On Relative Preclosedness of Strongly Compact (Countably p-Compact) Sets

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***Abstract*— In this paper, we study the preclosedness of strongly compact (countably p-compact) subsets of subspaces of strongly p-normal spaces. Consequences of the result for unions of specific sets are given. Examples are given to illustrate the results.**
***Keywords*— preclosed, p*closed, pre-R_1, strongly compact, countably p-compact, pre-accumulation, p-convergent, pre-sequential, strongly p-normal, net.**

I. Introduction

In [3] Ganster answered the question posed by Katetov as to when preopen sets form a topology. In fact he proved that, for space X having its unique Hewitt representation $X = F \cup G$, where F is closed and resolvable and G is open and hereditarily irresolvable, the preopen sets of X form a topology if and only if closure of G is open and each singleton in the interior of F is preopen in X. In this paper, we will call such spaces in which preopen sets form a topology as spaces having *Strong Hewitt Representation*. On the other hand, in [4] Garg and Singh took up the question of closedness of a compact (countably compact) set in S_2 (sequential, S_2) and normal (sequential, normal) spaces. Since normality is not hereditary, in [5] Garg and Singh further generalized the results to closedness of a compact (countably compact) set in subspaces (sequential subspaces) of normal spaces. In [10], Noorie and A. Singh obtained necessary and sufficient conditions for p*closedness of a strongly compact (countably p-compact) set in pre-R_1 (pre-sequential, pre- R_1), p*normal (pre-sequential, p*normal) spaces and sufficient conditions for p*closedness of a strongly compact (countably p-compact) set in strongly p-normal (pre-sequential, strongly p – normal) and p-normal (pre-sequential, p-normal) spaces. Among others, the following results have been proved in [10]:

Theorem 1.1 [10]:
For a strongly compact (countably p-compact) subset K of a pre- R_1 (pre-sequential, pre- R_1) space X, the following conditions are equivalent:
(i) K is p*closed;
(ii) either K or K^C is a union of p*closed sets;
(iii) both K and K^C are unions of p*closed sets.

Theorem 1.2 [10]:
In a p-normal (pre-sequential, p-normal) space X, strongly compact (countably p-compact) set K is p*closed if K is a union of closed sets and K^C is of the form $G \cup C$, where G and C are arbitrary preopen and closed sets respectively.

Theorem 1.3 [10]:
In a strongly p-normal (pre-sequential, strongly p-normal) space X, strongly compact (countably p-compact) set K is p*closed if K is a union of preclosed sets and K^C is of the form $G \cup F$, where G and F are arbitrary preopen and preclosed sets respectively.

In this paper, results of [10] are generalized to subspaces (pre-sequential subspaces) of spaces having Strong Hewitt Representation, obtaining necessary and sufficient conditions for preclosedness of strongly compact (countably p-compact) subset of the preopen (presequential, preopen) subspace Y in strongly p-normal spaces [Theorem 2.9, Theorem 2.11]. Also sufficient conditions for relative preclosedness of arbitrary union of preclosed sets in strongly p-normal spaces [Corollary 2.10 (a), (b)] are obtained.

A subset A of a space X is *preclosed* [8] if closure of interior of A is contained in A. The complement of a preclosed set is called a *preopen* set and preclosure [2] is the intersection of all preclosed sets containing A and is denoted by *pcl(A)*. A point $x \in X$ is a *pre-accumulation (p-convergent)* [6] of a net in X if the net is frequently (eventually) in every preopen set containing x. A subset A of space X is said to be *p*closed* [10] if no net in A p-converges to a point of A^C. A space X is, (i) *strongly compact* [10] (*countably p-compact*) [13] if every preopen (countable preopen) cover of X has finite subcover, (ii) *p-normal* [11] if for each pair of disjoint closed sets of X, there exist disjoint preopen sets containing them, (iii) *strongly p-normal*

[10] if for each pair of disjoint preclosed sets of X, there exist disjoint preopen sets containing them. (iv) pre-R_1 [1] if for points *x, y* in *X* with distinct preclosures there exist disjoint preopen sets containing pcl({x}) and pcl({y}), (v) *pre-T_2* [7] if for each pair of distinct points x and y of X, there exists a pair of disjoint preopen sets, one containing *x* and the other containing *y*, (vi) *pre-sequential* [10] if for every non-preclosed subset A of X there is a sequence $\{x_n\}$ in A which p-converges to a point of A^C, (vii) *resolvable* [3] if it is the disjoint union of two dense subsets, (viii) *hereditarily irresolvable* [3] if it does not contain a non-empty resolvable set. Also we will call a set *pre-F_σ*, if it is countable union of preclosed sets. A subset A of X is *strongly compact* (*countably p-compact*) *relative to X* [9] if every preopen (countable preopen) cover of A by preopen sets of X has finite subcover.

