On the Zeros of Lacunary Polynomials

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Abstract : In this paper we find the bounds for the zeros of some special lacunary polynomials. Our results generalize some known results in the distribution of zeros of polynomials.

Mathematics Subject Classification: 30 C 10, 30 C 15
Keywords and Phrases: Coefficient, Polynomial, Zero

1. Introduction and Statement of Results

A classic result on the zeros of polynomials is the following theorem due to Cauchy[2]:

Theorem A: Let $P(z)=a_0+a_1z+a_2z^2+......+a_{n-1}z^{n-1}+a_nz^n, a_n \neq 0$ be a complex polynomial. Then all the zeros of P(z) lie in the closed disk $|z| \le r$, where r is the positive root of the equation

$$|a_0|+|a_1|z+......+|a_{n-1}|z^{n-1}-|a_n|z^n=0.$$

Another classical result due to Cauchy [2] is the following:

Theorem B: : Let $P(z)=a_0+a_1z+a_2z^2+......+a_{n-1}z^{n-1}+a_nz^n, a_n \neq 0$ be a complex polynomial. Then all the zeros of P(z) lie in the closed disk $|z| \le 1+M$, where

$$M=\max\left|\frac{a_j}{a_n}\right|, j=0,1,2,......,n-1.$$

Similar results due to Dehmer [1] are the following:

Theorem C: Let $P(z)=a_0+a_1z+a_2z^2+......+a_{n-1}z^{n-1}+a_nz^n, a_n \neq 0$ be a complex polynomial. Then all the zeros of P(z) lie in the closed disk $|z| \le \max(1,k)$, where $k \neq 1$ is the positive root of the equation

$$z^{n+1}-(1+M)z^n+M=0$$

and $M=\max\left|\frac{a_j}{a_n}\right|, j=0,1,2,......,n-1.$

Theorem D: Let $P(z)=a_0+a_1z+a_2z^2+......+a_{n-1}z^{n-1}+a_nz^n, a_n \neq 0$ be a complex polynomial. Then all the zeros of P(z) lie in the closed disk $|z| \le \max(1,k)$, where $k \neq 1$ is the positive root of the equation

$$z^{n+2}-(1+M')z^{n+1}+M'=0$$

and $M'=\max\left|\dfrac{a_{n-j}-a_{n-j-1}}{a_n}\right|, j=0,1,2,......,n; a_{-1}=0$.

The following result is the famous Enestrom-Kakeya Theorem [2]:

Theorem E: Let $P(z)=a_0+a_1z+a_2z^2+......+a_{n-1}z^{n-1}+a_nz^n, a_n\neq 0$ be a polynomial with real coefficients such that

$$a_n\geq a_{n-1}\geq......\geq a_1\geq a_0>0.$$

Then all the zeros of P(z) lie in the closed disk $|z|\leq 1$.

In this paper , we consider some special lacunary polynomials and see what happens to Theorems A,B,C, D and E. In fact, we prove the following results:

Theorem 1: Let $P(z)=a_0z^n+a_pz^{n-p}+a_{p+1}z^{n-p-1}+......+a_{n-1}z+a_n, a_p\neq 0, p<n$**,** be a polynomial of degree n. Then all the zeros of P(z) lie in $|z|\leq k$, where k >1 is the root of the equation

$$k^p-k^{p-1}+M=0$$

and $M=\max\left|\dfrac{a_j}{a_0}\right|, j=p,p+1,.......,n.$.

Remark 1: Taking p=1in Theorem 1, we get k=1+M so that Theorem 1 reduces to Theorem A.

Theorem 2: Let $P(z)=a_0+a_1z+a_2z^2+......+a_pz^p+a_nz^n, a_p\neq 0, 1\leq p\leq n-1$**,** be a polynomial of degree n. Then all the zeros of P(z) lie in $|z|\leq\max(1,k)$, where k $\neq 1$is the positive root of the equation

$$z^{n+1}-z^n-Mz^{p+1}+M=0$$

and $M=\max\left|\dfrac{a_j}{a_n}\right|, j=0,1,2,......,p$.

Remark 2: Taking p=n-1in Theorem 1, we get Theorem B of Dehmer.

Theorem 3: Let $P(z)=a_0+a_1z+a_2z^2+......+a_pz^p+a_nz^n, a_p\neq 0, 1\leq p\leq n-1$**,** be a polynomial of degree n. Then all the zeros of P(z) lie in $|z|\leq k$, where k is the greatest positive root of the equation

$$z^{n+2}-2z^{n+1}+z^n-M'z^{p+1}+M'=0$$

and $M'=\max\left|\dfrac{a_j-a_{j-1}}{a_n}\right|, j=0,1,2,......,p; a_{-1}=0$.

