Appell Transforms Associated With Expansions in Terms of Generalized Heat Polynomials

B.B.Waphare

MIT ACSC, Alandi, Tal: Khed, Dist: Pune, Pin: 412 105, Maharashtra, India.

Abstract: In this paper we have discussed the regions of convergence of the series $\sum_{n=0}^{\infty} a_n\, p_{n,\alpha,\beta}(x,t)$ and $\sum_{n=0}^{\infty} b_n\, W_{n,\alpha,\beta}(x_1 t)$.We have also discussed the class H^* of generalized temperature functions having the Huygens property. Finally expansions for generalized temperature functions in terms of polynomials are derived and criteria for the expansions of functions in terms of $W_{n,\alpha,\beta}(x,t)$ is also established.

Keywords: Generalized heat polynomials, Appell transforms, generalized temperature function, Huygens property

1. INTRODUCTION

Rosenbloom and Widder [8] discussed expansions of solutions $u(x,t)$ of the heat equation $u_{xx} = u_t$ in series of polynomial solutions $v_n(x,t)$ and of their Apell transforms $W_n(x,t)$. It is our goal to extend this study by considering the generalized heat equation.

$$\Delta_x u(x,t) = \frac{\partial}{\partial t} u(x,t), \tag{1.1}$$

where $\Delta_x f(x) = f''(x) + \frac{4\alpha}{x} f'(x)$, α a fixed positive number, and by seeking criteria for representing solutions of (1.1) in either of the form

$$\sum_{n=0}^{\infty} a_n\, P_{n,\alpha,\beta}(x,t) \tag{1.2}$$

or

$$\sum_{n=0}^{\infty} b_n\, W_{n,\alpha,\beta}(x,t) \tag{1.3}$$

where $P_{n,\alpha,\beta}(x,t)$ is the polynomial solution of (1.1) given explicitly by

$$P_{n,\alpha,\beta}(x,t) = \sum_{k=0}^{n} 2^{2k} \binom{n}{k} \frac{\Gamma(3\alpha+\beta+n)}{\Gamma(3\alpha+\beta+n-k)} x^{2n-2k} t^k ,$$

and $W_{n,\alpha,\beta}(x,t)$ is its Appell transform. We note that $P_{n,0}(x,t)=v_{2n}(x)$ the ordinary heat polynomials of even order defined in [8, p.222]. Also, $P_{n,0}(x,-1)=H_{2n}(x/2)$ the Hermite polynomials of even order defined in [1; p. 222].

We establish the fact that, in general, the series (1.2) converges in a strip $|t|<\sigma$, whereas (1.3) converges in a half plane $\sigma<t<\infty$. The representation of $u(x,t)$ by (1.2)

for $|t|<\sigma$ is found to be valid if and only if $u(x,t)$ has Huygens property defined in §4, in that strip. These points up the analogy between expansions in terms of generalized heat polynomials for functions with the Huygens property and expansions in a Taylor series for analytic functions. The Huygens property is not sufficient for representing a function by (1.3), and, in this case, an additional integrability assumption is required.

2. SOME DEFINITIONS AND PRELIMINARY RESULTS

The generalized heat polynomial $P_{n,\alpha,\beta,}(x,t)$ is a polynomial of degree $2n$ in x and n in t given by

$$P_{n,\alpha,\beta}(x,t)=\sum_{k=0}^{n}2^{2k}\binom{n}{k}\frac{\Gamma(3\alpha+\beta+n)}{\Gamma(3\alpha+\beta+n-k)}x^{2n-2k}t^{k}, \tag{2.1}$$

where $\left(\alpha-\beta+\frac{1}{2}\right)$ a fixed positive number. For all values of its variables, the generalized heat polynomials is readily shown to satisfy the generalized heat equation.

$$\Delta_x u(x,t)=\frac{\partial}{\partial t}u(x,t) \tag{2.2}$$

A C^2-function $u(x,t)$ belongs to class H, for $a<t<b$, and is called a generalized temperature function if and only if it is a solution of (2.2). The fundamental solution of (2.2) is the function $G(x;t)$, where

$$G(x,y;t)=\left(\frac{1}{2t}\right)^{3\alpha+\beta}e^{-\left(x^2+y^2/4t\right)}g\left(\frac{xy}{2t}\right), \tag{2.3}$$

with

$$g(z)=2^{\alpha-\beta}\,\Gamma(3\alpha+\beta)\,z^{-(\alpha-\beta)}I_{\alpha-\beta}(z),$$

$I_\lambda(z)$ being the Bessel function of order λ of imaginary argument and $G(x;t)=G(x,0;t)$. Properties of $G(x,y;t)$ are studied in detail in [2]. Corresponding to $P_{n,\alpha,\beta}(x,t)$ is its Appell transform $W_{n,\alpha,\beta}(x,t)$ given by

$$W_{n,\alpha,\beta}(x,t)=G(x,t)P_{n,\alpha,\beta}\left(\frac{x}{t},-\frac{1}{t}\right),t>0. \tag{2.4}$$

It follows from (2.1) that

$$W_{n,\alpha,\beta}(x,t)=t^{-2n}G(x,t)P_{n,\alpha,\beta}(x,-t) \tag{2.5}$$

$W_{n,\alpha,\beta}(x,t)$ likewise satisfies the generalized heat equation (2.2).

An integral representation for the generalized heat polynomial is given by the following result.

Lemma 2.1: For $0 \le x < \infty,\ t > 0,$

$$P_{n,\alpha,\beta}(x,t)=\int_0^\infty y^{2n}G(x,y;t)\,d\mu(y),\ d\mu(x)=2^{-(\alpha-\beta)}\left[\Gamma(3\alpha+\beta)\right]^{-1}x^{4\alpha}\,dx \tag{2.6}$$

Proof: We have

$$\int_0^\infty y^{2n}G(x,y;t)\,d\mu(y)=\left(\frac{1}{2t}\right)e^{-x^{2/4t}}\,x^{-(\alpha-\beta)}\int_0^\infty e^{-y^2/4t}\,y^{2n+3\alpha+\beta}\,I_{\alpha-\beta}\left(\frac{xy}{2t}\right)dy,$$

and the result follows by [10; p.394].

By Theorem 5.3 of [2], it is clear that the integral (2.6) represents an analytic function of x since

$$P_{n,\alpha,\beta}(x,-t)=(-1)^n P_{n,\alpha,\beta}(ix,\ t), \tag{2.7}$$

the following corollary is immediate.

Corollary 2.2: For $0 \le x < \infty,\ t > 0,$

$$P_{n,\alpha,\beta}(x,-t)=(-1)^n\int_0^\infty y^{2n}G(ix,y;t)\,d\mu(y). \tag{2.8}$$

We next establish that x^{2n} is the Poisson-Hankel integral transform of $P_{n,\alpha,\beta}(x,-t)$.

Lemma 2.3: For $0 \le x < \infty,\ t > 0,$

$$x^{2n}=\int_0^\infty G(x,y;t)P_{n,\alpha,\beta}(y,-t)\,d\mu(y). \tag{2.9}$$

Proof : By (2.1) and an appeal to (2.6), we have

$$\int_0^\infty G(x,y;t)P_{n,\alpha,\beta}(y,-t)\,d\mu(y)$$

$$=\sum_{k=0}^{n}(-1)^k\,2^{2k}\binom{n}{k}\frac{\Gamma(3\alpha+\beta+n)}{\Gamma(3\alpha+\beta+n-k)}t^k\,P_{n-k,\alpha,\beta}(x,t). \tag{2.10}$$

Another application of definition (2.1) and an inversion of the order of summation on the right of (2.10) yields

$$\sum_{m=o}^{n} 2^{2m}\binom{n}{m}\frac{\Gamma(3\alpha+\beta+n)}{\Gamma(3\alpha+\beta+n-k)}x^{2n-2m}\,t^m\sum_{k=o}^{m}(-1)^k\binom{m}{k}. \qquad (2.11)$$

As the inner sum of (2.11) vanishes when $m \neq 0$, and = 1 when $\mathrm{m}=0$, the result follows.

Now we derive a generating function for the generalized heat polynomials.

