Common Fixed Point Theorems for Compatible Mappings in Metric Spaces

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Abstract

The aim of this paper to establish unique common fixed point theorems for compatible mappings in complete metric spaces and also illustrate the main theorem through a example.

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1.INTRODUCTION

In 1976, Jungck [5] investigated an inter-dependence between commuting mappings and fixed points .Singh [13] further generalized the above result and proved unique common fixed point theorem two continuous and commuting mappings S and T from a complete metric space (X,d) .Ranganathan [12] has further generalized the result of Jungck [5] which gives criteria for the existence of a fixed point.Fisher[3] also proved the unique common fixed point theorem for two commuting mapping.Some common fixed point theorem of three and four commuting mapping were proved by Fisher[4],Khan and Imdad[8] and Lohani and Badshah[10] kang and kim[9].

In 1982 Sessa[14] defined weak commutativity in the theorem of Jungck[5] and its various generalizations by introducing the weak commutativity.Jungck[6] introduced again more generalized commutativity,the so called compatibility,which is more general than that of weak commutativity.After that Jungck[7] coined the term of compatible mappings in order to generalize the concept of weak commutativity.compatible mapping received much attention in recent years(see 1,2,11,15,16).

The main purpose of this paper is to present fixed point results for four self maps satisfying a new contractive condition by using the concept of compatible maps in a complete metric space. Which is generalization of the result of Badshah,Chauhan and Sharma[1] by using another type of rational expression. To illustrate our main theorems, an example is also given.

2. Definition 2.1: If S and T are mappings from a metric space (X, d) into itself, are called commuting on X, if

$$d(STx,TSx)=0 \quad \text{for all } x \text{ in } X.$$

Definition 2.2: According to Sessa [14] two self maps S and T defined on metric space (X,d) are said to be weakly commuting maps if and only if $d(STx,TSx)\le d(Sx,Tx)$, for all $x\in X$.

Definition 2.3: If S and T are mappings from a metric space (X, d) into itself, are called compatible on X, if $\lim_{m\to\infty} d(STx_m,TSx_m)=0$,whenever $\{x_m\}$ is a sequence in X such that $\lim_{m\to\infty} Sx_m=\lim_{m\to\infty} Tx_m=x$ for some point x in X.

Clearly , S and T are compatible mappings on X, then $d(STx,TSx)=0$,when $d(Sx,Tx)=0$ for some point x in X.

Note that weakly commuting mappings are compatible, but the converse is not necessarily true.

3. Lemma 3.1[6]: Let S and T be compatible mappings from a metric space (X, d) into itself. Suppose that $\lim_{m\to\infty} Sx_m = \lim_{m\to\infty} Tx_m = x$ for some point x in X.

Then $\lim_{m\to\infty} TSx_m = Sx$, if S is continuous.

Now, let P,Q, S and T be mappings from a complete metric space (X, d) into itself satisfying the conditions

$$S(X)\subset Q(X), T(X)\subset P(X) \qquad \ldots(3.1)$$

and

$$d(Sx,Ty) \leq \begin{cases} \dfrac{\alpha\{[d(Px,Sx)]^3+[d(Qy,Ty)]^3\}}{[d(Px,Sx)]^2+[d(Qy,Ty)]^2} + \dfrac{\beta\{[d(Px,Qy)]^2+[d(Px,Ty)]^2\}}{[2d(Px,Qy)+3d(Px,Ty)]} + \gamma[d(Px,Sx)+d(Px,Ty)] & ;\ if\ D_1\neq 0, D_2\neq 0 \\ 0 & ;\ if\ D_1=0, D_2=0 \end{cases}$$

$$\ldots(3.2)$$

Where $D_1 = [d(Px,Sx)]^2 + [d(Qy,Ty)]^2$ and

$$D_2 = [2d(Px,Qy) + 3d(Px,Ty)]$$

For all $x, y \in X$ where $\alpha,\beta,\gamma \geq 0$ and $\alpha + 2\beta + \gamma <$ 1. Then for an arbitrary point $x_0 \in X$,by (3.1) we choose a point $x_1$in X such that $Qx_1 = Sx_0$and for this point x_1,there exists a point $x_2$in X such that $Px_2 = Tx_1$and so on. Proceeding in the similar manner, we can define a sequence $\{y_m\}$in X such that

$$y_{2m+1} = Qx_{2m+1} = Sx_{2m} \text{ and } y_{2m} = Px_{2m} = Tx_{2m-1} \qquad \ldots(3.3)$$

Lemma 3.2[7]: Let P, Q, S and T be mappings from a metric space (X, d) into itself satisfying the conditions (3.1) and (3.2).Then the sequence $\{y_m\}$defined by (3.3) is a Cauchy sequence in X.

