A Note on Convolution Conditions

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ABSTRACT: In this note, we define the subclasses $\Omega_\lambda^\delta(A,B)$ and $\Omega^\delta(A,B)$ of analytic functions using the fractional derivative Ω^δ. Some interesting sufficient conditions involving coefficients inequalities for functions belonging to the above classes are derived.

Key words and phrases: Fractional derivative, Convolution condition, Janowski class.

1. INTRODUCTION

Let A denote the class of analytic functions in the open unit disc $U = \{z;\ |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

For $-1 \le B < A \le 1$, let $P(A,B)$ [2] denote the class of functions which are of the form

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)},$$

where is a bounded analytic function satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$. The Fractional derivative of order δ of an analytic function f is defined by

$$D_z^\delta f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^\delta} dt,\ 0 \le \delta < 1,$$

(1.2)

f is an analytic function in a simply connected region of the z-plane containing the origin and multiplicity of $(z-t)^{-\delta}$ is removed by requiring $\log(z-t)$ to be real when $(z-t)$ is greater than zero. Clearly $f(z) = \lim_{\delta\to 0} D_z^\delta f(z)$ and $f'(z) = \lim_{\delta\to 1} D_z^\delta f(z)$.

For the analytic function f of the form (1.1), Srivastava and Owa[5] introduced the operator Ω^δ which is defined by

$$\Omega^\delta f(z) = \Gamma(2-\delta) z^\delta D_z^\delta f(z) = z + \sum_{n=2}^{\infty} k(n,\delta) a_n z^n$$

where $k(n,\delta) = \frac{n!\Gamma(2-\delta)}{\Gamma(n+1-\delta)}$, $0 \le \delta < 1$.

Using the fractional derivative now we define following subclasses of A.

Let $\boldsymbol{\Omega_\lambda^\delta(A,B)}$ denote the class of analytic functions f obeying the condition

$$\frac{e^{i\lambda} \frac{z(\Omega^\delta f(z))'}{\Omega^\delta f(z)} - i\sin\lambda}{\cos\lambda} \in P(A,B)$$

where $-1 \le B < A \le 1, 0 \le \delta < 1, \lambda \ge 0$ and $z \in U$.

For parametric values $\delta = 0$ and $\delta = 1$ we get the classes $S_\lambda^*(A,B)$ and $K_\lambda(A,B)$ studied by Ganesan [1] respectively. For $\lambda = 0$ the class $\boldsymbol{\Omega_\lambda^\delta(A,B)}$ reduces to the class $\boldsymbol{\Omega^\delta(A,B)}$ consisting of analytic functions f such that

$$\frac{z(\Omega^\delta f(z))'}{\Omega^\delta f(z)} \in P(A,B),$$

where $-1 \le B < A \le 1, 0 \le \delta < 1$, and $z \in U$. For $\delta = 0$ and $\delta = 1$ we get the classes $S^*(A,B)$ and $K(A,B)$ studied by Ganesan [1] respectively.

To prove the main the result we need the following preliminaries.

Lemma1.1

[3] *A function* $p(z) \in P(A,B)$, *if and only if* $p(z) \neq \frac{1+A\zeta}{1+B\zeta}, z \in U, \zeta \in \mathbb{C}, |\zeta| = 1.$

Lemma1.2

A function $f \in A$ *is in the class* $\boldsymbol{\Omega_\lambda^\delta(A,B)}$ *if and only if*

$$1 + \sum_{n=2}^{\infty} D_n z^{n-1} \neq 0 \qquad (1.3)$$

where

$D_n = \frac{[(n-1)+(Bn-\gamma)\zeta]}{(B-A)\zeta} K(n,\delta) a_n$, *and* $\gamma = (A\cos\lambda + iB\sin\lambda)e^{-i\lambda}$

.

Proof: A function $f \in A$ is in the class $\boldsymbol{\Omega_\lambda^\delta(A,B)}$ *if and only if*

$$\frac{e^{i\lambda}\frac{z(\Omega^\delta f(z))'}{\Omega^\delta f(z)} - i\sin\lambda}{\cos\lambda} \neq \frac{1+A\zeta}{1+B\zeta}.$$

That is

$$(1+B\zeta)\left(z\left(\Omega^\delta f(z)\right)'\right) \neq (1+\gamma\zeta)\left(\Omega^\delta f(z)\right).$$

This simplifies into

$$(B-A)\zeta z + \sum_{n=2}^{\infty} k(n,\delta)[(n-1) + (Bn-\gamma)\zeta]a_n z^n \neq 0. \qquad (1.4)$$

i.e.,

$$1 + \sum_{n=2}^{\infty} \frac{[(n-1)+(Bn-\gamma)\zeta]}{(B-A)\zeta z} k(n,\delta) a_n z^n \neq 0.$$

Hence the proof.

For parametric values $\delta = 0$ and $\delta = 1$ *in Lemma* 1.2 we get the characterization theorems for the classes $S_\lambda^*(A,B)$ and $K_\lambda(A,B)$ in [3] respectively. As a special case of Lemma 1.2 we get the following result.

Lemma1.3[]

A function $f \in A$ *is in the class* $\boldsymbol{\Omega^\delta(A,B)}$ *if and only if*

$$1 + \sum_{n=2}^{\infty} D_n z^{n-1} \neq 0 \qquad (1.3)$$

where $D_n = \frac{[(n-1)+(Bn-A)\zeta]}{(B-A)\zeta} K(n,\delta) a_n$

For parametric values $\delta = 0$ and $\delta = 1$ *in Lemma* 1.3 we get the characterization theorems for the classes $S^*(A,B)$ and $K(A,B)$ in [3] respectively.