Lemma 2.4: For $o \le x < \infty,\ -\infty < t < \infty,\ z < 1/4t,$

$$\left(\frac{1}{1-4zt}\right)^{3\alpha+\beta} e^{\left(x^2 z/1-4\,zt\right)} = \sum_{n=o}^{\infty}\frac{z^n}{n!}P_{n,\alpha,\beta}(x,t). \qquad (2.12)$$

Proof: By (2.6), we have, for $t>0$,

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}P_{n,\alpha,\beta}(x,t)=\int_0^{\infty}G(x,y;t)\,d\mu(y)\sum_{n=0}^{\infty}\frac{(zy^2)^n}{n!}$$

$$=\int_0^{\infty}e^{zy^2}\,G(x,y;t)\,d\mu(y)$$

$$=\left(\frac{1}{1-4zt}\right)^{3\alpha+\beta} e^{\left(x^2 z/1-4zt\right)},\quad z<\frac{1}{4t},$$

where the interchange of summation and integration is valid since

$$\int_0^{\infty}e^{zy^2}\,e^{-y^{2/4t}}\,g\left(\frac{xy}{zt}\right)d\mu(y)<\int_0^{\infty}e^{-y^2(1/4t-z)}\,e^{xy/2t}\,d\mu(y)<\infty,\ \ z<\frac{1}{4t}$$

For $t=0$, the Lemma is obvious and for $t<0$ it may be established similarly by an appeal to (2.8).

From (2.12), a corresponding generating function for $W_{n,\alpha,\beta}(x,t)$ is derived on appealing to (2.5).

Lemma 2.5: For $0 \le x < \infty,\ z<\frac{1}{4t},\ t>0,$

$$G(x;\,t+4z)=\sum_{n=o}^{\infty}\frac{z^n}{n!}W_{n,\alpha,\beta}(x,t)\ . \qquad (2.13)$$

We may expand $G(x;\,t+4z)$ in a power series in z to get

$$G(x;\,t+4z)=\sum_{n=o}^{\infty}\frac{z^n}{n!}2^{2n}\left(\frac{\partial}{\partial t}\right)^n G(x,t). \qquad (2.14)$$

By comparing (2.13) and (2.14), we conclude that

$$W_{n,\alpha,\beta}(x,t)=2^{2n}\left(\frac{\partial}{\partial t}\right)^{n} G(x,t). \tag{2.15}$$

But

$$\left(\frac{\partial}{\partial t}\right)^{n} G(x;t)=\left(\frac{\partial}{\partial t}\right)^{n}\int_{o}^{\infty} h(xu)\, e^{-tu^2}\, d\mu(u),\ h(x)=g(ix),$$

$$=(-1)^{n}\int_{o}^{\infty} u^{2n}\, h(xu)\, e^{-tu^2}\, d\mu(u), \tag{2.16}$$

so that we have the following integral representation for $W_{n,\alpha,\beta}(x,t)$.

Lemma 2.6: For $t>0,\ 0\le x<\infty,$

$$W_{n,\alpha,\beta}(x,t)=\int_{0}^{\infty}(-1)^{n}(2u)^{2n}\, h(xu)e^{-tu^2}\, d\mu(u). \tag{2.17}$$

Another useful generating function for $P_{n,\alpha,\beta}(x,t)$ is given by the following

Lemma 2.7: For $o\le x<\infty,\ -\infty<t<\infty,$ all complex z,

$$e^{4z^2t}\, g(2xz)=\sum_{n=0}^{\infty}\frac{\Gamma(3\alpha+\beta)}{\Gamma(3\alpha+\beta+n)}.\frac{z^{2n}}{n!}P_{n,\alpha,\beta}(x,t). \tag{2.18}$$

Proof: For $t>0,$ by (2.6) and the definition of $g(z),$ we have

$$\sum_{n=0}^{\infty}\frac{\Gamma(3\alpha+\beta)}{\Gamma(3\alpha+\beta+n)}.\frac{z^{2n}}{n!}P_{n,\alpha,\beta}(x,t)=\int_{0}^{\infty}G(x,y;t)\,d\mu(y)\sum_{n=0}^{\infty}\frac{\Gamma(3\alpha+\beta)(zy)^{2n}}{n!\Gamma(3\alpha+\beta+n)}$$

$$=\int_{0}^{\infty} g(2zy)\, G(x,y;t)\,d\mu(y)$$

$$=e^{4tz^2}\, g(2xz),$$

where the interchange of summation and integration is valid since

$$\int_{0}^{\infty} e^{-y^2/4t}\, g\left(\frac{xy}{2t}\right)g(2|z|y)\,d\mu(y)\le\int_{0}^{\infty}e^{-y^2/4t+xy/2t+2|z|y}\,d\mu(y),$$

$$<\infty,\ t>0.$$

Again, for $t<0,$ the Lemma is similarly proved by an appeal to (2.8).

The dual for $W_{n,\alpha,\beta}(x,t),$ a consequence of (2.5) and (2.18) is the following

Lemma 2.8: For $t>0,\ 0\le x<\infty,$ and all complex z,

$$G(x,z;t)=\sum_{n=o}^{\infty}k_n W_{n,\alpha,\beta}(x,t)z^{2n}, \tag{2.19}$$

where

$$k_n=\Gamma(3\alpha+\beta)/\left[2^{4n}\,n!\,\Gamma(3\alpha+\beta+n)\right]. \tag{2.20}$$

An important property of the sets $P_{n,\alpha,\beta}(x,t)$ and $W_{n,\alpha,\beta}(x,t)$ is that they form a biorthogonal sytem. We determine this fact.

Theorem 2.9: For $t>0$,

$$\int_0^{\infty} W_{n,\alpha,\beta}(x,t)P_{m,\alpha,\beta}(x,-t)\,d\mu(x)=\delta_{mn}\,k_n, \tag{2.21}$$

where k_n is given by (2.20).

Proof: We have

$$P_{n,\alpha,\beta}(x,-t)=(-1)^n\,2^{2n}\,n!\,t^n\,L_n^{\alpha-\beta}\left(\frac{x^2}{4t}\right), \tag{2.22}$$

where $L_n^a(x)$ is the generalized Laguerre polynomial, [1; p.188]. The result then follows immediately from the definitions and the fact that

$$\int_0^{\infty} e^{-x}x^a L_m^a(x)L_n^a(x)dx=\delta_{mn}\frac{\Gamma(a+n+1)}{\Gamma(n+1)}. \qquad \text{(Sec [1; p.188]).}$$

Next, we establish a fundamental generating function for the biorthogonal set

$$P_{n,\alpha,\beta}(x,-t),W_{n,\alpha,\beta}(x,t).$$

Theorem 2.10: For $0\le x,\ y<\infty,\ |t|<s,\ s>0,$

$$G(x,y;s+t)=\sum_{n=o}^{\infty}k_n W_{n,\alpha,\beta}(y,s)P_{n,\alpha,\beta}(x,t), \tag{2.23}$$

where k_n is defined in (2.20).

Proof: By (2.17), for $s>0$, we have

$$\sum_{n=0}^{\infty}k_n P_{n,\alpha,\beta}(x,t)\,W_{n,\alpha,\beta}(y,s)=\sum_{n=0}^{\infty}k_n P_{n,\alpha,\beta}(x,t)(-1)^n 2^{2n}\int_0^{\infty}u^{2n}h(yu)\,e^{-su^2}\,d\mu(u)$$

$$=\int_0^{\infty}h(yu)e^{-su^2}\,d\mu(u)\sum_{n=0}^{\infty}(-1)^n(2u)^{2n}k_n P_{n,\alpha,\beta}(x,t).$$

By (2.18), the right hand side becomes

$$\int_0^\infty h(yu)\, e^{-su^2}\, e^{-u^2 t}\, h(xu)\, d\mu(u) = G(x,y;s+t),\ s+t>0\,.$$

The interchange of summation and integration is valid since

$$\int_0^\infty |h(yu)| e^{-su^2}\, d\mu(u) \sum_{n=0}^\infty (2u)^{2n} k_n \left|P_{n,\alpha,\beta}(x,t)\right|$$

$$\le \int_0^\infty e^{-su^2}\, d\mu(u) \sum_{n=0}^\infty (2u)^{2n} k_n\, P_{n,\alpha,\beta}(x,|t|)$$

$$\le \int_0^\infty e^{-su^2}\, e^{u^2|t|}\, g(xu)\, d\mu(u)$$

$$\le \int_0^\infty e^{-(s-|t|)u^2}\, e^{xu}\, d\mu(u) < \infty,\ |t|<s,$$

and the Theorem is proved.