4. Main Result:

Theorem 4. 1: Let P, Q, S and T be mappings from a metric space (X, d) into itself satisfying the conditions (3.1) and (3.2) and suppose that

One of P, Q, S and T is continuous, …(4.1.1)

Pairs (S, P) and(T, Q) are compatible on X …(4.1.2)

Then P, Q, S and T have a unique common fixed point in X.

Proof: Let $\{y_m\}$be the sequence in X defined by (3.3) .By lemma 3.2, $\{y_m\}$is a Cauchy sequence and hence converges to some point u in X. Consequently ,the subsequences $\{Sx_{2m}\}, \{Px_{2m}\}, \{Tx_{2m-1}\}$and $\{Qx_{2m+1}\}$of sequence $\{y_m\}$also converges to u.

Now suppose that P is continuous. Since S and P are compatible on X, lemmas 3.1 gives that

p^2x_{2m} and $Spx_{2m} \to pu$ as $m\to\infty$.

Consider …(3.2)

$$d(SPx_{2m},Tx_{2m-1}) \leq \frac{\alpha\{[d(P^2x_{2m},SPx_{2m})]^3+[d(Qx_{2m-1},Tx_{2m-1})]^3\}}{[d(P^2x_{2m},SPx_{2m})]^2+[d(Qx_{2m-1},Tx_{2m-1})]^2} + \frac{\beta\{[d(P^2x_{2m},Qx_{2m-1})]^2+[d(P^2x_{2m},Tx_{2m-1})]^2\}}{[2d(P^2x_{2m},Qx_{2m-1})]+[3d(P^2x_{2m},Tx_{2m-1})]} + \gamma[d(P^2x_{2m},SPx_{2m})+d(P^2x_{2m},Tx_{2m-1})]$$

Letting $m\to\infty$ and using above results we get

$$d(Pu,u) \leq \alpha[d(Pu,Pu)+d(u,u)] + \beta[d(Pu,u)]+d(Pu,u)] + \gamma[d(Pu,Pu)]+d(Pu,u)]$$

$$d(Pu,u) \leq 2\beta d(Pu,u) + \gamma d(Pu,u)$$

$(1-2\beta-\gamma)d(Pu,u) \leq 0$ so that $Pu = u$.

Again consider

$$d(Su,Tx_{2m-1}) \le$$
$$\frac{\alpha\{[d(Pu,Su)]^3+[d(Qx_{2m-1},Tx_{2m-1})]^3\}}{[d(Pu,Su)]^2+[d(Qx_{2m-1},Tx_{2m-1})]^2}$$
$$+\frac{\beta\{[d(Pu,Qx_{2m-1})]^2+[d(Pu,Tx_{2m-1})]^2\}}{[2d(Pu,Qx_{2m-1})]+[3d(Pu,Tx_{2m-1})]}$$
$$+\gamma[d(Pu,Su)+d(Pu,Tx_{2m-1})]$$
$$\le \alpha[d(Pu,Su)+d(Qx_{2m-1},Tx_{2m-1})]$$
$$+\beta[d(Pu,Qx_{2m-1})+d(Pu,Tx_{2m-1})]$$
$$+\gamma[d(Pu,Su)+d(Pu,Tx_{2m-1})]$$

Letting $m \to \infty$ and using above results we get

$$d(Su,u) \le \alpha[d(u,Su)+d(u,u)]$$
$$+\beta[d(u,u)]+d(u,u)]$$
$$+\gamma[d(u,Su)]+d(u,u)]$$
$$d(Su,u) \le \alpha d(u,Su)+\gamma d(u,Su)$$

$(1-\alpha-\gamma)d(u,Su) \le 0$ so that $Su = u$.