Remark1.4: It follows from the normalization conditions $a_0 = 0$ *and* $a_1 = 1$ that $\boldsymbol{D_0 = \frac{-(1+\gamma\zeta)}{(B-A)}(1-\delta) = 0}$, **and**

$$\boldsymbol{D_1 = \frac{(1+B\zeta)-(1+\gamma\zeta)}{(B-A)\zeta} a_1 = 1}.$$

Remark1.5: The assertion (1.3), of Lemma 1.2 is equivalent to

$$\frac{1}{z}\left(f(z) * \frac{\zeta z(B-\gamma) + (1+\gamma\zeta)z^2}{(1-z)^2} k(n,\delta)\right) \neq 0,$$

$|z| < R, |\zeta| < 1.$

For $\lambda = \delta = 0$; $\lambda = 0, \delta = 1$, we get the assertions given by Ganesan [1].

2. Main Results

Theorem2.1: If $f \in A$ satisfies the following condition

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}(j-1)k(j,\delta)\binom{\beta}{k-j}a_j\right)\binom{\eta}{n-k}\right| + \left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}(jB-\gamma)k(j,\delta)\binom{\beta}{k-j}a_j\right)\binom{\eta}{n-k}\right|\right] < B-A \qquad (2.1)$$

where $\gamma = (A\cos\lambda + iB\sin\lambda)e^{-i\lambda}$, with $\beta, \eta \in \mathbb{R}$ *and* $-1 \le B < A \le 1$ then $f(z) \in \Omega_\lambda^\delta(A,B)$.

Proof: We note that $(1-z)^\beta \neq 0$ *and* $(1-z)^\eta \neq 0, (\beta,\eta \in \mathbb{R})$.

$$\left(1 + \sum_{n=2}^{\infty} D_n z^{n-1}\right)(1-z)^\beta (1-z)^\eta \neq 0,$$

$$(z \in U, \beta, \eta \in \mathbb{R}). \qquad (2.2)$$

If the (2.2) holds true, then we have $1 + \sum_{n=2}^{\infty} D_n z^{n-1} \neq 0$, which is the relation (1.3) of Lemma 1.2. Equation (1.3) is equivalent to

$$\left(1+\sum_{n=2}^{\infty} D_n z^{n-1}\right)\left(1+\sum_{n=0}^{\infty}(-1)^n b_n z^n\right)\left(1+\sum_{n=0}^{\infty} c_n z^n\right) \neq 0 \qquad (2.3)$$

where $b_n = \binom{\beta}{n}$ and $c_n = \binom{\eta}{n}$.

Using the Cauchy product of the first two factor, we can write the expression (2.3), as

$$\left(1+\sum_{n=2}^{\infty} C_n z^{n-1}\right)\left(1+\sum_{n=0}^{\infty} c_n z^n\right) \neq 0 \qquad (2.4)$$

where $C_n = \sum_{j=1}^{n}(-1)^{n-j} D_j b_{n-j}$.

Further, by applying the Cauchy product again in (2.4), we find that

$$1+\sum_{n=2}^{\infty}\left(\sum_{k=1}^{n} C_k c_{n-k}\right) z^{n-1} \neq 0.$$

Equivalently, we have

$$1+\sum_{n=2}^{\infty}\left[\left(\sum_{k=1}^{n}(-1)^{k-j} D_j b_{k-j}\right) c_{n-k}\right] z^{n-1} \neq 0, \qquad (z \in U).$$

If $f \in A$ satisfies the following inequality

$$\sum_{n=2}^{\infty}\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j} D_j b_{k-j}\right) c_{n-k} z^{n-1}\right| \leq 0, \quad (z \in U).$$

That is if

$$\frac{1}{\zeta(B-\gamma)}\sum_{n=2}^{\infty}\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}[(j-1)+(jB-\gamma)\zeta] k(j,\delta) a_j b_{k-j}\right) c_{n-k}\right|$$

$$\leq \frac{1}{(B-\gamma)}\sum_{n=2}^{\infty}\left(\left|\sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{k-j}(j-1) k(j,\delta) a_j b_{k-j}\right] c_{n-k}\right| + \left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}(jB-\gamma) k(j,\delta) a_j b_{k-j}\right) c_{n-k}\right|\right)$$

$$\leq 1$$

for $-1 \leq B < A \leq 1,\ \zeta \in \mathbb{C}, |\zeta| = 1$, then $f \in \Omega_\lambda^\delta(A,B)$ which establishes the result. For particular choices of the parameter δ we get the Theorem 4.2 and Theorem 4.3 in [3] respectively.

For $A = 1-2\alpha,\ B = -1,\ \delta = 0$ and $\delta = 1$ we get coefficient conditions for functions in the classes $S_\lambda^*(\alpha)$ and $K_\lambda(\alpha)$ in [1] with suitable modifications. As a special case of Theorem 2.1 yields the following result.

Corollary2.2: If $f \in A$ satisfies the following condition

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}(j-1)k(j,\delta)\binom{\beta}{k-j}a_j\right)\binom{\eta}{n-k}\right| + \left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}(jB-A)k(j,\delta)\binom{\beta}{k-j}a_j\right)\binom{\eta}{n-k}\right|\right] < B-A$$

$\beta,\ \eta \in \mathbb{R}$ and $-1 \leq B < A \leq 1$ then $f(z) \in \Omega_\lambda^\delta(A,B)$.

For particular choices of the parameter δ we get the Theorem 3.1 and Theorem 3.5 in [3] respectively.

For $A = 1-2\alpha,\ B = -1,\ \delta = 0$ and $\delta = 1$ we get coefficient conditions for functions in the classes $S^*(\alpha)$ and $K(\alpha)$ in [1] with suitable modifications.

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