3. REGION OF CONVERGENCE

In order to establish the regions of convergence of the series

$$\sum_{n=0}^\infty a_n\, P_{n,\alpha,\beta}(x,t) \tag{3.1}$$

and

$$\sum_{n=0}^\infty b_n\, W_{n,\alpha,\beta}(x,t) \tag{3.2}$$

we need some preliminary results.

Lemma 3.1: Let the series (3.1) converge at (x_0,t_0), $x_0 \ge 0, t_0 > 0$. Then

$$a_n = \mathrm{O}\left(\frac{e}{4nt_0}\right)^n, n\to\infty. \tag{3.3}$$

Proof: The assumed convergence of the series (3.1) at (x_0,t_0) implies that the general term tends to zero as $n\to\infty$. Hence we have $a_n = \mathrm{O}\left(\frac{1}{P_{n,\alpha,\beta}(x_0,t_0)}\right), n\to\infty;$ (3.4)

However, by the definition of $P_{n,\alpha,\beta}(x_0,t_0)$, it is clear that

$$P_{n,\alpha,\beta}(x_0,t_0) \ge P_{n,\alpha,\beta}(_{0,}t_0) = 2^{2n}\frac{\Gamma(3\alpha+\beta+n)}{\Gamma(3\alpha+\beta)}\, t_0^n\,. \tag{3.5}$$

By applying (3.5) to (3.4) and using Stirling's formula, we establish the Lemma.

We need an asymptotic estimate of $P_{n,\alpha,\beta}(x,-t)$ for our next result. To this end we note that by [1; p. 199], we have for $n \to \infty$,

$$L_n^{\alpha-\beta}\left(\frac{x^2}{4t}\right) = \pi^{-1/2} e^{x^2/8t} \left(\frac{x^2}{4t}\right)^{-\alpha} n^{-\beta} \cos\left(\frac{n^{1/2}x}{t^{1/2}} - \alpha\pi\right) + \mathrm{O}\left(n^{-\alpha-2\beta}\right),$$

uniformly for $0 < \in \le \left(x^2/4t\right) \le \omega < \infty$. Now, by (2.22) and Stirling's formula, it follows that

$$P_{n,\alpha,\beta}(x,-t) = (-1)^n 2^{1/2} \left(\frac{4nt}{e}\right)^{n+\alpha} e^{x^2/8t+\alpha} x^{-2\alpha} \left[\cos\left(\frac{n^{1/2}x}{t^{1/2}} - \alpha\pi + \mathrm{O}\left(n^{-1/2}\right)\right)\right].$$

Thus we may derive the following:

Lemma 3.2: If the series (3.1) converges for each point (x,t) on the line segment $0 < a \le x \le b,\ t = t_0 < 0$, then

$$a_n = o\left(\frac{e}{4n|t_0|}\right)^{n+\alpha}, n \to \infty. \tag{3.7}$$

Proof: We give a proof by contradiction. Let

$$c_n = \left(\frac{4n\tau}{e}\right)^{n+\alpha},\ n \to \infty,$$

and assume that $\lim_{n\to\infty} a_n c_n \ne 0$. Then there exists a subsequence of positive integers m for which, for some number A,

$$|a_m c_m| > A > 0.$$

Now the convergence of the series (3.1) for $a \le x \le b$ implies that

$$\lim_{m\to\infty} \frac{a^2 m\, P_{m,\alpha,\beta}^2(x,t_0)}{a_m^2 c_m^2} = 0.$$

Further, by (3.6), we have

$$\left|\frac{P_{m,\alpha,\beta}(x,-r)}{c_m}\right| = 2^{1/2} e^{x^2/8t+\alpha}.\ x^{-2\alpha} \left[\cos\left(\frac{m^{1/2}x}{\tau^{1/2}} - \alpha\pi\right) + \mathrm{O}\left(m^{-1/2}\right)\right]$$

$$\le 2^{1/2} e^{b^2/8t+\alpha} a^{-4\alpha} \left[1 + \mathrm{O}\left(m^{-1/2}\right)\right]$$

$$\le K\, m^{1/2},$$

where K is a constant independent of x in $a \le x \le b$. Hence $(1/c_m)P_{m,\alpha,\beta}(x,-\tau)$ is uniformly bounded in $a \le x \le b$. An appeal to the Lebesgue limit theorem thus yields

$$\lim_{m\to\infty} \int_a^b \frac{P^2_{m,\alpha,\beta}(x,t_o)}{c_m^2}\,dx = 0. \tag{3.8}$$

But by (3.6), we have

$$\int_a^b \frac{P^2_{m,\alpha,\beta}(x,t_0)}{c_m^2}\,dx$$

$$= 2e^{2\alpha}\int_a^b e^{x^2/4\tau}\, x^{-4\alpha}\cos^2\left(\frac{m^{1/2}x}{\tau^{1/2}} - \alpha\pi\right) + \mathrm{O}\left(m^{-1/2}\right)$$

$$= e^{2\alpha}\int_a^b e^{x^2/4\tau}\, x^{-4\alpha}\,dx + e^{2\alpha}\int_a^b e^{x^2/4\tau}\, x^{-4\alpha}\cos\left(\frac{2m^{1/2}x}{\tau^{1/2}} - 2\alpha\pi\right)dx + \mathrm{O}\left(m^{-1/2}\right)$$

$$= e^{2\alpha}\int_a^b e^{x^2/4\tau}\, x^{-4\alpha}\,dx + e^{2\alpha}\cos(2\alpha\pi)\int_a^b e^{x^2/4\tau}\, x^{-4\alpha}\cos\left(\frac{2m^{1/2}x}{\tau^{1/2}}\right)dx$$
$$+ e^{2\alpha}\sin(2\alpha\pi)\int_a^b e^{x^2/4\tau}\, x^{-4\alpha}\sin\left(\frac{2m^{1/2}x}{\tau^{1/2}}\right)dx + \mathrm{O}\left(m^{-1/2}\right).$$

By the Riemann-Lebesgue theorem, both the second and third integrals on the right vanish as $m \to \infty$. It, therefore follows by (3.8) that

$$e^{2\alpha}\int_a^b e^{x^2/4\tau}\, x^{-4\alpha}\,dx = 0,$$

which is clearly a contradiction. Hence our assumption that $\lim_{n\to\infty} a_n c_n \ne 0$ is false and the Lemma is proved.

Next we need to determine bounds for the generalized heat polynomials. We first establish the following inequality.

Lemma 3.3: For $0 \le x < \infty,\ t > 0,\ \delta > 0,$

$$\left|P_{n,\alpha,\beta}(x,t)\right| \le \left(1 + \frac{t}{\delta}\right)^{3\alpha+\beta}\left[\frac{4n(t+\delta)}{e}\right]^n e^{x^2/4\delta},\ n = 1, 2, \tag{3.9}$$

Proof: This is an immediate consequence of (2.6) and the elementary inequality

$$x^{2n}\, e^{-\left[x^2/4(t+\delta)\right]} \le \left[\frac{4(t+\delta)n}{e}\right]^n.$$

By [1; p. 207], (3.6) and Stirling's formula, we also have the following result.

Lemma 3.4: For $0 \le x < \infty$, $t > 0$ and some constant A,

$$\left|P_{n,\alpha,\beta}(x,t)\right| \le A\left[\frac{4t(n+\alpha-\beta)}{e}\right](n+\alpha-\beta)^{2\alpha} e^{x^2/8t}. \qquad (3.10)$$

Further, (3.10) and (2.5) yields a bound for $W_{n,\alpha,\beta}(x,t)$.

Lemma 3.5: For $0 \le x < \infty$, $t > 0$ and some constant A,

$$\left|W_{A,\alpha,\beta}(x,t)\right| \le A\left[\frac{4(n+\alpha-\beta)}{et}\right]^n \frac{(n+\alpha-\beta)}{t^{3\alpha+\beta}} e^{-x^2/8t}. \qquad (3.11)$$

Now we are ready to establish the region of convergence of the series (3.1).

Theorem 3.6: If

$$\overline{\lim_{n\to\infty}} \left|a_n\right|^{1/n} \frac{4n}{e} = \frac{1}{\sigma} < \infty, \qquad (3.12)$$

then the series (3.1) converges absolutely in the strip $|t| < \sigma$ and does not converge everywhere in any including strip.

