Some Results on Super Harmonic Mean Graphs

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Abstract - Let G be a graph with p vertices and q edges. Let f: V(G) → {1, 2, …, p + q} be a injective function. For a vertex labeling f, the induced edge labeling f*(e = uv) is defined by f*(e) = $\left\lceil \frac{2f(u)f(v)}{f(u)+f(v)} \right\rceil$ or $\left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor$. Then f is called a Super harmonic mean labeling if f(V(G))∪{f(e) / e ϵ E(G)} = {1, 2, …, p + q}. A graph which admits Super harmonic mean labeling is called Super harmonic mean graphs. In this paper, we investigate Super harmonic mean labeling of some graphs.

Key words - **Graph, Super harmonic mean labeling, Super harmonic mean graphs**

I. INTRODUCTION

We begin with simple, finite, connected and undirected graph G(V, E) with p vertices and q edges. For a detailed survey of graph labeling we refer to Gallian[1]. For all other standared terminology and notations we follow Harary[2]. S. Somasundram and R. Ponraj introduced mean labeling of graphs in [3]. R. Ponraj and D. Ramya introduced Super mean labeling of graphs in [4]. S. Somasundram and S.S. Sandhya introduced the concept Harmonic mean labeling in [5] and studied their behavior in [6, 7, 8]. S. Sandhya and C. David Raj introduced Super harmonic labeling in [9]. In this paper, we investigate Super harmonic mean labeling of some graphs. We now give the following definitions which are useful for the present investigation.

Definition 1.1. Let f: V(G) → {1, 2, …, p + q} be a injective function. For a vertex labeling f, the induced edge labeling f*(e = uv) is defined by f*(e) = $\left\lceil \frac{2f(u)f(v)}{f(u)+f(v)} \right\rceil$ or $\left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor$. Then f is called a Super harmonic mean labeling if f(V(G))∪{f(e) / e ϵ E(G)} = {1, 2, …, p + q}. A graph which admits Super harmonic mean labeling is called Super harmonic mean graphs.

Definition 1.2. The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as the graph G obtained by taking one copy of G_1(which has p_1 vertices) and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 to every vertices in the i^{th} copy of G_2.

Definition 1.3. The graph $P_n \odot K_1$ is called Comb.

Definition 1.4. The graph $C_n \odot K_1$ is called crown.

Definition 1.5. The prism D_n, $n \geq 3$ is a trivalent graph which can be defined as the Cartesian product $P_2 \times C_n$ of a path on two vertices with a cycle on n vertices. We denote a graph obtained by attaching P_2 at each vertex of outer cycle of D_n by (D_n; P_2).

II. SUPER HARMONIC MEAN LABELING FOR CONNECTED GRAPHS

Theorem 2.1 nP_m is a Super harmonic mean graph.

Proof. Let $v_{i,j}$, $1 \le i \le n$, $1 \le j \le m$ be the vertices of nP_m. Then its edge set is $E = \{u_{i,j}u_{i,j+1} / 1 \le i \le n, 1 \le j \le m-1\}$. Define a function $f: V(nP_m) \to \{1, 2, \ldots, p+q\}$ by

$$f(v_{i,j}) = (2m-1)(i-1) + 2j - 1, \ 1 \le i \le n, \ 1 \le j \le m$$

Then the induced edge labels are

$$f^*(v_{i,j}v_{i,j+1}) = (2m-1)(i-1) + 2j, \ 1 \le i \le n, \ 1 \le j \le m-1;$$

Thus f provides a Super harmonic mean labeling for nP_m.

Example 2.2. A Super harmonic mean labeling of $4P_7$ is shown in figure 2.1.