Throughout, by a space X we shall mean a topological space. In a space X, A^C will denote the complement of A for any subset A of X. $\mathbf{Z}^+$ will denote the set of all positive integers. G, F and C, respectively, will stand for arbitrary preopen, preclosed and subsets of X. G_Y, F_Y and C_Y respectively, will stand for relatively preopen, relatively preclosed and relatively closed subsets of the subspace Y of X. For a set $S \subset Y$, $cl_Y(S)$, $pcl_Y(S)$, and $Y-S$, respectively will denote the closure, preclosure and complement of the set S in the subspace Y of X.

The following results will be used in the next section.

Lemma 1.4 [6]**:**

A space X is strongly compact if and only if every net in X pre-accumulates to some point of X.

Lemma 1.5 [7]**:**

If $A \subset Y \subset X$ and Y is preopen in X then, A is preopen in Y if and only if A is preopen in X.

Lemma 1.6 [12]**:**

If $A \subset Y \subset X$ and Y is α-set in X then $pcl_Y(A) = pcl_X(A) \cap Y$.

Lemma 1.7 [1]:

For a space X, the following conditions are equivalent:

(i) X is pre-R_1;

(ii) X is pre-T_2.

II. RESULTS

Proof of Lemma 2.1 follows from Lemma 1.5 and Lemma 1.6.

Lemma 2.1:

If $A \subset Y \subset X$, A is preclosed in Y and Y is preopen in X then, $A = pcl_X(A) \cap Y$.

Remark:

From now onwards, throughout this section, the space X is assumed to have Strong Hewitt Representation.

With the assumption that preopen sets of space X, form a topology and a directed set, the well known relationship between adherent points of a set and nets has the following analogous form.

Lemma 2.2:

For $A \subset X$, $x \in pcl(A)$ if and only if there exists a net in A which p-converges to x.

The following Lemma gives another characterization of the preclosed sets.

Lemma 2.3:

A subset A of X is p*closed if and only if it is preclosed.

Proof: Since every preclosed set is p*closed [10]. The converse follows with the help of Lemma 2.2.

We now obtain a necessary and sufficient condition for the preclosedness of a countably p-compact set in a strongly p-normal space in Theorem 2.4 and sufficient condition for the equality of the union of preclosures and the preclosure of the union of countable families of sets in strongly p-normal spaces are also obtained in Corollary 2.5 below.

Theorem 2.4:

Let X be a strongly p-normal space and K a countably p-compact subset of X. Then K is preclosed in X if and only if K is a pre-F_σ and K^C is of the form $G \cup F$.

Proof: Let K = $\bigcup_{i=1}^{\infty} F_n$, where each F_n is preclosed in X, and $K^C = G \cup F$. We prove K^C is preopen in X. For any $x \in K^C$, if $x \in G$, then $x \in G \subset K^C$, where G is preopen in X. Otherwise, $x \in F$. Since X is strongly p-normal, for each n, there exist disjoint preopen sets U_n and V_n in X containing F_n and F respectively. Then $\{U_n\}_{n=1}^{\infty}$ is a countable preopen cover of K and therefore, there exists a positive integer n, such that, $K \subset \bigcup_{i=1}^{n} U_n$ and $F \subset \bigcap_{i=1}^{n} V_n$. Then $U = \bigcup_{i=1}^{n} U_n$ and $V = \bigcap_{i=1}^{n} V_n$ are disjoint preopen sets such that $x \in V \subset U^C \subset K^C$. Therefore, K^C is preopen and hence K is preclosed.