Proof: If $\sigma = 0$, the strip reduces to a straight line, and if $\sigma = \infty$, the strip is the whole plane. Suppose that σ is a finite positive number. If $0 < \theta < 1$, then (3.12) implies the existence of a number $N = N(\theta)$ such that

$$\left|a_n\right| \le \left(\frac{e}{4n\sigma\theta}\right)^n, n \ge N. \qquad (3.13)$$

Now, for $t > 0$, we have by (3.13) and (3.9),

$$\sum_{n=0}^{\infty}\left|a_n P_{n,\alpha,\beta}(x,t)\right| \le \sum_{n=0}^{\infty}\left(\frac{e}{4n\sigma\theta}\right)^n \left(1+\frac{t}{\delta}\right)^{3\alpha+\beta}\left[\frac{4n(t+\delta)}{e}\right]^n e^{x^2/4\delta}$$

$$= \left(1+\frac{t}{\delta}\right)^{3\alpha+\beta} e^{x^2/4\delta} \sum_{n=o}^{\infty}\left(\frac{t+\delta}{\sigma\theta}\right)^n.$$

Since the dominating series on the right converges for $t+\delta < \sigma\theta$, and since δ may be taken arbitrarily close to 0 and θ arbitrarily close to 1, it follows that the series (3.1) converges absolutely for $0 \le t < \sigma$.

Suppose now that $t < 0$. Then by (3.13) and (3.10), we have

$$\sum_{n=0}^{\infty}\left|a_n P_{n,\alpha,\beta}(x,t)\right| \le A\sum_{n=0}^{\infty}\left(\frac{e}{4n\sigma\theta}\right)^n \left[\frac{4t(n+\alpha-\beta)}{e}\right]^n (n+\alpha-\beta)^{2\alpha} e^{x^2\delta t},$$

and again, the dominating series on the right converges for $-\sigma\theta < t < 0$, or since θ may be taken arbitrarily close to 1, for $-\sigma < t < 0$.

To prove the second part of the theorem, assume that the series (3.1) converges everywhere in the including strip $-\sigma < t < \sigma'$, $\sigma' > \sigma$, or in $-\sigma' < t < \sigma$, $\sigma' > \sigma$. In the former case, for some t_0, $\sigma' > t_0 > \sigma$, we apply Lemma 3.1 to get

$$a_n = \Box\left(\frac{e}{4n\, t_0}\right)^n, n \to \infty.$$

Hence

$$\overline{\lim_{n\to\infty}}\ \frac{4n}{e}\left|a_n\right|^{1/n} \le \frac{1}{t_0} < \frac{1}{\sigma},$$

contradicting the hypothesis. In the later case, we apply Lemma 3.2 with $-\sigma' < t_0 < -\sigma$, to obtain

$$a_n = \Box\left(\frac{e}{4n\left|t_0\right|}\right)^{n+\alpha}, n \to \infty.$$

whence

$$\overline{\lim_{n\to\infty}}\ \frac{4n}{e}\left|a_n\right|^{1/n} \le \frac{1}{\left|t_0\right|} < \frac{1}{\sigma},$$

equally contradicting the hypothesis and to so the theorem is proved.

Next we show that within its strip of convergence, the series (3.1) represents a generalized temperature function. To state the result fully, we need the following definition.

Definition 3.7: An even entire function

$$f(x) = \sum_{n=0}^{\infty} a_n\, x^{2n}$$

belongs to the class (ρ,τ) or has growth (ρ,τ) if and only if

$$\lim_{n\to\infty} \frac{n}{e\rho}\left|a_n\right|^{\rho/n} \le \tau. \qquad (3.14)$$

Theorem 3.8: Let

$$u(x,t) = \sum_{n=0}^{\infty} a_n\, P_{n,\alpha,\beta}(x,t), \qquad (3.15)$$

the series converging for $|t| < \sigma$. Then $u(x,t) \in H$ there and $u(x,0)$ is an even function of growth $(1, 1/4\sigma)$.

Proof: Since $P_{n,\alpha,\beta}(x,t)\in H$, $u(x,t)$ will also be a generalized temperature function if we can justify differentiating under the summation sign. To this end, we must show that

$$\sum_{n=0}^{\infty} a_n \frac{\partial}{\partial t} P_{n,\alpha,\beta}(x,t) \tag{3.16}$$

converges uniformly. This will be so if the series (3.16) is dominated by a convergent series independent of x and t.

Now, for $o<t_0<\sigma$, we have, by Lemma 3.1,

$$a_n = \mathrm{O}\left(\frac{e}{4nt_0}\right)^n,\ n\to\infty. \tag{3.17}$$

Further, by Lemma 3.3, it follows that, for $t>0$, $\delta>0$,

$$P_{n-1,\alpha,\beta}(x,t) \le \left(1+\frac{t}{\delta}\right)^{3\alpha+\beta}\left[\frac{4(n-1)(t+\delta)}{e}\right]^{n-1} e^{x^2/4\delta}.$$

With these bounds, we have

$$\sum_{n=0}^{\infty} a_n \frac{\partial}{\partial t} P_{n,\alpha,\beta}(x,t)$$

$$=\sum_{n=0}^{\infty} a_n\, 4n(n+3\alpha+\beta) P_{n-1,\alpha,\beta}(x,t)$$

$$\le K\sum_{n=0}^{\infty}\left(\frac{e}{4nt_o}\right)^n 4n(n+3\alpha+\beta)\left(1+\frac{t}{\delta}\right)^{3\alpha+\beta}\left[\frac{4(n-1)(t+\delta)}{e}\right]^{n-1} e^{x^2/4t}$$

$$\le K\frac{(t+\delta)^{\alpha-\beta}}{\delta^{3\alpha+\beta}} e^{1+x^2/4\delta}\sum_{n=0}^{\infty}(n+3\alpha+\beta)\left[\frac{(n-1)}{n}\right]^{n-1}\left(\frac{t+\delta}{t_0}\right)^n.$$

The series on the right converges for $|t+\delta|<t_0$ and since δ may be taken arbitrarily close to o and to arbitrarily close to $\sigma, u(x,t)\in H$ throughout the strip $|t|<\sigma$.

Finally, we have from (3.15) and the definition of $P_{n,\alpha,\beta}(x,t)$,

$$u(x,0)=\sum_{n=o}^{\infty} a_n\, x^{2n},$$

so that $u(x,0)$ is an even function. Moreover by (3.17),

$$\overline{\lim_{n\to\infty}}\ \frac{n}{e}|a_n|^{1/n} \le \frac{1}{4t_0},$$

so that $u(x,0)$ is of growth $(1,1/4t_0)$ and allowing t_0 to tend to σ. We derive the desired result.

Corollary 3.9: For each fixed t, $0<t<\sigma$, the series (3.1) converges uniformly in any compact region of the complex x plane.

Proof: Let $x=x_1+ix_2$. Now, by (2.6), we have for $|x_1|\le R$,

$$\left|P_{n,\alpha,\beta}(x,t)\right| \le \left(\frac{1}{2t}\right)^{3\alpha+\beta} \left|e^{-x^2/4t}\right| \int_0^\infty e^{R|y|/2t}\, e^{-y^2/4t}\, y^{2n}\, d\mu(y).$$

Hence, for $|x_1| \le R, |x_2| \le R,$ we have

$$\left|\sum_{n=0}^{\infty} a_n P_{n,\alpha,\beta}(x,t)\right| \le \left(\frac{1}{2t}\right)^{3\alpha+\beta} e^{R^2/4t} \int_0^\infty e^{R|y|/2t}\, e^{-y^2/4t}\, d\mu(y) \sum_{n=0}^{\infty} |a_n|\, y^{2n},$$

where the interchange of summation and integration is valid if

$$\int_0^\infty e^{R|y|/2t}\, e^{-y^2/4t}\, d\mu(y) \sum_{n=0}^{\infty} |a_n|\, y^{2n} < \infty. \tag{3.18}$$

But, by the theorem, $u(x,0)$ has growth $(1,1/4\sigma)$, and hence, so does $\sum_{n=0}^{\infty} |a_n|\, x_n^2$, so that, for

$0<t<\sigma'<\sigma,$

$$\sum_{n=0}^{\infty} |a_n|\, y^{2n} < M \;\; e^{y^2/4\sigma'},$$

For a suitable constant m. Thus we have

$$\int_0^\infty e^{R|y|/2t}\, e^{-y^2/4t}\, d\mu(y) \sum_{n=0}^{\infty} |a_n|\, y^{2n} < M \int_0^\infty e^{R|y|/2t}\, e^{-y^2/4t + y^2/4\sigma}\, d\mu(y) < \infty$$

Now by the Weierstrass M-test, we have thus established that the series (3.1) converges uniformly in very square of the complex plane.