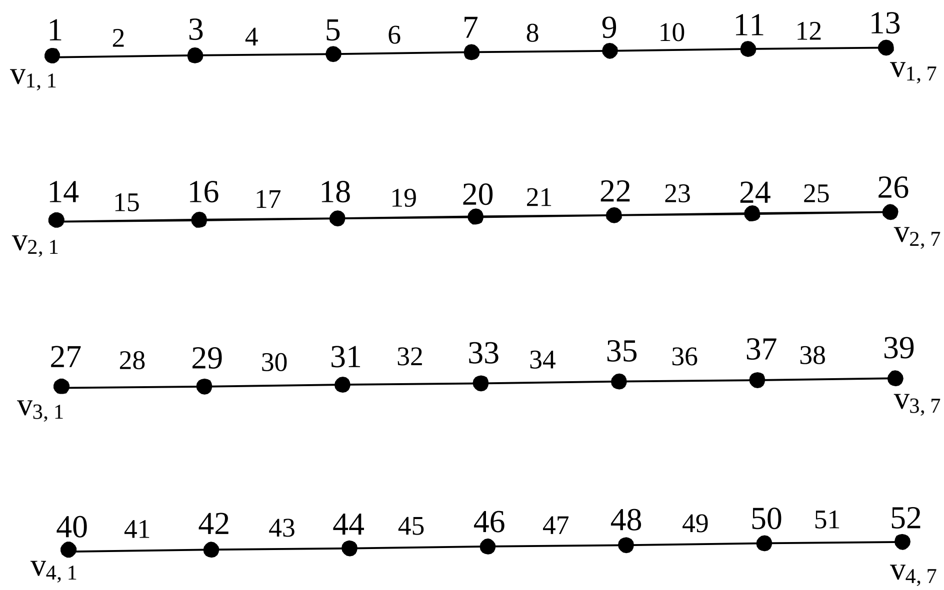

Fig. 2.1 $4P_7$

Theorem 2.3. $nK_{1,3}$ is a Super harmonic mean graph.

Proof. Let u_i, $u_{i,j}$, $1 \le i \le n$, $1 \le j \le 3$ be the vertices of $nK_{1,3}$ in which u_i is the central vertex of $K_{1,3}$. Its edge set is $E = \{u_iu_{i,j} / 1 \le i \le n, 1 \le j \le 3\}$. Define a function $f: V(nK_{1,3}) \to \{1, 2, \ldots, p+q\}$ by

$$f(u_1) = 7; \ f(u_i) = 7i - 2; \ 2 \le i \le n;$$

$$f(u_{1,1}) = 1; \ f(u_{1,2}) = 3; \ f(u_{1,3}) = 5;$$

$$f(u_{i,j}) = 7(i-1) + j, \ 2 \le i \le n, \ 1 \le j \le 2;$$

$$f(u_{i,3}) = 7i, \ 2 \le i \le n.$$

Then the induced edge labels are

$$f(u_1u_{1,1}) = 2; \ f(u_1u_{1,2}) = 4; \ f(u_1u_{1,3}) = 6;$$

$$f(u_iu_{i,j}) = 7i - 5 + j; \ 2 \le i \le n, \ 1 \le j \le 2;$$

$f(u_i u_{i,3}) = 7i - 1$, $2 \le i \le n$.

Thus both vertices and edges together get distinct labels from {1, 2, …, p+q}. Hence $nK_{1,3}$ is a Super harmonic mean graph.

Example 2.4. A Super harmonic mean labeling of $4K_{1,3}$ is given in figure 2.2.

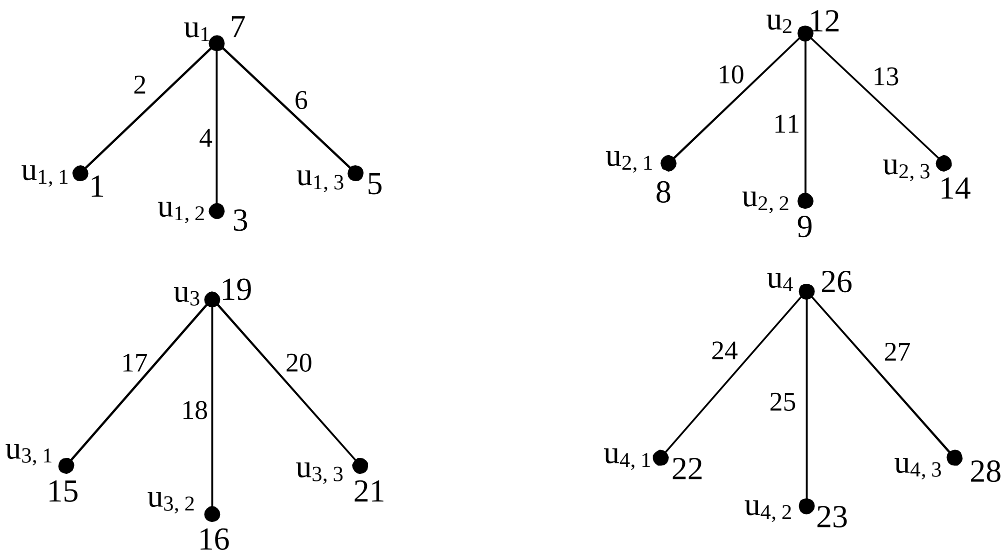

Figure 2.2. $4K_{1,3}$

Theorem 2.5. $(D_n; P_2)$ is a Super harmonic mean graph.