Since $S(X) \subset Q(X)$ and hence there exist a point v in X, such that

$$u = Su = Qv.$$

$$d(u,Tv) = d(Su,Tv)$$
$$\le \frac{\alpha\{[d(Pu,Su)]^3+[d(Qv,Tv)]^3\}}{[d(Pu,Su)]^2+[d(Qv,Tv)]^2}$$
$$+\frac{\beta\{[d(Pu,Qv)]^2+[d(Pu,Tv)]^2\}}{[2d(Pu,Qv)]+[3d(Pu,Tv)]}$$
$$+\gamma[d(Pu,Su)+d(Pu,Tv)]$$
$$\le \frac{\alpha\{[d(u,u)]^3+[d(u,Tv)]^3\}}{[d(u,u)]^2+[d(u,Tv)]^2}$$
$$+\frac{\beta\{[d(u,u)]^2+[d(u,Tv)]^2\}}{[2d(u,u)]+[3d(u,Tv)]}$$
$$+\gamma[d(u,u)+d(u,Tv)]$$

$$d(u,Tv) \le \alpha d(u,Tv)+\frac{1}{3}\beta d(u,Tv)+\gamma d(u,Tv)$$
$$(1-\alpha-\frac{\beta}{3}-\gamma)d(u,Tv) \le 0$$

So that $u = Tv$.

Since T and Q are compatible on X and $Qv = Tv = u$ and $d(QTv,TQv) = 0$ and hence $Qu = QTv = TQv = Tu$.

Moreover by (3.2) ,we obtain

$$d(u,Qu) = d(Su,Tu)$$
$$\le \frac{\alpha\{[d(Pu,Su)]^3+[d(Qu,Tu)]^3\}}{[d(Pu,Su)]^2+[d(Qu,Tu)]^2}$$
$$+\frac{\beta\{[d(Pu,Qu)]^2+[d(Pu,Tu)]^2\}}{[2d(Pu,Qu)]+[3d(Pu,Tu)]}$$
$$+\gamma[d(Pu,Su)+d(Pu,Tu)]$$
$$\le \frac{\alpha\{[d(u,u)]^3+[d(Qu,Qu)]^3\}}{[d(u,u)]^2+[d(Qu,Qu)]^2}$$
$$+\frac{\beta\{[d(u,Qu)]^2+[d(u,Qu)]^2\}}{[2d(u,Qu)]+[3d(u,Qu)]}$$
$$+\gamma[d(u,u)+d(u,Qu)]$$
$$d(u,Qu) \le \beta[d(u,Qu)+d(u,Qu)]+\gamma d(u,Qu)$$
$$d(u,Qu) \le 2\beta d(u,Qu)+\gamma d(u,Qu)$$

$(1-2\beta-\gamma)d(u,Qu) \le 0$ So that $Qu = u$.

Therefore u is a common fixed point of P, Q, S and T.

Similarly, we can also complete the proof, when Q is continuous. Since S and P are compatible on X, it follows from lemma 3.1 that

S^2x_{2m} and $PSx_{2m} \to Su$ as $m \to \infty$.

By (3.2),we have

$$d(S^2x_{2m},Tx_{2m-1}) \le$$
$$\frac{\alpha\{[d(PSx_{2m},S^2x_{2m})]^3+[d(Qx_{2m-1},Tx_{2m-1})]^3\}}{[d(PSx_{2m},S^2x_{2m})]^2+[d(Qx_{2m-1},Tx_{2m-1})]^2}$$
$$+\frac{\beta\{[d(PSx_{2m},Qx_{2m-1})]^2+[d(PSx_{2m},Tx_{2m-1})]^2\}}{[2d(PSx_{2m},Qx_{2m-1})]+[3d(PSx_{2m},Tx_{2m-1})]}$$
$$+\gamma[d(PSx_{2m},S^2x_{2m})+d(PSx_{2m},Tx_{2m-1})]$$

$$d(S^2x_{2m},Tx_{2m-1}) \le$$
$$\alpha[d(PSx_{2m},S^2x_{2m})+d(Qx_{2m-1},Tx_{2m-1})]$$
$$+\beta[d(PSx_{2m},Qx_{2m-1})+d(PSx_{2m},Tx_{2m-1})]$$
$$+\gamma[d(PSx_{2m},S^2x_{2m})+d(PSx_{2m},Tx_{2m-1}).$$

Letting $m \to \infty$ and using above results we get

$$d(Su,u) \le \alpha[d(Su,Su)+d(u,u)] + \beta[d(Su,u)]+d(Su,u)] + \gamma[d(Su,Su)]+d(Su,u)]$$

$$d(u,Su) \le 2\beta d(Su,u)+\gamma d(Su,u)$$

$(1-2\beta-\gamma)d(Su,u) \le 0$ so that $Su = u.$

Hence by (3.1), there exists a point w in X, such that $u = Su = Qw$.