Now we determine the region of convergence for the series (3.2).

Theorem 3.10: Let

$$\overline{\lim_{n\to\infty}} \frac{4n}{e} |b_n|^{1/n} = \sigma < \infty, \tag{3.19}$$

Then the series (3.2) converges absolutely in the half plane $t>\sigma$, and does not converge everywhere in any including half plane.

Proof: If $0<\theta<1$, then (3.19) implies the existence of a number $N=N(\theta)$ such that

$$|b_n| \le \left(\frac{e\sigma}{4n\theta}\right)^n, \quad n \ge N. \tag{3.20}$$

Hence this bound and (3.11) yields

$$\left|\sum_{n=0}^{\infty} b_n W_{n,\alpha,\beta}(x,t)\right| \le \frac{Ae^{-x^2/8t}}{t^{3\alpha+\beta}} \sum_{n=0}^{\infty} \left(\frac{e\sigma}{4n\theta}\right)^n \left[\frac{4(n+\alpha-\beta)}{et}\right]^n (n+\alpha-\beta)^{2\alpha}.$$

It is clear that the dominating series converges for $\theta t > \sigma$ and since θ may be chosen arbitrarily close to 1, we have established the absolute convergence of the series (3.2) for $t > \sigma$.

To prove that the half plane of convergence cannot be extended, assume that the series (3.2) converges everywhere in a half plane including $t > \sigma$. Then it follows that the series

$$\sum_{n=0}^{\infty} b_n W_{n,\alpha,\beta}(x,t_0) = G(x;\, t_0) \sum b_n t_0^{-2n} P_{n,\alpha,\beta}(x,-t_0)$$

converges for all x and some $t_0 < \sigma$. By Lemma 3.2, we have $b_n = \mathrm{O}\left(\frac{e t_0}{4n}\right)^{n+\alpha}, n \to \infty$, or

$$\overline{\lim_{n\to\infty}} \frac{4n}{e} |b_n|^{1/n} \le t_0 < \sigma,$$

contradicting hypothesis (3.19). Since $W_{n,\alpha,\beta}(x,t)$ is not defined for $t \le o$, the second fact of the theorem is meaning-less if $\sigma = 0$.

The result corresponding to Theorem 3.8 for the series (3.2) has a similar proof and is stated as follows.

Theorem 3.11: Let

$$u(x,t) = \sum_{n=0}^{\infty} b_n W_{n,\alpha,\beta}(x,t),$$

the series converging for $t > \sigma \ge 0$. Then $u(x,t) \in H$ there.

4. THE HUYGENS PROPERTY

Significant in our theory is the class H^* of generalized temperature functions which have the Huygens property. We define the members of the class H^* as follows. :

Definition 4.1: A function $u(x,t) \in H^*$, or has the Huygens property, for $a < t < b$, if and only if $u \in H$ and

$$u(x,t)=\int_0^\infty G(x,y;\,t-t')\,u\,(y,t')\,d\mu(y), \qquad (4.1)$$

for every $t,\ t',\ a<t'<t<b,$ the integral converging absolutely.

It is proved in [5] that any function $u(x,t)$ having a Poisson-Hankel-Stieltjes integral representation,

$$u(x,t)=\int_0^\infty G(x,y;t)\,d(y), \qquad (4.2)$$

belongs to H^* in the strip of absolute convergence of (4.2). In particular, as established in [2], every positive generalized temperature function has a representation (4.2) and is thus a function of H^*.

It is of importance in our theory that the generalized heat polynomials and their Appell transforms have the Huygens property. Since for $0\le t<\infty,\ P_{n,\alpha,\beta}(x,t)$ is a positive generalized temperature function, it is a member of H^* in that interval. Moreover, $P_{n,\alpha,\beta}(x,-t)\in H$ for $-\infty<t<0$ and by an appeal to (2.8), it may readily be shown that $P_{n,\alpha,\beta}(x,t)$ satisfies the relation (4.1) for $-\infty<t<0,$ so that $P_{n,\alpha,\beta}(x,t)\in H^*$ for $-\infty<t<\infty.$ That $W_{n,\alpha,\beta}(x,t)\in H^*$ for $0<t<\infty$ follows from the fact that it is a member of H and by an application of (2.17), relation (4.1) may be established.

Note that we may use the fact that $P_{n,\alpha,\beta}(x,t)\in H^*$ to verify equation (2.9). For, by (4.1), we have, for $-\infty<-t<t'<\infty,$

$$P_{n,\alpha,\beta}(x,t')=\int_0^\infty G(x,y;t'+t)\,P_{n,\alpha,\beta}(y,-t)\,d\mu(y),$$

and taking $t'=0$, we obtain the desired relation.

Following [5], we can prove that following result.

Theorem 4.2: Let $g(x,t)\in H^*$ for $a<t<b,$ and $h(x,t)\in H^*$ for $a<-t<b.$ If

$$\int_0^\infty |g(y,t)|\,d\mu(y)\int_0^\infty G(y,u;\,t'-t)\,|h(u,-t')|\,d\mu(u)<\infty, \qquad (4.3)$$

for $a<t<t'<b,$ then, for $a<t<b,$

$$\int_0^\infty g(y,\,t)\,h\,(y,-t)\,d\mu(y) \qquad (4.4)$$

is a constant.

Special cases of this theorem are of interest to us.

Corollary 4.3: If $u(x,-t)\in H^*$ for $0<t<\infty$, then

$$\int_0^\infty u(x,-t)\,W_{n,\alpha,\beta}(x,t)\,d\mu(x) \tag{4.5}$$

is a constant.

Corollary 4.4: If $u(x,t)\in H^*$ for $0<t<\infty$, then

$$\int_0^\infty u(x,t)\,P_{n,\alpha,\beta}(x,-t)\,d\mu(x) \tag{4.6}$$

is a constant.

Note that in each of these corollaries, hypothesis (4.3) is satisfied by an appeal to (3.11) and (3.10) respectively.

It was shown in [5] that a function with the Huygens property has a complex integral representation. Indeed, we established the following result.

Theorem 4.5: If $u(x,t)\in H^*$ for $a<t<b$, then for $a<t<t'<b$

$$u(x,t)=\int_0^\infty G(ix,\ y;\ t'-t)\,u(iy,\ t')\,d\mu(y). \tag{4.7}$$

This enables us to extend Theorem 4.2 further to include the following special case.

Theorem 4.6: If $u(x,t)\in H^*$ for $0<t<\infty$, then

$$\int_0^\infty u(ix,t)\,W_{n,\alpha,\beta}(x,t)\,d\mu(x) \tag{4.8}$$

is a constant.

We conclude this section with two simple Lemmas.

Lemma 4.7: Let $u(x,t)\in H^*$ for $|t|<\sigma$. Then for $-\sigma<t'<0$,

$$\int_0^\infty e^{y^2/8t'}\,|u(y,\ t')|\,d\mu(y)<\infty. \tag{4.9}$$

Proof: Clearly, for $0<-t'<c$, we have

$$\left[\frac{1}{2(c-t')}\right]^{3\alpha+\beta}\int_0^\infty e^{y^2/8t'}\,|u(y,t')|\,d\mu(y).$$

$$\leq\int_0^\infty G(y;\ c-t')\,|u(y,t')|\,d\mu(y)$$

and the dominating integral converges since $u(x,t)\in H^*$.

Lemma 4.8: Let $u(x,t)\in H^*$ for $a<t<b$. Then, for $a<t'<t<b$,

$$u(ix,\, t)=\mathrm{O}\left(e^{\left[x^2/4(t-t')\right]}\right),\ \ x\to\infty. \tag{4.10}$$

Proof: By Theorem 5.3 of [2] and the fact that $u(x,t)\in H^*$, we have

$$\left|u(ix,t)\right|=\int_0^\infty\left|G(ix,y;t-t')\right|\left|u(y,t')\right|d\mu(y),\leq e^{\left[x^2/4(t-t')\right]}\int_0^\infty G(y;\,t-t')\left|u(y,\,t')\right|d\mu(y),$$

$a<t'<t<b.$

Now the result is immediate when we note that the dominant integral converges as $u(x,t)\in H^*$.