Proof. Let u_i and v_i be the vertices of inner and outer cycle of $(D_n; P_2)$ respectively in which u_i and v_i are adjacent, $1 \le i \le n$. Let w_i be a vertex which is joined with v_i, $1 \le i \le n$. Its edge set is $E = \{u_i u_{i+1}, u_n u_1, v_i v_{i+1}, v_n v_1 / 1 \le i \le n-1\} \cup \{u_i v_i, v_i w_i / 1 \le i \le n\}$. Define a function $f: V(D_n; P_2) \to \{1, 2, \ldots, p+q\}$ by

$f(u_1) = 7$; $f(u_2) = 14$; $f(u_i) = 7i - 2$, $3 \le i \le n$;

$f(v_1) = 3$; $f(v_2) = 11$; $f(v_i) = 7i$, $3 \le i \le n$;

$f(w_1) = 1$; $f(w_2) = 8$; $f(w_i) = 7i - 5$, $3 \le i \le n$.

Then the induced edge labels are

$f^*(u_1u_2) = 10$; $f^*(u_2u_3) = 17$; $f^*(u_iu_{i+1}) = 7i + 1$, $3 \le i \le n-1$;

$$f^*(u_nu_1) = \begin{cases} 12 \; if \; n \le 6 \\ 13 \; if \; n > 6 \end{cases};$$

$f^*(u_1v_1) = 5$;

$$f^*(u_2v_2) = \begin{cases} 13 \; if \; n \le 6 \\ 12 \; if \; n > 6 \end{cases};$$

$f^*(u_iv_i) = 7i - 1;\ 3 \le i \le n;$

$f^*(v_1v_2) = 4;\ f^*(v_2v_3) = 15;\ f^*(v_iv_{i+1}) = 7i + 3,\ 3 \le i \le n - 1;\ f^*(v_nv_1) = 6;$

$f^*(v_1w_1) = 2;\ f^*(v_2w_2) = 9;\ f^*(v_iw_i) = 7i - 3,\ 3 \le i \le n.$

Thus the vertices and edges together get distinct labels from{1, 2, …, p+q}. Thus f provides a Super harmonic mean labeling for $(D_n; P_2)$.

Example 2.6. A Super harmonic mean labeling of $(D_7; P_2)$ is shown in figure 2.3.

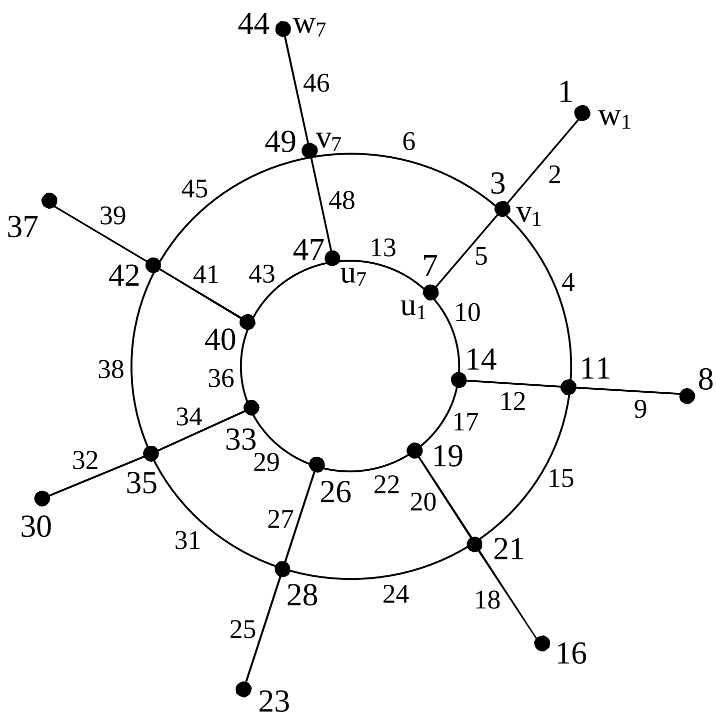

Fig. 2.3 $(D_7; P_2)$

III. SUPER HARMONIC MEAN LABELING FOR DISCONNECTED GRAPHS

In this section, we prove $C_m \cup P_n$, $(P_m \odot K_1) \cup C_n$, $(C_m \odot K_1) \cup P_n$, $(C_m \odot K_1) \cup C_n$ and $(C_m \odot K_1) \cup (P_n \odot K_1)$ are Super harmonic mean graphs.