Corollary 2.5:
In a strongly p-normal space X,
(a) a countable union of preclosed sets is preclosed, if it is countably p-compact and is of the form $G \cap F$;
(b) if ε is a family of subsets of X such that $\cup\{pcl\ (E) : E \in \varepsilon\}$, in particular $\cup \{E : E \in \varepsilon\}$, is countably p-compact and is of the form $G \cap F$, then $pcl\ (\cup \{E : E \in \varepsilon\}) = \cup\{pcl\ (E) : E \in \varepsilon\}$.

Lemma 2.6:
For a preopen subspace Y of X, the preopen sets of Y form a topology.
Proof: Preopen sets of Y form a topology if they satisfy finite intersection property. Let $\{U_\alpha\}$ be preopen sets of Y. By Lemma 1.5 since Y is preopen in X each U_α is also preopen in X. Since, finite intersection of preopen sets is preopen in X and the finite intersection will also be preopen in Y again by Lemma 1.5.

Lemma 2.7:
If $A \subset Y \subset X$ and Y is preopen set in X then, A is strongly compact (countably p-compact) relative to X if and only if A is strongly compact (countably p-compact) relative to Y.
Proof: Necessary condition is obvious and the sufficient part follows from Lemma 1.5 and Lemma 2.6.

Proof of Theorem 2.8 follows from Lemma 1.5 and Lemma 1.7.
Theorem 2.8:
Every preopen set Y of a pre-R_1 X is also pre-R_1.

We now obtain the necessary and sufficient conditions for the preclosedness of a strongly compact (countably p-compact) subset of the preopen subspace (presequential, preopen subspace) of a strongly p-normal space in Theorem 2.9 and Corollary 2.10 as a generalization of Theorem 1.3 above in subspaces.
Theorem 2.9:
Let K be a strongly compact (countably p-compact) subset of the preopen subspace (presequential, preopen subspace) Y of a strongly p-normal space X. Then K is relatively preclosed in Y,
(a) if and only if K is a union of relatively preclosed subsets of Y and Y – K is of the form $G_Y \cup F$,
(b) if K is a union of preclosed subsets of X and Y − K is of the form $G_Y \cup F_Y$;
(c) if K is a union of preclosed subsets of X and Y − K is a union of relatively preclosed subsets of Y ;
(d) if K is a union of relatively preclosed subsets of Y and Y − K is a union of preclosed subsets of X.
*Proof: **(a)*** Since the necessary condition is obvious we prove the sufficient part. Let $K = \cup_\alpha V_\alpha$, where each V_α is a preclosed set in Y. Since Y is preopen set by Lemma 2.1, $K = \cup_\alpha (F_\alpha \cap Y)$, where each F_α is a preclosed set in X. By Lemma 2.3, if K is not relatively preclosed in Y then there exists a net $\{x_\lambda\}$ (a sequence $\{x_n\}$) in K such that x_λ ($\{x_n\}$) p-converges to point *a* and *a* is in $Y - K$. Then as K is strongly compact (countably p-compact) in Y, Lemma 1.4 implies that the net$\{x_\lambda\}$ (the sequence $\{x_n\}$) has a pre-accumulation point *b* in K. By Lemma 2.6 the net$\{x_\lambda\}$ (the sequence $\{x_n\}$) in K pre-accumulates relative to X to the point b. Therefore, there exists an α such that $b \in F_\alpha$ and $a \notin F_\alpha$. Thus *a* and *b* belong to the disjoint preclosed sets *F* and F_α of X and since X is strongly p-normal it follows that they have disjoint preopen sets U and V of X containing them respectively, and since Y is preopen in X and preopen sets of X form a topology $U \cap Y$ and $V \cap Y$ are disjoint preopen sets of Y (Lemma 1.5) containing a and b respectively, contradicting to the fact that x_λ p-converges to *a* (x_n p-converges to *a*) and *b* is a pre-accumulation point of $\{x_\lambda\}(\{x_n\})$. Hence K must be preclosed in Y and (a) follows.
The proofs of (b) - (d) are similar to that of part (a).