5. HEAT POLYNOMIAL EXPANSIONS

In this section we derive expansions for generalized temperature functions in terms of the polynomials considered.

Theorem 5.1: A necessary and sufficient condition that

$$u(x,t)=\sum_{n=0}^{\infty}a_n\,P_{n,\alpha,\beta}(x,t), \tag{5.1}$$

the series converging for $|t|<\sigma$, is that $u(x,t)\in H^*$. The coefficients a_n have either of the determinations

$$a_n=\frac{u^{(2n)}(0,0)}{(2n)!} \tag{5.2}$$

or

$$a_n=k_n\int_0^\infty u(y,-t)W_{n,\alpha,\beta}\,(y,\,t)\;d\mu(y),\qquad 0<t<\sigma, \tag{5.3}$$

where k_n is defined in (2.20).

Proof : To prove the sufficiency of the condition, we assume that $u(x,t)\in H^*$ for $|t|<\sigma$. Then, we have

$$u(x,t)=\int_0^\infty G(x,y;\,t-t')\;u(y,t')d\mu(y),\quad -\sigma<t'<t<\sigma\,. \tag{5.4}$$

Let us choose $t'<0$. Now, (2.23) gives us

$$G(x,y;t-t')=\sum_{n=0}^{\infty}k_n\,W_{n,\alpha,\beta}(y,-t')P_{n,\alpha,\beta}(x,t), \tag{5.5}$$

and substituting this in (5.4), we find that

$$u(x,t)=\sum_{n=0}^{\infty} k_n P_{n,\alpha,\beta}(x,t)\int_0^{\infty} u(y,t')W_{n,\alpha,\beta}(y,-t')\,d\mu(y), \tag{5.6}$$

provided that the interchange of summation and integration is valid.
This will be so if

$$J=\sum_{n=0}^{\infty} k_n \left|P_{n,\alpha,\beta}(x,t)\right| \int_0^{\infty} \left|u(y,t')W_{n,\alpha,\beta}(y,-t')\right| d\mu(y) < \infty.$$

But, for $t' < 0 < t$, we have, by (3.9) and (3.11) (5.7)

$$J \le \frac{A}{(-t')^{3\alpha+\beta}}\left(1+\frac{t}{\delta}\right)^{3\alpha+\beta} e^{x^2/4\delta}\sum_{n=0}^{\infty} k_n\left[\frac{4^2 n(t+\delta)(n+\alpha-\beta)}{-e^2 t'}\right]^n (n+\alpha-\beta)^{2\alpha}\int_0^{\infty}\left|u(y,t')\right| e^{y^2/\delta t'}d\mu(y). \tag{5.8}$$

Lemma 4.7 assures the convergence of the integral to the right of (5.8), and the ratio test establishes the fact that the series converges for $t+\delta<|t'|$. Further for $t'<t<0,$ we have by (3.10) and (3.11),

$$J \le \frac{A^2}{(-t')^{3\alpha+\beta}} e^{-x^2/\delta t}\sum_{n=0}^{\infty} k_n\left[\frac{4^2 t(n+\alpha-\beta)^2}{e^2 t'}\right]^n (n+\alpha-\beta)^{4\alpha}\int_0^{\infty}\left|u(y,t')\right| e^{y^2/\delta t'} d\mu(y),$$

with the series converging for $|t|<|t'|$. Thus since δ may be chosen arbitrarily small, we have established the absolute convergence of the series (5.6) for $|t|<|t'|$. By taking t' arbitrarily close to $-\sigma$, the series (5.6) converges to $u(x,t)$ for $|t|<\sigma$ as required. By taking

$$a_n = k_n\int_0^{\infty} u(y,-t)W_{n,\alpha,\beta}(y,t)\,d\mu(y), \quad 0<t<\sigma,$$

in (5.6), we have the second determination (5.3), and by Corollary 4.3, we know that a_n is independent of t. To derive the first determination (5.2), we note that $u(x,0)=\sum_{k=0}^{\infty} a_n x^{2n}$,

which is a power series, so that $a_n = \dfrac{u^{2n}(0,0)}{(2n)!}$.

To prove the necessity of the condition, we now assume that (5.1) holds for $|t|<\sigma$. Let c be a number , $0<c<\sigma$, and consider

$$\int_0^{\infty} G(x,y;t+c)u(y,-c)\,d\mu(y). \tag{5.9}$$

We shall establish that this integral is equal to $u(x,t)$ for $|t|<c$. To this end, substitute the series (5.1) for $u(y,-c)$ in (5.9). This yields, provided term wise integration is valid,

$$\sum_{n=0}^{\infty} a_n \int_0^{\infty} G(x,y;t+c) P_{n,\alpha,\beta}(y,-c)\, d\mu(y) = \sum_{n=0}^{\infty} a_n P_{n,\alpha,\beta}(x,t) = u(x,t) \tag{5.10}$$

where we have used the fact that $P_{n,\alpha,\beta}(x,t) \in H^*$. We now justify our computation.

We have, by (3.10)

$$\sum_{n=0}^{\infty} |a_n| \int_0^{\infty} G(x,y;t+c) \left|P_{n,\alpha,\beta}(y,-c)\right| d\mu(y) \tag{5.11}$$

$$\leq A \sum_{n=0}^{\infty} |a_n| \left[\frac{4c(n+\alpha-\beta)}{e}\right]^n (n+\alpha-\beta)^{2\alpha} \int_0^{\infty} G(x,y;t+c)\, e^{y^2/8c}\, d\mu(y).$$

The integral on the right of inequality (5.11) clearly converges for $|t|<c$. Further by Lemma 3.1,

$$a_n = \mathrm{O}\left(\frac{e}{4nt_0}\right)^n, \quad n \to \infty,$$

for any positive $t_o < \sigma$. If we choose $t_o > c$, the dominating series of (5.11) converges. Hence we have

$$u(x,t) = \int_0^{\infty} G(x,y;\, t+c)\, u(y,-c)\, d\mu(y), \quad |t|<c. \tag{5.12}$$

But we have shown that the integral (5.12) converges absolutely for every t and c with $-\sigma < -c < t < \sigma$, and so by the definition of class H^*, $u(x,t) \in H^*$ for $|t|<\sigma$.

An example illustrating the theorem is given by

$$u(x,t) = e^{a^2 t}\, g(ax), \tag{5.13}$$

Here we have, for $-\infty < t < \infty$,

$$e^{a^2 t}\, g(ax) = \sum_{n=0}^{\infty} k_n (2a)^{2n} P_{n,\alpha,\beta}(x,t). \tag{5.14}$$

In addition to (5.2) and (5.3), we also establish a complex determination of the coefficients, as given in the following.

Theorem 5.2 : Let

$$u(x,t) = \sum_{n=0}^{\infty} a_n P_{n,\alpha,\beta}(x,t), \tag{5.15}$$

the series converging for $|t|<\sigma$. Then

$$a_n = (-1)^n k_n \int_0^\infty u(ix, t)\, W_{n,\alpha,\beta}(x,t)\, d\mu(x),\ 0 < t < \sigma, \tag{5.16}$$

where k_n is defined in (2.20).