Theorem 3.1. $C_m \cup P_n$ is a Super harmonic mean graph.

Proof. Let $u_1u_2 \ldots u_mu_1$ be the cycle C_m and $v_1v_2 \ldots v_n$ be the path P_n. Then $C_m \cup Pn$ has edge set $E = \{u_iu_{i+1}, u_nu_1 / 1 \le i \le m - 1\} \cup \{v_iv_{i+1} / 1 \le i \le m - 1\}$. Define a function f: $V(C_m \cup P_n) \to \{1, 2, \ldots, p + q\}$ by

$f(u_1) = 3$; $f(u_i) = 2(i + 1)$, $2 \le i \le m$;

$f(v_1) = 1$; $f(v_i) = 2m + 2i – 1$, $2 \le i \le n$.

Then the induced edge labels are

$f^*(u_1u_2) = 4$; $f^*(u_iu_{i+1}) = 2i + 3$, $2 \le i \le m – 1$; $f^*(u_1u_m) = 5$;

$f^*(v_1v_2) = 2$; $f^*(v_iv_{i+1}) = 2m + 2i$, $2 \le i \le n – 1$.

Thus the vertices and edges together get distinct labels from $\{1, 2, \ldots, p+q\}$. Thus f provides a Super harmonic mean labeling for $C_m \cup P_n$.

Example 3.2. A Super harmonic mean labeling of $C_9 \cup P_7$ is shown in figure 3.1.

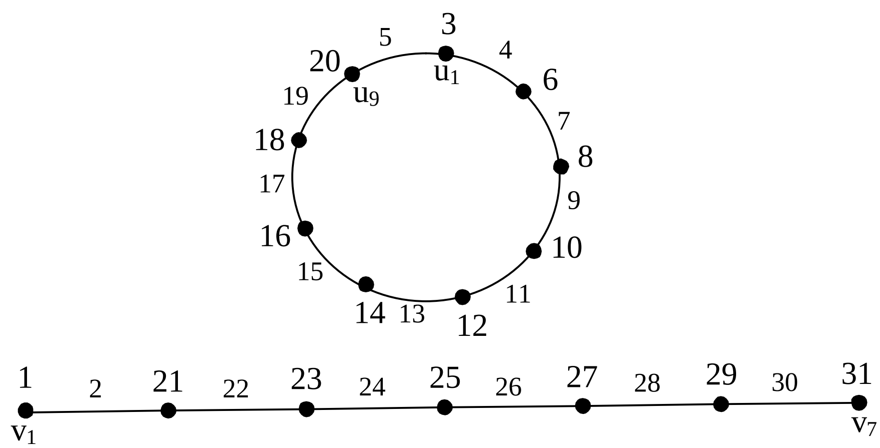

Fig. 3.1 $C_9 \cup P_7$

Theorem 3.3. $(P_m \odot K_1) \cup C_n$ is a Super harmonic mean graph.

Proof. Let $u_1u_2 \ldots u_m$ be the path P_m. Add vertices v_i such that v_i is adjacent to u_i, $1 \le i \le m$. The resultant graph is $P_m \odot K_1$. Let $w_1w_2 \ldots w_nw_1$ be the cycle C_n. Let $G = (P_m \odot K_1) \cup C_n$ whose edge set is $E = \{u_iu_{i+1} / 1 \le i \le m – 1\} \cup \{ w_iw_{i+1}, w_nw_1 / 1 \le i \le n – 1\} \cup \{u_iv_i / 1 \le i \le m\}$. Define a function $f: V(G) \rightarrow \{1, 2, \ldots, q + 1\}$ by

$f(w_1) = 3$; $f(w_i) = 2(i + 1)$, $2 \le i \le n$;

$f(u_i) = 2(m + 1) + 4i + 3$, $1 \le i \le m$;

$f(v_1) = 1$; $f(v_i) = 2(m + 2) + 4i – 2$, $2 \le i \le m$.