Corollary 2.10:
Let K be a strongly compact (countably p-compact) subset of the preopen subspace (presequential, preopen subspace) Y of a strongly p-normal space X. Then

(a) a union K of relatively preclosed subsets of Y is relatively preclosed in Y , if K is strongly compact (countably p- compact) and $Y - K$ is either of the form $G_Y \cup F$ or is a union of preclosed subsets of X;
(b) a union $K \subset Y$ of preclosed subsets of X is relatively preclosed in Y if K is strongly compact (countably p- compact) and $Y - K$ is either of the form $G_Y \cup F_Y$ or is a union of relatively preclosed subsets of Y;
(c) if ε is a family of subsets of X such that $K = \cup \{pcl_Y(E) : E \in \varepsilon\}$ is strongly compact (countably p-compact) and $Y - K$ is of the form $G_Y \cup F$ or is a union of preclosed subsets of X, then $pcl_Y(\cup \{E : E \in \varepsilon\}) = \cup\{pcl_Y(E) : E \in \varepsilon\}$;
(d) if ε is a family of subsets of X such that $K = \cup \{pcl_Y(E) : E \in \varepsilon\}$ is strongly compact (countably p-compact) and $Y - K$ is of the form $G_Y \cup F_Y$ or is a union of relatively preclosed subsets of Y, then $pcl_Y(\cup \{E : E \in \varepsilon\}) = \cup\{pcl_Y(E) : E \in \varepsilon\}$.

We now obtain the necessary and sufficient conditions for the preclosedness of a countably p-compact subset of the preopen subspace of a strongly p-normal space as a generalization of Theorem 2.4 above in subspaces.

Theorem 2.11:
Let K be a countably p-compact subset of the preopen subspace Y of a strongly p-normal space X. Then K is relatively preclosed in Y
(a) if and only if K is a relatively pre-F_σ set in Y and $Y - K$ is of the form $G_Y \cup F$,
(b) if any one of the following conditions holds:
(i) K is an pre-F_σ in X and $Y - K$ is of the form $G_Y \cup F_Y$;
(ii) K is an pre-F_σ in X and $Y - K$ is a union of relatively closed subsets of Y;
(iii) K is a relatively pre-F_σ set in Y and $Y - K$ is a union of closed subsets of X.

Proof: (a) Since necessity is obvious for any set K, we need prove only the sufficient part. Let $K = \bigcup_{i=1}^{\infty} P_n$, where each P_n is preclosed in X . Since Y is preopen set by Lemma 2.1, we have $K = \bigcup_{i=1}^{\infty}(F_n \cap Y)$, where each F_n is a preclosed set in X and $Y - K = G_Y \cup F$. We prove that $Y - K$ is relatively preopen in Y. For any $x \in Y - K$, if $x \in G_Y$, then $Y - K = G_Y \cup F$ implies $x \in G_Y \subset Y - K$. If $x \in F$, then since X is strongly p-normal and each F_n is disjoint from F, there exist, for each positive integer n, disjoint preopen sets U_n and V_n in X containing F_n and F respectively. Then $\{U_n\}_{n=1}^{\infty}$ is a countable preopen cover of K and therefore, there exists a positive integer n, such that, $K \subset \bigcup_{i=1}^{n} U_n$ and $F \subset \bigcap_{i=1}^{n} V_n$. Then $U = \bigcup_{i=1}^{n} U_n$ and $V = \bigcap_{i=1}^{n} V_n$ are disjoint preopen sets such that $x \in V \subset U^C \subset Y - K$. It follows that $x \in V \cap Y \subset Y - K$, where $V \cap Y$ is relatively preopen in Y (Lemma 1.5). Therefore, $Y - K$ is relatively preopen and hence K is preclosed in Y.
(b) The proof is similar to that of part (a) above.

III. EXAMPLES

Example 3.1 below shows that in a space which does not have strong Hewitt representation,
(i) a preopen subspace of strongly p-normal need not be strongly p-normal and (ii) a preopen subspace of pre-R_1 need not be pre-R_1.

Example 3.1:
Let $X = \mathbf{Z}^+$**,** together with the topology, $T = \{G \subset X \mid G = \varnothing \text{ or } \{1, 2\} \subset G\}$.Then (X,T) strongly p-normal space which in not normal and a pre-R_1space which is not R_1 where preopen sets of X do not form a topology. Let Y be any set containing {1} but not containing {2}.Y is a preopen subspace of X which is neither strongly p-normal space nor a pre-R_1 space.

