sets iff $Rcl(A) \cap A^c$ contain the non-zero closed set in X.

Proof: Let A be a gr^*-closed subset of X. Also if possible let M be a closed subset of X such that $M \subseteq Rcl(A) \cap A^c$ i.e, $M \subseteq Rcl(A)$ and $M \subseteq A^c$. Since M is a closed subset of X, M^c is an open subset of $X \subseteq A$. A being gr^*-open subset of X, $Rcl(A) \subseteq M^c$. But $M \subseteq Rcl(A)$. So, we get a contradiction, which leads to the conclusion that $M=\phi$. So the condition is true. Conversely, Let $A \subseteq N$, N being an open subset of X. Then $N^c \subseteq A^c$, N^c is a closed subset of X. Let if possible $Rcl(A) \subseteq N$, Then $Rcl(A) \cap N^c$ is a non-zero closed subset of $Rcl(A) \cap A^c$, which is a contradiction. Hence A is a gr^*-closed subset of X.

Theorem 3.25. A subset A of X is gr^*-closed set in X iff $Rcl(A)-A$ contains no non-empty g-closed set in X.

Proof: Suppose that F is a non-empty g-closed subset if $Rcl(A)-A$. Now $F \subseteq Rcl(A)-A$. Then $F \subseteq Rcl(A) \cap A^c$. Therefore $F \subseteq Rcl(A)$ and $F \subseteq A^c$. Since F^c is g-open set and A is gr^*-closed, $Rcl(A) \subseteq F^c$. That is $F \subseteq Rcl(A)^c$. Since F^c is g-open set and A is

gr^*-closed, $Rcl(A)\subseteq F^c$. i.e $F\subseteq Rcl(A)^c$. Hence $F\subseteq Rcl(A)\cap[Rcl(A)]^c=\phi$. i.e $F=\phi$. Thus $Rcl(A)$-A contains no non-empty g-closed set. Conversely, assume that $Rcl(A)$-A contains no non-empty g-closed set. Let $A\subseteq U$, U is g-open. Suppose that $Rcl(A)$ is not contained in U. Then $Rcl(A)\cap U^c$ is a non-empty g-closed set and contained in $Rcl(A)$-A which is a contradiction. Therefore $Rcl(A)\subseteq U$ and hence A is gr^*-closed set.

Theorem 3.26. For each $x\in X$, either $\{x\}$ is g-closed or $\{x\}^c$ is gr^*-closed in X.

Proof: If $\{x\}$ is not g-closed, then the only g-open set containing $\{x\}^c$ is X. Thus $Rcl(x^c)$ is contained in X and hence $\{x\}^c$ is gr^*-closed in X.

Theorem 3.27. In a partition space, every gr^*-closed is g-closed set.

Proof: Let A be a gr^*-closed and $A\subseteq U$, where U is open. Since every open set is a g-open set and U is g-open. By hypothesis A is gr^*-closed set. Hence we have $Rcl(A)\subseteq U$. In partition space every closed set is open. Hence the class of r-closed sets coincides with the class of closed sets. Therefore we have $cl(A)\subseteq Rcl(A)\subseteq U$. Thus we have A is g-closed.

Theorem 3.28. In a partition space, every gr^*-closed is rg-closed set.

Proof: Let A be a gr^*-closed and $A\subseteq U$, where U is r-open. In partition space every closed set, the class of r-closed sets coincides with the class of closed sets (open sets) and the class of r-open sets also coincides with the class of closed sets (open sets). Therefore we have $(X,\tau)=RO(X,\tau)=RC(X,\tau)$. Hence we also get in a partition space every r-open set is a g open set. So we have U is a g –open set with $A\subseteq U$. By hypothesis A is gr^*-closed. Hence we have $Rcl(A)\subseteq U$. Thus we have A is rg-closed.

Theorem 3.29. A subset A of a topological space (X,τ) is gr^*-closed if and only if $Rcl(A)\subseteq$g kernel of A.

Proof: Necessity: Suppose that A is gr^*-closed in X. Suppose $x\in Rcl(A)$ but $x\notin$g kernel of A. Then there exists a g-open set $G\supseteq A$, such that $x\notin G$. But $Rcl(A)\subseteq G$, since A is gr^*-closed and G is g-open containing A. Hence $x\in Rcl(A)$ and $x\notin G$ is a contradiction. Therefore $Rcl(A)\subseteq$g kernel of A.

Sufficiency: Let $Rcl(A)\subseteq$g kernel of A. Let $A\subseteq U$ where U is g-open. Then g kernel of $A\subseteq U$, implies $Rcl(A)\subseteq U$. Hence A is gr^*-closed.