Then the edges are labeled with

$f^*(w_1w_2) = 4$; $f^*(w_iw_{i+1}) = 2i + 3$, $2 \le i \le n – 1$; $f^*(w_nw_1) = 5$;

$f^*(u_iu_{i+1}) = 2(m + 2) + 4i + 3$, $1 \le i \le m – 1$;

$f^*(u_1v_1) = 2$; $f^*(u_iv_i) = 2(m + 2) + 4i$, $2 \le i \le m$.

Therefore, f is a Super harmonic mean labeling for G. Hence G is a Super harmonic mean graph.

Example 3.4. A Super harmonic mean labeling of $(P_6 \odot K_1) \cup C_9$ is given in figure 3.2.

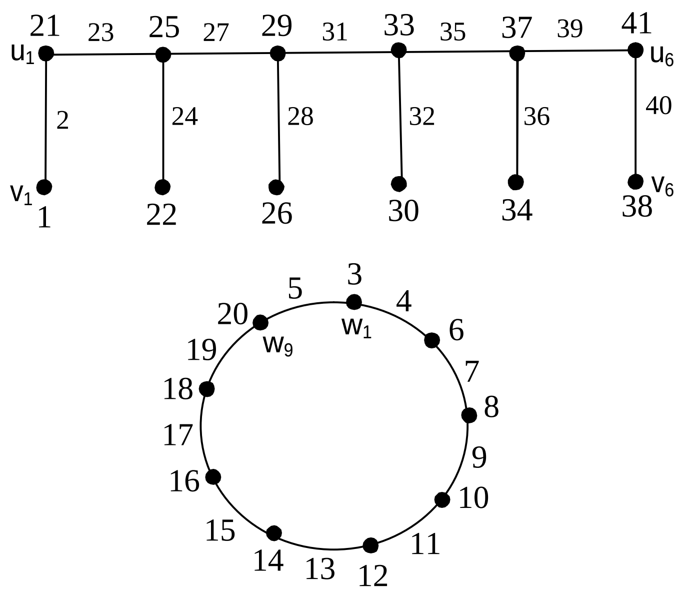

Fig. 3.2 $(P_6 \odot K_1) \cup C_9$

Theorem 3.5. $(C_m \odot K_1) \cup P_n$ is a Super harmonic mean graph.

Proof. Let $u_1u_2\ldots u_mu_1$ be the cycle C_m. Add vertices v_i such that v_i is adjacent to u_i, $1 \le i \le m$. The resultant graph is $C_m \odot K_1$. Let $w_1w_2\ldots w_n$ be the path P_n. Let $G = (C_m \odot K_1) \cup P_n$ whose edge set is $E = \{u_iu_{i+1}, u_mu_1 \,/\, 1 \le i \le m - 1\} \cup \{w_iw_{i+1} \,/ 1 \le i \le n - 1\} \cup \{u_iv_i/$ $1 \le i \le m - 1\}$. Define a function $f: V(G) \to \{1, 2, \ldots, p + q\}$ by

$f(u_1) = 3$; $f(u_i) = 4i$, $2 \le i \le m$;

$f(v_1) = 1$; $f(v_2) = 6$; $f(v_i) = 4i - 3$, $3 \le i \le m$;

$f(w_i) = 4m + 2i - 1$, $1 \le i \le n$.

Then the edges are labeled with

$f^*(u_1u_2) = 4$; $f^*(u_iu_{i+1}) = 4i + 2$, $2 \le i \le m - 1$; $f^*(u_mu_1) = 5$;

$f^*(u_1v_1) = 2$; $f^*(u_iv_i) = 4i - 1$, $2 \le i \le m$;

$f^*(w_iw_{i+1}) = 4m + 2i$, $1 \le i \le n - 1$.

In the view of the above labeling pattern, f provides a Super harmonic mean labeling for G. Hence G is a Super harmonic mean graph.

Example 3.6. A Super harmonic mean labeling of $(C_6 \odot K_1) \cup P_9$ is given in figure 3.3.