$$d(S^2x_{2m},Tw) \le \frac{\alpha\{[d(PSx_{2m},S^2x_{2m})]^3+[d(Qw,Tw)]^3\}}{[d(PSx_{2m},S^2x_{2m})]^2+[d(Qw,Tw)]^2} + \frac{\beta\{[d(PSx_{2m},Qw)]^2+[d(PSx_{2m},Tw)]^2\}}{[2d(PSx_{2m},Qw)]+[3d(PSx_{2m},Tw)]} + \gamma[d(PSx_{2m},S^2x_{2m})+d(PSx_{2m},Tw)].$$

$$d(S^2x_{2m},Tw) \le \alpha[d(PSx_{2m},S^2x_{2m})+d(Qw,Tw)] + \beta[d(PSx_{2m},Qw)+d(PSx_{2m},Tw)] + \gamma[d(PSx_{2m},S^2x_{2m})+d(PSx_{2m},Tw)]$$

Letting $m \to \infty$ and using above results we get

$$d(Su,Tw) \le \alpha[d(Su,Su)+d(u,Tw)] + \beta[d(Su,u)+d(Su,Tw)] + \gamma[d(Su,Su)+d(Su,Tw)].$$

$$d(Su,Tw) \le \alpha d(u,Tw)+\beta d(u,Tw)+\gamma d(u,Tw)$$

$(1-\alpha-\beta-\gamma)d(u,Tw) \le 0$ so that $u = Tw.$

Since T and Q are compatible on X and $Qw = Tw = u$ $d(QTw,TQw) = 0$ and hence $Qw = QTw = TQw = Tu.$

Moreover by (3.2),we have

$$d(Sx_{2m},Tu) \le \frac{\alpha\{[d(Px_{2m},Sx_{2m})]^3+[d(Qu,Tu)]^3\}}{[d(Px_{2m},Sx_{2m})]^2+[d(Qu,Tu)]^2} + \frac{\beta\{[d(Px_{2m},Qu)]^2+[d(Px_{2m},Tu)]^2\}}{[2d(Px_{2m},Qu)]+[3d(Px_{2m},Tu)]} + \gamma[d(Px_{2m},Sx_{2m})+d(Px_{2m},Tu)].$$

$$\le \alpha[d(Px_{2m},Sx_{2m})+d(Qu,Tu)] + \beta[d(Px_{2m},Qu)+d(Px_{2m},Tu)] + \gamma[d(Px_{2m},Sx_{2m})+d(Px_{2m},Tu)].$$

Letting $m \to \infty$ and using above results we get

$$d(u,Tu) \le \alpha[d(u,u)+d(Qu,Tu)] + \beta[d(u,Qu)+d(u,Tu)] + \gamma[d(u,u)+d(u,Tu)].$$

$$d(u,Tu) \le \alpha d(Qu,Qu)+2\beta d(u,Tu)+\gamma d(u,Tu).$$

$(1-2\beta-\gamma)d(u,Tu) \le 0$ so that $u = Tu.$

Since $T(X) \subset P(X)$,there exists a point z in X such that $u = Tu = Pz.$

$$d(Sz,u) = d(Sz,Tu)$$

$$\le \frac{\alpha\{[d(Pz,Sz)]^3+[d(Qu,Tu)]^3\}}{[d(Pz,Sz)]^2+[d(Qu,Tu)]^2} + \frac{\beta\{[d(Pz,Qu)]^2+[d(Pz,Tu)]^2\}}{[2d(Pz,Qu)]+[3d(Pz,Tu)]} + \gamma[d(Pz,Sz)+d(Pz,Tu)].$$

$$\le \alpha[d(u,Sz)+d(Qu,Tu)] + \beta[d(u,Qu)+d(u,Tu)] + \gamma[d(u,Sz)+d(u,Tu)]$$

$$d(Sz,u) \le \alpha[d(u,Sz)+d(u,u)] + \beta[d(u,u)+d(u,u)] + \gamma[d(u,Sz)+d(u,u)].$$

$$d(Sz,u) \le \alpha d(u,Sz)+\gamma d(u,Sz)$$

$(1-\alpha-\gamma)d(Sz,u) \le 0.$ so that $Sz = u.$

Since S and P are compatible on X and $Sz = Pz = u$, $d(PSz,SPz) = 0$ and hence $Pu = PSz = SPz = Su.$ Therefore, u is a common fixed point of P ,Q ,S and T. Similarly,we can complete the proof, when T is continuous.

Finally in order to prove the uniqueness of u , suppose u and z, $u \ne z$, are common fixed points of P ,Q ,S and T.