Example 3.2 below shows, (i) a strongly p-normal space which is not pre-R_1, (ii) a preopen subspace of strongly p-normal space need not be strongly p-normal even if the space has strong Hewitt representation, (iii) "$G_Y \cup F$" in cannot be replaced by "$G_Y \cup F_Y$" or "$G \cup F_Y$" in Theorem 2.9(a) and Corollary 2.10(a).

Example 3.2 ([c.f. 15; 8.1 Problem 1 and 13; Example 27]):
Let $Y = N \cup x_1 \cup x_2$, such that x_1, x_2 are two distinct points and N is any infinite set neither containing x_1 nor x_2 . We topologize Y with topology T by calling any subset of N open and calling any set containing x_1 or x_2 open if and only if it contains all but finite number of points in N. Let $p \notin X$ and $X^* = Y \cup p$, with the topology $T^* = \{G \subset X^* : G \in T \text{ or } G = X^*\}$. Then (X*, T*) is a strongly p-normal space but (Y, T) is not strongly p-normal space. Let $K = G \cup x_1$, where G contains all but finite number of points in N, K is a strongly compact (countably p-compact) set which is union of relatively closed and preclosed sets of Y but K^C is not of the of the form $G_Y \cup F$ or $G_Y \cup C$ and K is not preclosed.

Example 3.3 below shows that if a space X has Strong Hewitt representation then subspace Y of X need not have Strong Hewitt representation.

Example 3.3: Let X = {a,b,c,d}**,** together with the topology T = { ∅,{a},{a, b, c},X}. Then preopen sets of (X, T) form a topology. Further, Y = {b,c,d} is a preclosed subspace of X the preopen sets of which do not form a topology.

REFERENCES

[1] M. Caldas, S. Jafari and T. Noiri, "Characterizations of pre-R_0 and pre-R_1 topological spaces", *Topology Proceedings*, **25,** 2000, 17-30.
[2] S. N. El-Deeb, I. A. Hasanein, A. S. Mashhour and T. Noiri, "On p-regular spaces", *Bull. Math. Soc. Sci. Math. R.S. Roumanie*, **27,** 1983, 311-315.
[3] M. Ganster, "Preopen sets and resolvable spaces", *Kyngpook, Math. J.*, **27(2)**, 1987, 135-143.
[4] G. L. Garg and N. Singh, "When a compact (countably compact) set is closed", *Acta. Math. Hungar.*, **94(3),** 2002, 233-239.
[5] G. L. Garg and N. Singh, "When a compact (countably compact) set is closed II", *Acta. Math. Hungar.*, **105(4)**, 2004, 331-337.
[6] S. Jafari and T. Noiri, "More on strongly compact spaces", *Missouri J. Math. Sci.*, **19(1),** 2007, 52-61.
[7] A. Kar and P. Bhattacharyya, "Some weak separation axioms", *Bull Calcutta Math.Soc.*, **82,** 1990, 415-422.
[8] A. S. Mashhour, M. E. Abd El-Monsef, and S. N. El-Deeb, "On precontinuous and weak precontinuous mappings", *Proc. Math. Phys. Soc. Egypt*, **53,** 1982, 47-53.
[9] A. S. Mashhour, M. E. Abd El-Monsef, I. A. Hasanein and T. Noiri, "Strongly compact spaces", *Delta J. Sci*., **8 (1),** 1984, 30–46.
[10] N. S. Noorie and A. Singh, "Charachterizations of pre-R_0, pre-R_1 spaces and p*closedness of strongly compact (countably p-compact) sets", *Journal of Advanced Studies in Topology*, **5(1)**, 2014, 50-56.
[11] Paul and Bhattacharyya, "On p-normal spaces", *Soochow J. Math.*, **21(3),** 1995, 273-289.
[12] Paul and Bhattacharyya, "On Pre-Urysohn Space", *Bull, Malaysian Math. Soc. (Second Series)*, **22,** 1999, 23-34.
[13] L. A. Steen and J. A. Seebach, Jr., *Counterexamples in topology*, Holt, Rinehart and Winston, Inc., 1970.
[14] S. S. Thakur and P. Paik, "Countably p-compact spaces", *Sci. Phys. Sci.*, **1(1),** 1989, 48–51.
[15] A. Wilansky, *Topology for Analysis*, Xerox College Publishing, Lexington, Massa-chusetts, Toronto, 1970.