Proof: Since $u(x,t)$ has the series expansion (5.15), the preceding theorem assures its membership in H^*. Hence we have

$$u(x,t) = \int_0^\infty G(x,y;t)\, u(y,0)\, d\mu(y),\ 0 < t < \sigma, \tag{5.17}$$

the integral converging absolutely. Now for any fixed t in this integral, the integral (5.17) defines $u(x,t)$ as an analytic function of x. Further by Corollary 3.9, for each fixed t, $0 < t < \sigma$, the series (5.15) define $u(x,t)$ as an analytic function in any compact region of the complex $x-$plane. Since the two functions are equal for real x, by analytic continuation, the expansion of the series must also hold for complex x and we have, by (2.7),

$$u(ix, t) = \sum_{n=0}^\infty (-1)^n\ a_n\, P_{n,\alpha,\beta}(x,-t). \tag{5.18}$$

Now, by (4.7) and Theorem 5.3 of [2], we find that

$$u(ix,t) = \int_0^\infty G(x,y;\, t'-t)\, u(iy,t')\, d\mu(y),\ 0 < t < t' < \sigma. \tag{5.19}$$

Further (2.23) gives us

$$G(x,y;\, t'-t) = \sum_{n=0}^\infty k_n W_{n,\alpha,\beta}(y,t')\, P_{n,\alpha,\beta}(x,-t). \tag{5.20}$$

Substituting (5.20) in (5.19), we obtain

$$u(ix,t) = \sum_{n=0}^\infty k_n\, P_{n,\alpha,\beta}(x,-t) \int_0^\infty u(iy,t')\, W_{n,\alpha,\beta}(y,t')\, d\mu(y) \tag{5.21}$$

provided that term wise integration is justifiable. But, by (3.10) and (3.11)

$$\sum_{n=0}^\infty k_n \left|P_{n,\alpha,\beta}(x,-t)\right| \int_0^\infty \left|u(iy,t')\right| W_{n,\alpha,\beta}(y,t')\, d\mu(y)$$

$$\le \frac{A^2}{(t')^{3\alpha+\beta}}\, e^{x^2/8t} \sum_{n=0}^\infty k_n \left[\frac{4^2\, t_o\,(n+\alpha-\beta)^2}{e^2\, t'}\right]^n (n+\alpha-\beta)^{4\alpha} \tag{5.22}$$

$$\times \int_0^\infty \left|u(iy,\, t')\right| e^{-y^2/8t'}\, d\mu(y).$$

Now by Lemma 4.8,

$$u(iy,t)=O\left(e^{\left[y^2/4(t-a)\right]}\right), y\to\infty,$$

where a may be chosen arbitrarily close to $-\sigma$. It follows that the integral of the dominant series of (5.22) converges for $0<t'<-a$ and hence for $0<t'<\sigma$. Since $t<t'$, the series itself converges. We have thus established the validity of (5.21). Now, comparing coefficients in the expansions (5.18) and (5.21), we find that a_n has the determination (5.16) required and it is independent of t by Theorem 4.6.

6. EXPANSION IN TERMS OF THE APPELL TRANSFORM

Criterion for the expansions of functions in terms of $W_{n,\alpha,\beta}(x,t)$ are derived in this section. We find that membership in H^* is no longer a sufficient condition in this case. Before we establish the needed modification for a theorem corresponding to Theorem 5.1, we need a series representation theorem with conditions of a different nature.

Theorem 6.1: A necessary and sufficient condition that

$$u(x,t)=\sum_{n=0}^{\infty} b_n\, W_{n,\alpha,\beta}(x,t), \tag{6.1}$$

the series converging for $0\le\sigma<t,$ is that

$$u(x,t)=\int_0^{\infty} h(xy)\, e^{-ty^2}\,\phi(y)\,d\mu(y), \tag{6.2}$$

where $\phi(y)$ is an even entire function of growth $(1,\sigma)$ and

$$b_n=(-1)^n\,\frac{\phi^{(2n)}(0)}{2^{2n}(2n)!}. \tag{6.3}$$

Proof: To prove sufficiency, assume that (6.2) holds with $\phi(y)$ as described. Now, let

$$\phi(y)=\sum_{n=0}^{\infty} c_n\, y^{2n}, \tag{6.4}$$

and substitute the series for $\phi(y)$ in (6.2). Hence if we may interchange the order of summation and integration, we obtain

$$u(x,t)=\sum_{n=0}^{\infty} c_n\int_0^{\infty} h(xy)\, e^{-ty^2}\, y^{2n}\,d\mu(y), \tag{6.5}$$

or by (2.17)

$$u(x,t)=\sum_{n=0}^{\infty}\frac{(-1)^n c_n}{2^{2n}}W_{n,\alpha,\beta}(x,t). \quad (6.6)$$

From (6.4), we get

$$c_n=\frac{\phi^{(2n)}(0)}{(2n)!} \quad , \quad (6.7)$$

and hence taking

$$b_n=\frac{(-1)^n c_n}{2^{2n}}, \quad (6.8)$$

we have the determination (6.3). It remains to prove the validity of the termwise integration. Since $\phi(y)$ is of growth $(1,\sigma)$, the same is true of $\sum_{n=o}^{\infty}|c_n|y^{2n}$, and for a suitable constant M , and any $\sigma'>\sigma$, we have

$$\sum_{n=0}^{\infty}|c_n|\ y^{2n}<M\ e^{\sigma' y^2},\ 0<y<\infty.$$

Hence

$$\int_0^{\infty}e^{-ty^2}\ d\mu(y)\sum_{n=0}^{\infty}|c_n|y^{2n}<M\int_0^{\infty}e^{-ty^2+\sigma' y^2}\ d\mu(y),$$

which converges for $t>\sigma'$. Thus condition is sufficient. Conversely, assume that (6.1) holds for $0\le\sigma<t$. Choose a number $c>\sigma.$ Now, by (2.5)

$$\sum_{n=0}^{\infty}b_n W_{n,\alpha,\beta}(x,t)=G(x,t)\sum_{n=0}^{\infty}b_n\ t^{-2n}\ P_{n,\alpha,\beta}(x,-t), \quad (6.9)$$

so that the series at the right of (6.9) converges for all x on the line $t=c.$ We thus may apply Lemma 3.2 to obtain

$$b_n=\mathrm{O}\left(\frac{ec}{4n}\right)^{n+\alpha},\ n\to\infty. \quad (6.10)$$

By (2.17), we have

$$\begin{aligned}u(x,t)&=\sum_{n=0}^{\infty}b_n\ W_{n,\alpha,\beta}(x,t)\\&=\int_0^{\infty}h(xu)\ e^{-tu^2}\ d\mu(u)\sum_{n=0}^{\infty}(-1)^n b_n\ (2u)^{2n}\\&=\int_0^{\infty}h(xu)\ e^{-tu^2}\ \phi(u)\ d\mu(u),\end{aligned}$$

where

$$\phi(u)=\sum_{n=0}^{\infty} c_n u^{2n}, \tag{6.11}$$

with

$$c_n=(-1)^n\ 2^{2n}\ b_n, \tag{6.12}$$

provided that

$$\int_0^{\infty} e^{-tu^2}\, d\mu(u) \sum_{n=0}^{\infty} |b_n|\ (2u)^{2n}<\infty. \tag{6.13}$$

As a consequence of (6.10) and (6.12), we have

$$\overline{\lim_{n\to\infty}} \frac{2^2 n}{e} |b_n|^{1/n} = \overline{\lim_{n\to\infty}} \frac{n}{e} |c_n|^{1/n}.$$

That is, $\phi(x)$ is of growth $(1,c)$ for every $c>\sigma$, and hence also of growth $(1,\sigma)$. Thus the integral (6.13) converges for $t>\sigma$ and $u(x,t)$ has the required representation.

An example illustrating the theorem is obtained by taking

$$u(x,t)=G\left(x;\ x-\frac{1}{2}\right), \tag{6.14}$$

$$=\int_0^{\infty} e^{-tu^2}\, h(xu)\, e^{u^2/4}\, d\mu(u),\quad t>\frac{1}{4}.$$

Here

$$\phi(u)=e^{u^2/4}$$

an even function of growth $(1,1/4)$ and

$$b_n=\frac{(-1)^n}{2^{4n}\, n!}\quad ,$$

so that we have

$$G\left(x;\, t-\frac{1}{4}\right)=\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{4n} n!} W_{n,\alpha,\beta}(x,t),\ t>1/4,$$

verifying equation (2.13).

To derive the principal representation theorem, we need two preliminary lemmas:

Lemma 6.2: Let $E(x)$ be an even function for $-\infty<x<\infty$. Then

$$\int_0^\infty E(y)h(xy)\,d\mu(y)=x^{-(\alpha-\beta)}\int_{-\infty}^\infty E(y)\;y^{3\alpha+\beta}\,H^{(1)}_{(3\alpha+\beta)}(xy)\,dy\,. \tag{6.15}$$

Proof: The proof is substantially that of Lemma 3.1 of [4] and is omitted.