4. gr^*-OPEN SETS

Definition 4.1. A subset A of a topological space X is called gr^*-open set if A^c is gr^*-closed.

Theorem 4.2. A subset A of a topological space (X,τ) is gr^*-open if and only if $B\subseteq Rint(A)$ where B is g closed in X and $B\subseteq A$.

Proof: Necessity: Suppose $B\subseteq Rint(A)$ where B is g-closed in (X,τ) and $B\subseteq A$. Let $A^C\subseteq M$, where M is g-open. Hence $M^C\subseteq A$, where M^C is g-closed. Hence by assumption $M^C\subseteq Rint(A)$, which implies $[Rint(A)]^C\subseteq M$. Therefore $Rcl(A^C)\subseteq M$. Thus A^C is gr^*-closed, implies A is gr^*-open.

Sufficiency: Let A is gr^*-open in X with $N\subseteq A$, where N is g-closed. We have A^C is gr^*-closed with $A^C\subseteq N^C$ where N^C is g-open. Then we have $Rcl(A^C)\subseteq N^C$ implies $N\subseteq X-Rcl(A^C)=Rint(X-A^C)=Rint(A)$. Hence proved.

Theorem 4.3. Every r-open set is gr^*-open set.

Proof: Let A be a r-open set. Then X-A is r-closed. By Theorem 2.4, X-A is gr^*-closed. Hence A is gr^*-open set.

Theorem 4.4. If $Rint(A)\subseteq B\subseteq A$ and A is a generalized regular star open subset of (X,τ) then B is also a generalized regular star open subset of (X,τ).

Proof: $Rint(A)\subseteq B\subseteq A$ implies $A^C\subseteq B^C\subseteq Rcl(A^C)$. Given A^C is gr^*-closed. By Theorem 3.21 B^C is gr^*-closed. Therefore B is gr^*-open.

Theorem 4.5. If a subset A of a topological space (X, τ) is gr^*-open in X then F=X, whenever F is g-open and $Rint(A)\subseteq A^C\subseteq F$.

Proof: Let A be a gr^*-open and F be g-open, $Rint(A)\cup A^C\subseteq F$. This gives $F^C\subseteq(X-Rint(A))\cap A=Rcl(A^C)\cap A=Rcl(A^C) - A^C$. Since F^C is g-closed and A^C is gr^*-open by Theorem 3.25 we have $F^C=\emptyset$. Thus F=X.

Theorem 4.6. If a subset A of a topological space (X,τ) is gr^*- closed, then Rcl(A)–A is gr^*- open.

Proof: Let $A\subseteq X$ be a gr^*-closed and let F be g-closed such that $F\subseteq Rcl(A)–A$. Then by Theorem 3.25 $F=\emptyset$. So $\emptyset=F\subseteq Rint(Rcl(A)-A)$. This shows that A is gr^*-open set.

Theorem 4.7. If $A\times B$ is a gr^*-open subset of $(X\times Y, \tau\times\sigma)$, iff A is a gr^*-open subset in(X,τ) and B is a gr^*-open subset in (Y,σ).

Proof: Let if possible $A\times B$ is a gr^*-open subset of $(X\times Y, \tau\times\sigma)$. Let H be a closed subset of (X,τ) and G be a closed subset of (Y,σ) such that $H\subseteq A$, $G\subseteq B$. Then $H\times G$ is closed in $(X\times Y, \tau\times\sigma)$ such that $H\times G\subseteq A\times B$. By assumption $A\times B$ is a gr^*-open subset of $(X\times Y, \tau\times\sigma)$ and so $H\times G\subseteq Rint(A\times B)\subseteq Rint(A)\times Rint(B)$. i.e $H\subseteq Rint(A)$, $G\subseteq Rint(B)$ and hence A is a gr^*-open subset in (X,τ) and B is a gr^*-open subset in (Y,σ). Conversely, Let F be a closed subset of $(X\times Y, \tau\times\sigma)$ such that $F\subseteq A\times B$. For each $(x,y)\subseteq F$, $cl(X)\times cl(Y)\subseteq cl(F)=F\subseteq A\times B$. Then the two closed sets cl(X) and cl(Y) are contained in A and B respectively. By assumption $cl(X)\subseteq R\ int\ A$ and $cl(Y)\subseteq R\ int\ B$ hold. This implies that for each $(x,y)\subseteq F$, $(x,y)\in R\ int(A\times B)$. $A\times B$ is a gr^*-open subset of $(X\times Y, \tau\times\sigma)$. Hence the theorem.

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