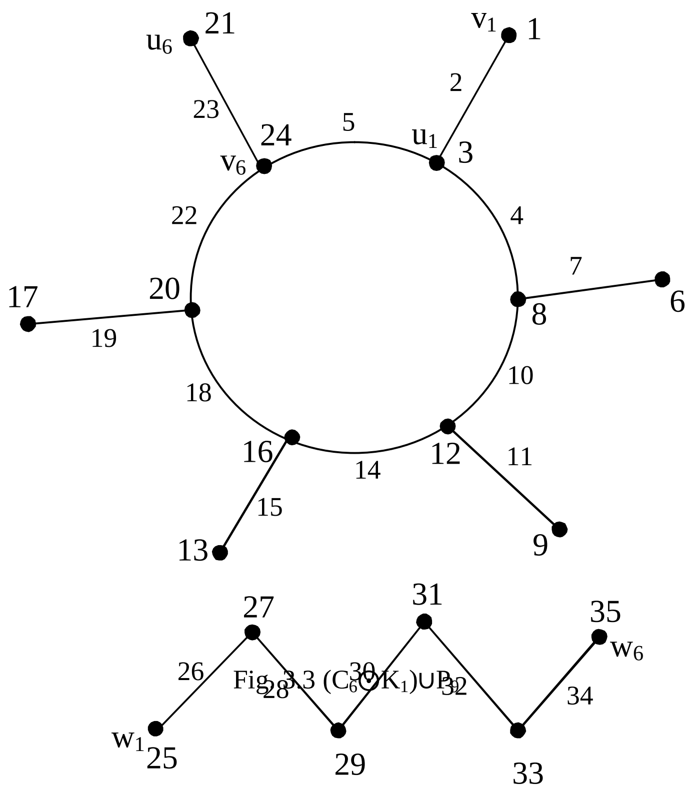

Fig. 3.3 $(C_6 \odot K_1) \cup P_6$

Theorem 3.7. $(C_m \odot K_1) \cup C_n$ is a Super harmonic mean graph.

Proof. Let $u_1u_2\ldots u_mu_1$ be the cycle C_m. Add vertices v_i such that v_i is adjacent to u_i, $1 \leq i \leq m$. The resultant graph is $C_m \odot K_1$. Let $w_1w_2\ldots w_nw_1$ be the cycle C_n. Let $G = (C_m \odot K_1) \cup C_n$ whose edge set is $E = \{u_iu_{i+1}, u_mu_1 \ / \ 1 \leq i \leq m-1\} \cup \{w_iw_{i+1}, w_nw_1 \ / \ 1 \leq i \leq n-1\} \cup \{u_iv_i \ / \ 1 \leq i \leq m\}$.

A Super harmonic mean labeling of $(C_m \odot K_1) \cup C_n$ when $m, n \leq 4$ are given in figures 3.4, 3.5, 3.6 and 3.7 respectively.