Theorem 4: Let $P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_p z^{n-p} + a_n z^n, a_p \neq 0, p < n$, be a polynomial of degree n such that

$$a_p \geq a_{p-1} \geq ...... \geq a_1 \geq a_0 .$$

Then all the zeros of P(z) lie in the closed disk

$$|z| \leq \frac{1}{|a_n|}[|a_n| + a_p - a_0 + |a_0|]$$

2. Proofs of Theorems

Proof of Theorem 1: For $|z| > 1$, $\frac{1}{|z|^j} < 1, \forall j = p, p+1, ..., n$, so that , by using the hypothesis, we have

$$|P(z)| = |a_0 z^n + a_p z^{n-p} + a_{p+1} z^{n-p-1} + ... + a_{n-1} z + a_n|$$

$$\geq |z|^n [1 - \{\left|\frac{a_p}{a_0}\right|.\frac{1}{|z|^p} + \left|\frac{a_{p+1}}{a_0}\right|.\frac{1}{|z|^{p+1}} + ... + \left|\frac{a_{n-1}}{a_0}\right|.\frac{1}{|z|^{n-1}} + \left|\frac{a_n}{a_0}\right|.\frac{1}{|z|^n}\}]$$

$$\geq |z|^n [1 - M\{\frac{1}{|z|^p} + \frac{1}{|z|^{p+1}} + ... + \frac{1}{|z|^n}\}]$$

$$> |z|^n [1 - M\{\frac{1}{|z|^p} + \frac{1}{|z|^{p+1}} + ... + \frac{1}{|z|^n} + ...\}]$$

$$= |z|^n [1 - \frac{M}{|z|^{p-1}(|z| - 1)}]$$

$$= \frac{|z|^n [|z|^p - |z|^{p-1} - M]}{|z| - 1}$$

$$> 0$$

if

$$|z|^p - |z|^{p-1} - M > 0.$$

This shows that the zeros of P(z) with modulus greater than 1 lie in the closed disk $|z| \leq k$, where k >1 is the root of the equation

$$k^p - k^{p-1} + M = 0 .$$

Since the zeros of P(z) of modulus less than or equal to 1 are already in $|z| \leq k$, the result follows.

Proof of Theorem 2: For $|z|>1$, we have $\frac{1}{|z|^{n-j}}<1, \forall j=0,1,2,......,p$, so that

$$|P(z)|=|a_n z^n + a_p z^p + a_{p-1}z^{p-1}+......+a_1 z + a_0|$$

$$\geq |a_n||z|^n - \{|a_p||z|^p + |a_{p-1}||z|^{p-1}+......+|a_1||z|+|a_0|\}$$

$$=|a_n|[|z|^n - |z|^p\{\frac{|a_p|}{|a_n|}+\frac{|a_{p-1}|}{|a_n|}.\frac{1}{|z|}+......+\frac{|a_1|}{|a_n|}.\frac{1}{|z|^{p-1}}+\frac{|a_0|}{|a_n|}.\frac{1}{|z|^p}\}]$$

$$\geq |a_n|[|z|^n - |z|^p\{M+\frac{M}{|z|}+\frac{M}{|z|^2}+......+\frac{M}{|z|^{p-1}}+\frac{M}{|z|^p}\}]$$

$$=|a_n|[|z|^n - |z|^p M\{1+\frac{1}{|z|}+\frac{1}{|z|^2}+......+\frac{1}{|z|^{p-1}}+\frac{1}{|z|^p}\}]$$

$$=|a_n|[|z|^n - \frac{M(|z|^{p+1}-1)}{|z|-1}]$$

$$=\frac{|a_n|[|z|^{n+1}-|z|^n - M|z|^{p+1}+M]}{|z|-1}$$

$$>0$$

if

$$|z|^{n+1}-|z|^n - M|z|^{p+1}+M>0.$$

Define

$$F(z)=z^{n+1}-z^n-Mz^{p+1}+M.$$

By using Descarte's Rule of Signs, F(z) has exactly two positive zeros k_1 and k_2 and $F(k_1)=0$ with Sign{F(0)}=1. Hence we conclude that

$$|F(z)|>0 \text{ for } |z|>\max(1,k).$$

Hence , it follows that all the zeros of P(z) lie in $|z|\leq \max(1,k)$, where k $\neq 1$is the positive root of the equation

$$z^{n+1}-z^n-Mz^{p+1}+M=0.$$

That proves Theorem 2.