Then by (3.2) ,we obtain

$$d(u,z) = d(Su,Tz)$$

$$\le \frac{\alpha\{[d(Pu,Su)]^3+[d(Qz,Tz)]^3\}}{[d(Pu,Su)]^2+[d(Qz,Tz)]^2} + \frac{\beta\{[d(Pu,Qz)]^2+[d(Pu,Tz)]^2\}}{[2d(Pu,Qz)]+[3d(Pu,Tz)]} + \gamma[d(Pu,Su)+d(Pu,Tz)].$$

$$\le \alpha[d(u,u)+d(z,z)] + \beta[d(u,z)+d(u,z)] + \gamma[d(u,u)+d(u,z)].$$

$(1-2\beta-\gamma)d(u,z) \le 0.$so that $u = z.$Therefore, u is a common fixed point of P ,Q ,S and $T.$

The following corollary follows from theorem 4.1.

Corollary 4.1: Let P ,Q ,S and T be mappings from a complete metric space (X,d) into itself satisfying the conditions (3.1) and (3.2).Then P ,Q ,S and T have a unique common fixed point in X.

Theorem 4.2: Let P ,Q ,S and T be mappings from a complete metric space (X,d) into itself satisfying the condition (4.1.1),for some positive integers s ,t, p and q, following condition are as follows:

$$S^s(X) \subset Q^q(X), T^t(X) \subset P^p(X) \qquad \ldots(4.2.1)$$

and

$$d(S^s x, T^t y) \le \begin{cases} \dfrac{\alpha\{[d(P^p x,S^s x)]^3+[d(Q^q y,T^t y)]^3\}}{[d(P^p x,S^s x)]^2+[d(Q^q y,T^t y)]^2} + \dfrac{\beta\{[d(P^p x,Q^q y)]^2+[d(P^p x,T^t y)]^2\}}{[2d(P^p x,Q^q y)+3d(P^p x,Q^q y)]} + \gamma[d(P^p x,S^s x)+d(P^p x,T^t y)] & ; \ if \ D_1 \ne 0, D_2 \ne 0 \\ 0 & ; \ if \ D_1 = 0, D_2 = 0 \end{cases} \qquad \ldots(4.2.2)$$

Where $D_1 = [d(P^p x, S^s x)]^2 + [d(Q^q y, T^t y)]^2$ and

$$D_2 = [2d(P^p x, Q^q y) + 3d(P^p x, T^t y)]$$

For all $x, y \in X$ where $\alpha, \beta, \gamma \ge 0$ and

$\alpha + 2\beta + \gamma <$ 1.Suppose that S and T are commuting with P and Q resepetively.Then P ,Q,S and T have a common fixed point in X.

Now, we give the example to justify our results.

Example: Let X = [0, 1) ,with $d(x, y) = |x - y|$

$$Px = Qx = \left\{ \frac{1}{5} - x \right. \qquad ; x \in [0,1)$$

$$Tx = Sx = \left\{ \frac{1}{10} \right. \qquad ; x \in [0,1)$$

In fact that $SP(0) = \frac{1}{10} = PS(0)$ so that $SPx = PSx$ on [0,1).

Similarly $QT(0) = \frac{1}{5} - \frac{1}{10} = \frac{1}{10}$ and $TQ(0) = \frac{1}{10}$ so that $QTx = TQx$ on [0,1).

Which shows that the pair (S,P) and (T,Q) are weakly compatible.

Let $x_n = \left(\frac{1}{10} - \frac{1}{10^n} \right)$ be a sequence in X converges to $\frac{1}{10}$ as $n \to \infty$.Hence , for such x_n sequences Sx_n, Tx_n, Px_n, Qx_n converges to $\frac{1}{10}$ as $n \to \infty$.

$$PSx_n \to \frac{1}{10}, SPx_n \to \frac{1}{10} \ as \ n \to \infty.$$

Therefore ,

$$\lim_{n\to\infty} d(SPx_n, PSx_n) = d\left(\frac{1}{10}, \frac{1}{10} \right) = 0$$

Showing that the pair (S,P) and (T,Q) are compatible. We can easily seen that condition (3.1),(3.2) and all the condition of theorem 4.1 is satisfied, then from above example (4.2.1) is clear that $\frac{1}{10}$ is a fixed point.

Remark: In the above example ,the mappings S,T,P and Q are continuous and the pair (S,P) and (T,Q) , compatible and weakly compatible.

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