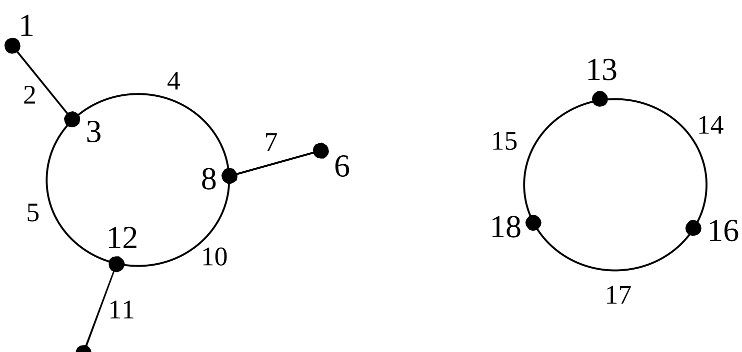

Fig. 3.4 $(C_3 \odot K_1) \cup C_3$

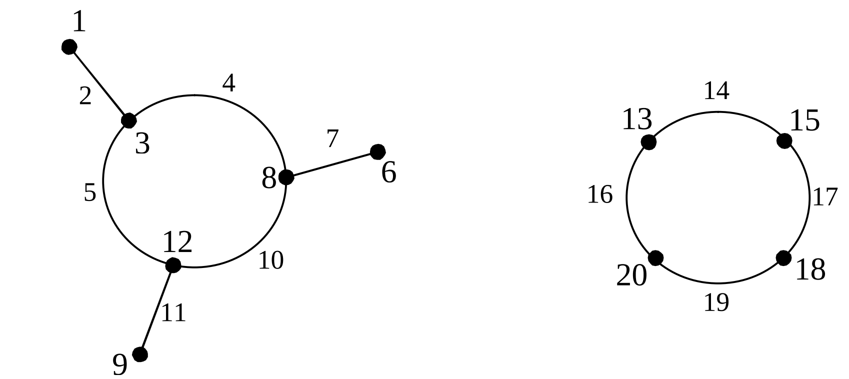

Fig. 3.5 $(C_3 \odot K_1) \cup C_4$

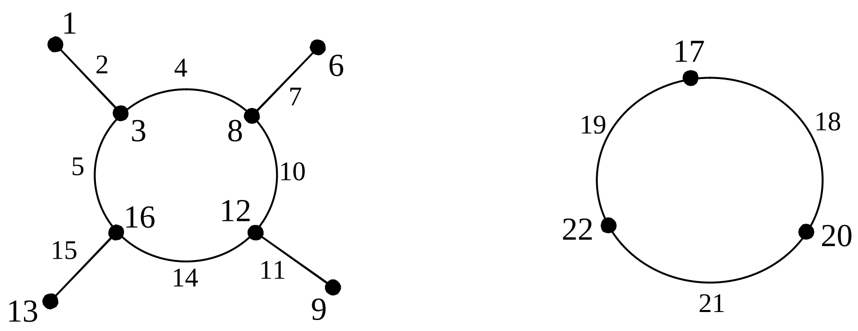

Fig. 3.6 $(C_4 \odot K_1) \cup C_3$

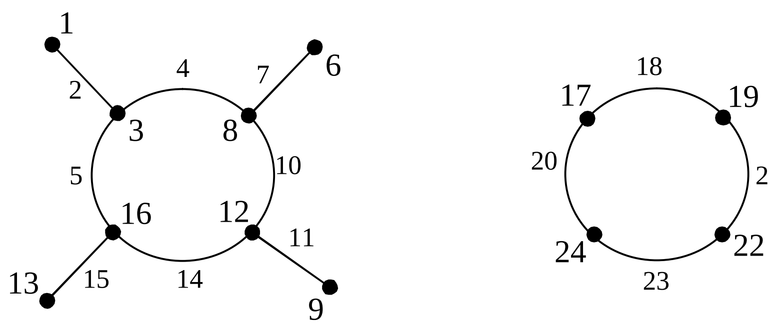

Fig. 3.7 $(C_4 \odot K_1) \cup C_4$

Assume that m, n > 4. Define a function f: $V(G) \to \{1, 2, \ldots, p + q\}$ by

$f(u_1) = 3$; $f(u_i) = 4i + 3$, $2 \le i \le m$;

$f(v_1) = 1$; $f(v_2) = 9$; $f(v_i) = 4i$, $3 \le i \le m$;

$f(w_1) = 4$; $f(w_i) = 4m + 2i$, $2 \le i \le n$.

Then the edges are labeled with

$f^*(u_1u_2) = 5$; $f(u_iu_{i+1}) = 4i + 5$, $2 \le i \le m - 1$; $f^*(u_nu_1) = 6$;

$f^*(u_1v_1) = 2$; $f^*(u_iv_i) = 4i + 2$, $2 \le i \le m$;

$f^*(w_1w_2) = 7$; $f^*(w_iw_{i+1}) = 4m + 2i + 1$, $2 \le i \le n - 1$; $f^*(w_nw_1) = 8$.

In the view of the above labeling pattern, f provides a Super harmonic mean labeling for G. Hence G is a Super harmonic mean graph.

Example 3.8. A Super harmonic mean labeling of $(C_5 \odot K_1) \cup C_7$ is given in figure 3.8.

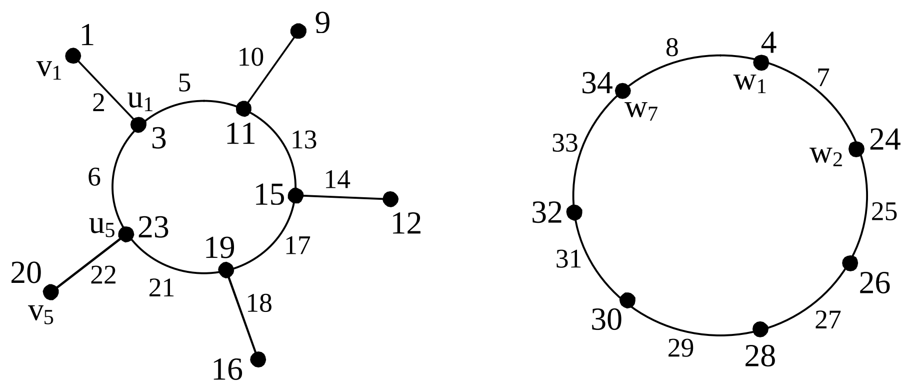

Figure 3.8. $(C_5 \odot K_1) \cup C_7$

Theorem 3.9. $(C_m \odot K_1) \cup (P_n \odot K_1)$ is a Super harmonic mean graph.