Lemma 6.3: Let

$$u(x,t)=\int_0^\infty h(xy)e^{-tv^2}\,\phi(y)\,d\mu(y), \tag{6.16}$$

where $\phi(y)$ is an even function of growth $(1,\sigma)$. Then, for each $c>\sigma$, there exists a constant $m(c)$ such that

$$|u(x,t)|\le m(c)\,x^{-2\alpha}.\,\frac{e^{-\left[x^2/4(t+c)\right]}}{\sqrt{t-c}},\; t>c. \tag{6.17}$$

Proof: Since $e^{-ty^2}\phi(y)$ is even, we may apply the preceding lemma to get

$$u(x,t)=x^{-(\alpha-\beta)}\int_{-\infty}^\infty e^{ty^2}\,\phi(y)\;y^{3\alpha+\beta}H^{(1)}_{\alpha-\beta}(xy)\,dy. \tag{6.18}$$

Now we know that for $0<x<\infty,\ 0\le\eta\le A,$

$$H^{(1)}_{\alpha-\beta}\;\left(x(\pm R+i\eta)\right)=\mathrm{O}\left(\frac{1}{xR}\right)^{1/2},R\to\infty, \tag{6.19}$$

and, by the growth property of $\phi(y)$, that

$$\left|\phi(\xi+i\eta)\right|\le \mathrm{N}(c)\;e^{c\left(\xi^2+\eta^2\right)}, \tag{6.20}$$

where $c>\sigma$ and $\mathrm{N}(c)$ depends only on c. Hence

$$\left|x^{-(\alpha-\beta)}\int_0^A e^{-t(\pm R+i\eta)^2}\;\phi(\pm R+i\eta)(\pm R+i\eta)^{3\alpha+\beta}\;H^{(1)}_{\alpha-\beta}\left(x(\pm R+i\eta)\right)d\eta\right|$$

$$\le e^{-tR^2+cR^2}\;\mathrm{O}\left(\frac{R}{x}\right)^{2\alpha}\int_0^A e^{t\eta^2+c\eta^2}\;d\eta$$

$$=o(1),\;\; R\to\infty.$$

Thus, we use Cauchy's theorem to shift the path of integration to get

$$u(x,t)=x^{-\alpha-\beta}\int_{-\infty}^\infty e^{-t(\xi+iA)^2}\;\phi(\xi+iA)(\xi+iA)^{3\alpha+\beta}\;H_{-(\alpha-\beta)}\left(x(\xi+iA)\right)\,d\xi. \tag{6.21}$$

Now we know that

$$H^{(1)}_{-(\alpha-\beta)}\, x(\xi+iA)=O\left(x^{-1/2}\, e^{-Ax}\right),\ x\to\infty, \tag{6.22}$$

uniformly for $-\infty<\xi<\infty,$ and we may readily establish the inequality

$$\left(\xi^2+A^2\right)^{\left(\frac{3\alpha+\beta}{2}\right)}\le e^{\in\left(\xi^2+A^2\right)}\left(\frac{6\alpha+2\beta}{4\in e}\right)^{\left(\frac{3\alpha+\beta}{2}\right)},\ \in>0.$$

Using (6.20), (6.22) and (6.23) in (6.21), we find that

$$|u(x,t)|\le N'(c)\,x^{-2\alpha}\, e^{(t+c+\in)A^2-Ax}\int_{-\infty}^{\infty} e^{(c+\in-t)\xi^2}\, d\xi$$

$$=N'(c)\,x^{-2\alpha}\, e^{(t+c+\in)A^2-Ax}\,\frac{\sqrt{\pi}}{\sqrt{t-c-\in}},\ t>c>\in.$$

Choose

$$A=\frac{x}{2(t+c+\in)}$$

to make the right hand side a minimum, and we have

$$|u(x,t)|\le N'(c)\,x^{-2\alpha}\, e^{-\left[x^2/4(t+c+\in)\right]}\,\frac{\sqrt{\pi}}{\sqrt{t-c-\in}}.$$

As $\in$ is arbitrary, the lemma is proved.

Now we are ready to establish our main theorem.

Theorem 6.4: A necessary and sufficient condition that

$$u(x,t)=\sum_{n=0}^{\infty} b_n\, W_{n,\alpha,\beta}(x,t), \tag{6.24}$$

the series converging for $t>\sigma\ge 0$ is that $u(x,t)\in H^*$ and that

$$\int_0^{\infty}|u(x,t)|\,e^{x^2/8t}\, d\mu(x)<\infty,\ \sigma<t<\infty. \tag{6.25}$$

The coefficients b_n have the determination

$$b_n=k_n\int_0^{\infty} u(y,t)\,P_{n,\alpha,\beta}\,(y,-t)\,d\mu(y),\ \sigma<t<\infty, \tag{6.26}$$

where k_n is defined in (2.20).

Proof: To prove sufficiency, we assume that $u(x,t)\in H^*$ for $t>\sigma\ge o$ and (6.25) holds. Then

$$u(x,t)=\int_0^{\infty} G(x,t;t-t')\,u(y,t')\,d\mu(y),\ \sigma<t'<t<\infty. \tag{6.27}$$

But, by (2.23)

$$G(x,y;t-t')=\sum_{n=0}^{\infty}k_n\ P_{n,\alpha,\beta}(y,-t')\,W_{n,\alpha,\beta}(x,t). \tag{6.28}$$

Inserting this in (6.27), we obtain

$$u(x,t)=\sum_{n=0}^{\infty}k_n W_{n,\alpha,\beta}(x,t)\int_0^{\infty}u(y,t')P_{n,\alpha,\beta}(y,-t')\,d\mu(y), \tag{6.29}$$

which is the result sought, provided that term wise integration is valid. That this is so follows from the fact that

$$\sum_{n=0}^{\infty}k_n\left|W_{n,\alpha,\beta}(x,t)\right|\int_0^{\infty}\left|u(y,t')\right|\left|P_{n,\alpha,\beta}(y,-t')\right|d\mu(y), \tag{6.30}$$

because of (3.10) and (3.11), is dominated by

$$A^2\frac{e^{-x^2/8t}}{t^{3\alpha+\beta}}\sum_{n=0}^{\infty}k_n\left[\frac{4^2(n+\alpha-\beta)^2\,t'}{e^2\,t}\right]^n(n+\alpha-\beta)^{4\alpha}$$

$$\times\int_0^{\infty}e^{y^2/8t'}\left|u(y,t')\right|\,d\mu(y).$$

The integral of (6.31) converges, by hypothesis, for $t'>\sigma$ and the series converges for $t'<t$, so that the expansion (6.29) is established. That the coefficients are independent of t is a result of Corollary 4.4. Conversely, assume that (6.24) holds for $t>\sigma\geq 0$. Then by Theorem 6.1, we have

$$u(x,t)=\int_0^{\infty}h\,(xy)\,e^{-ty^2}\,\phi(y)\,d\mu(y),$$

with $\phi(y)$ an even function of growth $(1,\sigma)$. By the preceding Lemma, it follows that

$$\left|u(x,t)\right|\ \leq M\ (c)\,x^{-2\alpha}\,\frac{e^{-\left[x^2/4(t+c)\right]}}{\sqrt{t-c}},\ \sigma<c<t<\infty. \tag{6.32}$$

Hence

$$\int_0^{\infty}\left|u(x,t)\right|e^{x^2/8t}\,d\mu(x)\leq\frac{M(c)}{\sqrt{t-c}}\int_0^{\infty}x^{-2\alpha}\,e^{\left(x^2/8t\right)-\left[x^2/4(t+c)\right]}\,d\mu(x)$$

$$<\infty\ ,\ c<t<\infty,$$

and (6.25) is established. To prove that $u(x,t)\in H^*$, we apply Lemma 6.1 of [5]. Since (6.32) holds, $u(x,t)$ is uniformly bounded for $t\geq c+\delta,\ \delta>0$. Hence $u(x,t)\in H^*$ for $t>c+\delta$ and, so also for $t>\sigma$, and the theorem is proved.

An example illustrating this theorem is obtained by taking

$$u(x,t)=G(x,b;t).$$

Here

$$\int_0^\infty \left|G(x,b;t)\right| e^{x^2/8t}\, d\mu(t) < \infty,\ \ 0<t<\infty,$$

and we have

$$G(x,b;t)=\sum_{n=0}^{\infty} k_n\, b^{2n}\, W_{n,\alpha,\beta}(x,t),$$

verifying equation (2.19).

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