Proof of Theorem 3: Consider the polynomial

$$F(z)=(1-z)P(z)=(1-z)(a_n z^n + a_p z^p + ...+a_1 z + a_0)$$

$$=-a_n z^{n+1}+a_n z^n + (a_p - a_{p-1})z^p + (a_{p-1}-a_{p-2})z^{p-1}+......+(a_1 - a_0)z + a_0.$$

For $|z|>1$, we have, by using the hypothesis,

$$|F(z)| \geq |a_n||z|^{n=1} - |a_n z^n + (a_p - a_{p-1})z^p + (a_{p-1} - a_{p-2})z^{p-1} + ...+ (a_1 - a_0)z + a_0|$$

$$\geq |a_n|[|z|^{n+1} - |z|^n - |z|^p\{\frac{|a_p - a_{p-1}|}{|a_n|} + \frac{|a_{p-1} - a_{p-2}|}{|a_n||z|} + ...+ \frac{|a_1 - a_0|}{|a_n||z|^{p-1}} + \frac{|a_0 - a_{-1}|}{|a_n||z|^p}\}]$$

$$\geq |a_n|[|z|^{n+1} - |z|^n - |z|^p\{M' + \frac{M'}{|z|} + ...+ \frac{M'}{|z|^{p-1}} + \frac{M'}{|z|^p}\}]$$

$$= |a_n|[|z|^{n+1} - |z|^n - |z|^p M'\{1 + \frac{1}{|z|} + ...+ \frac{1}{|z|^{p-1}} + \frac{1}{|z|^p}\}]$$

$$= |a_n|[|z|^{n+1} - |z|^n - \frac{(|z|^{p+1} - 1)M'}{|z| - 1}]$$

$$= \frac{|a_n|[|z|^{n+2} - 2|z|^{n+1} + |z|^n - M'|z|^{p+1} + M']}{|z| - 1}$$

$$> 0$$

if

$$|z|^{n+2} - 2|z|^{n+1} + |z|^n - M'|z|^{p+1} + M' > 0.$$

This shows that the zeros of F(z) with modulus greater than 1 lie the closed disk $|z| \leq k$, where k is the greatest positive root of the equation

$$z^{n+2} - 2z^{n+1} + z^n - M'z^{p+1} + M' = 0$$

Since z=1 is a zero of the above equation, it follows that the zeros of F(z) with modulus less than 1 also lie in $|z| \leq k$.

Since the zeros of P(z) are also the zeros of F(z) , it follows that all the zeros of P(z) lie in the closed disk $|z| \leq k$, where k is the greatest positive root of the equation

$$z^{n+2} - 2z^{n+1} + z^n - M'z^{p+1} + M' = 0,$$

thereby proving Theorem 3.

Proof of Theorem 4: Consider the polynomial

$$F(z) = (1 - z)P(z) = (1 - z)(a_n z^n + a_p z^p + ...+ a_1 z + a_0)$$

$$= -a_n z^{n+1} + a_n z^n + (a_p - a_{p-1})z^p + (a_{p-1} - a_{p-2})z^{p-1} + ...+ (a_1 - a_0)z + a_0.$$

For $|z| > 1$, we have, by using the hypothesis,

$$|F(z)| \geq |a_n||z|^{n+1} - |z|^n\{|a_n| + \frac{|a_p - a_{p-1}|}{|z|^{n-p}} + \frac{|a_{p-1} - a_{p-2}|}{|z|^{n-p+1}} + ...+ \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\}$$

$$> |z|^n[|a_n||z| - \{|a_n| + a_p - a_{p-1} + a_{p-1} - a_{p-2} + ...+ a_1 - a_0 + |a_0|\}]$$

$$= |z|^n[|a_n||z| - \{|a_n| + a_p - a_0 + |a_0|]$$

$$>0$$

if

$$|a_n||z| - \{|a_n| + a_p - a_0 + |a_0| > 0$$

i.e. $$|z| > \frac{1}{|a_n|}[|a_n| + a_p - a_0 + |a_0|].$$

This shows that the zeros of F(z) with modulus greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_n|}[|a_n| + a_p - a_0 + |a_0|].$$

Since the zeros of F(z) with modulus less than 1already lie in the above disk, it follows that all the zeros of F(z) lie lie in the closed disk

$$|z| \leq \frac{1}{|a_n|}[|a_n| + a_p - a_0 + |a_0|].$$

Since the zeros of P(z) are also the zeros of F(z), the result follows.

References

[1] M. Dehmer, On the Location of Zeros of Complex Polynomials, Journal Of Inequalities in Pure and Applied Mathematics, Vol. 7, Issue 1, article 26, 2006, 1-27.

[2]M.Marden, Geometry of Polynomials,Math. Surveys No. 3, Amer. Math Society(R.I. Providence) (1996).