Proof. Let $u_1u_2 \ldots u_mu_1$ be the cycle C_m and let v_i be the vertex which is joined to the vertex u_i, $1 \le i \le m$, of the cycle C_m. The resultant graph is $C_m \odot K_1$. Let $s_1s_2 \ldots s_n$ be the path P_n and let t_i be the vertex which is joined to the vertex s_i, $1 \le i \le n$, of the path P_n. The resultant graph is $P_n \odot K_1$. Let $G = (C_m \odot K_1) \cup (P_n \odot K_1)$. Define a function $f: V(G) \to \{1, 2, \ldots, p + q\}$ by

$f(u_1) = 3$; $f(u_i) = 4i$, $2 \le i \le m$;

$f(v_1) = 1$; $f(v_2) = 6$; $f(v_i) = 4i - 3$, $3 \le i \le m$;

$f(s_i) = 4m + 4i - 1$; $1 \le i \le n$;

$f(t_1) = 4m + 1$; $f(t_i) = 4(m - 1) + 4i$, $2 \le i \le n$.

Then the edges are labeled with

$f^*(u_1u_2) = 4$; $f^*(u_iu_{i+1}) = 4i + 2$, $2 \le i \le m - 1$; $f^*(u_mu_1) = 5$;

$f^*(u_1v_1) = 2$; $f^*(u_iv_i) = 4i - 1$, $2 \le i \le m$;

$f^*(s_is_{i+1}) = 4m + 4i + 1$, $1 \le i \le n - 1$;

$f^*(s_it_i) = 4m + 4i - 2$, $1 \le i \le n$.

Thus f provides a Super harmonic mean labeling for G. Hence G is a Super harmonic mean graph.

Example 3.10. A Super harmonic mean labeling of $(C_6 \odot K_1) \cup (P_7 \odot K_1)$ is given in figure 3.9.

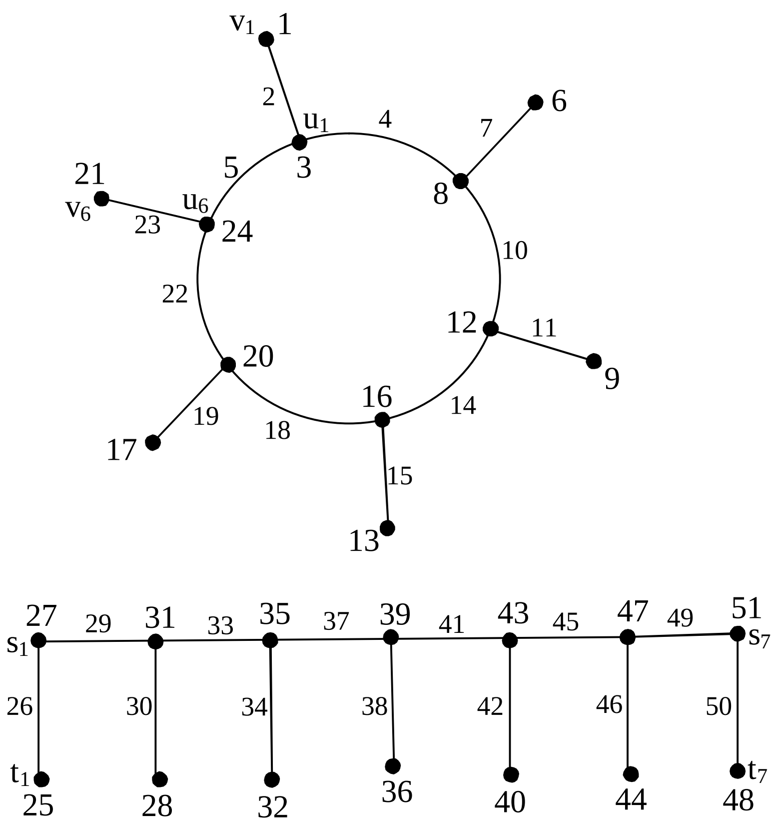

Fig. 3.9 $(C_6 \odot K_1) \cup (P_7 \odot K_1)$

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